Introduction to Quantum Mechanics

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Lecture - 02
Equivalence of the Heisenberg and the Schroedinger formulations - The x and p operators and the quantum condition

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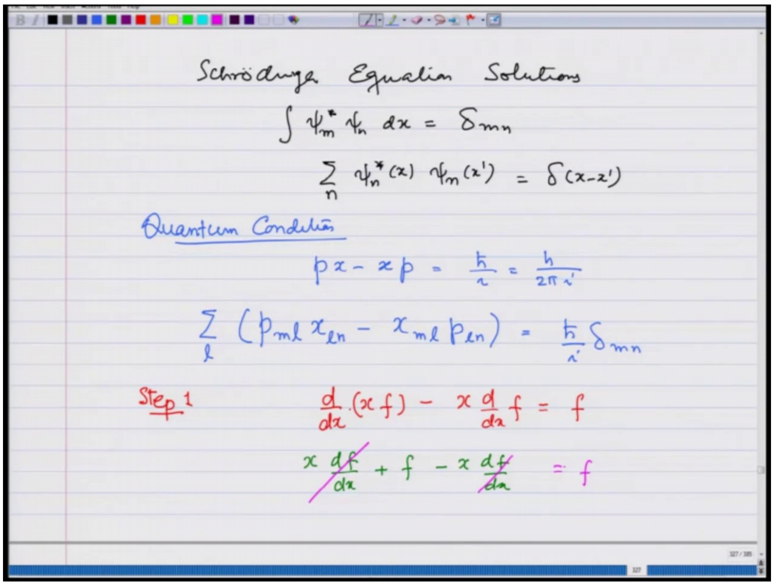

What you have learned in the previous lecture is that for the Schrodinger equation solutions, we have integration psi m star psi n d x equals delta m n. So, we are assuming we have normalized the wave functions and summation n psi n star x psi n x prime equals delta x minus x prime and we are going to use these 2 properties to show the equivalence of Heisenberg's and Schrodinger's approach and then see how to connect the 2.

So, remember the main thing in Heisenberg's approach is quantum condition which says that p x minus x p is equal to h cross over i which is h over 2 pi i in matrix form this means that p m l x l n minus x m l p l n; that means, I am taking the m n component of this matrix p x minus x p is summed over l is equal to h cross over i delta m n can we get the same condition from Schrodinger picture and what does it mean to have it coming from there. So, let us see that. So, to get this first recognize step one that if I take this operation d by d x times x f some function f minus x d by d x f this is equal to f alone.

So, how do we see that from the first term we get x d f by d x plus f and second term is minus x d f by d x and the first term and the last term cancel and you get f.

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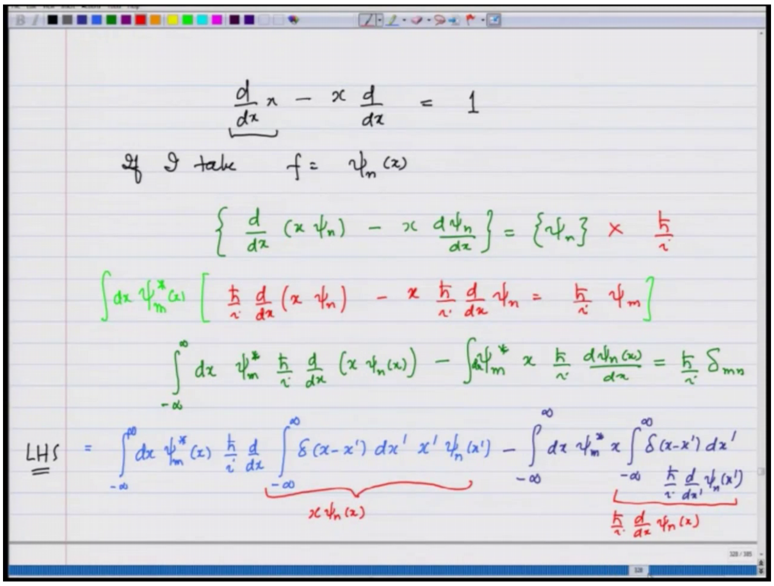

Therefore I can write that d by d x x operating on a function by that I mean d by d x will act on first on x and then f both this operating on a function means I will multiply x and f and then take the derivative minus x d by d x is like multiplying by one in particular, if I take f to be the n th I can function of the solution of the Schrodinger equation, I am going to have d by d x of x psi n minus x d psi n by d x is equal to psi n.

Let me now multiply both sides by h cross over i and write h cross over i d by d x x psi n minus x h cross over i d by d x operating on psi is equal to h cross over i psi n. Next, let me multiply both sides by psi m star x and integrate over x. So, that I am going to get on the left hand side, I am going to get integration minus infinity to infinity d x psi m star h cross over i d by d x operating on x psi and x minus psi m star integration d x x h cross over i d psi n x d x is equal to h cross over i delta m n this we have seen in the previous lecture that psi m actually orthogonal. So, orthogonality here is important.

Now, I am going to use something else and that is completeness. So, what I am going to do now write this as minus infinity to infinity, there is an integration over d x, I have psi n star x, I have h cross over i d by d x, I am going to write the inner integral inner portion as minus infinity to infinity delta x minus x prime d x prime x prime psi n x prime because of the delta function is nothing, but x psi n x, but I am doing it. So, that I can write delta x minus x prime using completeness in terms of size again.

And second term is therefore, going to be similarly minus integral minus infinity to infinity d x psi m star x integration minus infinity to infinity delta x minus x prime d x prime h cross over i d by d x prime of psi n x prime, I am only writing this left hand side. So, this is my left hand side again because this term in the second integral because of the delta function is nothing, but h cross over i d by d x of psi n x, all right. So, this is what we have done now I am going to use completeness.

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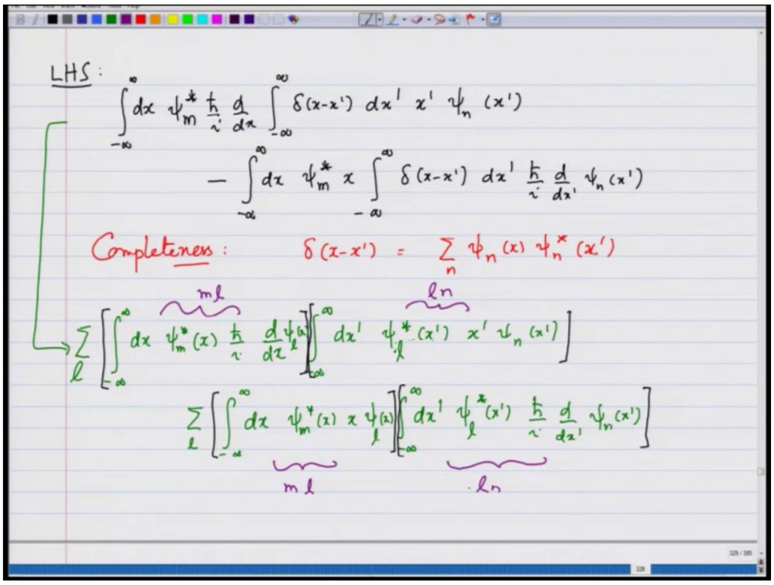

So, let me write this left hand side again. So, that we can compete the whole thing, right. Here I have integration d x psi n star h cross over i d by d x integration of delta x minus x prime d x prime x prime psi n x prime minus integration d x psi m star x integration. This is all minus infinity to infinity minus infinity to infinity minus infinity to infinity minus infinity to infinity delta x minus x prime d x prime h cross over i d by d x prime psi n x prime that is my left hand side and now use completeness to write delta of x minus x prime as summation psi n x psi n star x prime, I am going to substitute this for delta function. So, that the left hand side now becomes integration minus infinity to infinity d x psi m star that will be very careful, right. This variable inside is x h cross over i d by d x of integration minus infinity to infinity, there is a summation over let me change the index because I am using m and n. So, I am going to write this as n psi l x.

So, I can write psi l x outside because inside the integration it does not really matter and then I am going to write psi l star x prime x prime psi n x prime that is the first term for the second term i am going to write again summation over l minus infinity to infinity d x psi n star x x integration minus infinity to infinity d x prime i am going to put the x prime index inside. So, psi l x prime star h cross over i d by d x prime psi n star psi x prime and out here, I get psi l x, let me now put some brackets. So, that terms are easy to identify, I will put one bracket here, second bracket in the second term one bracket here and second one for the second integer, this first term is like a matrix element with m and l indices this term is also a matrix element with l and n indices.

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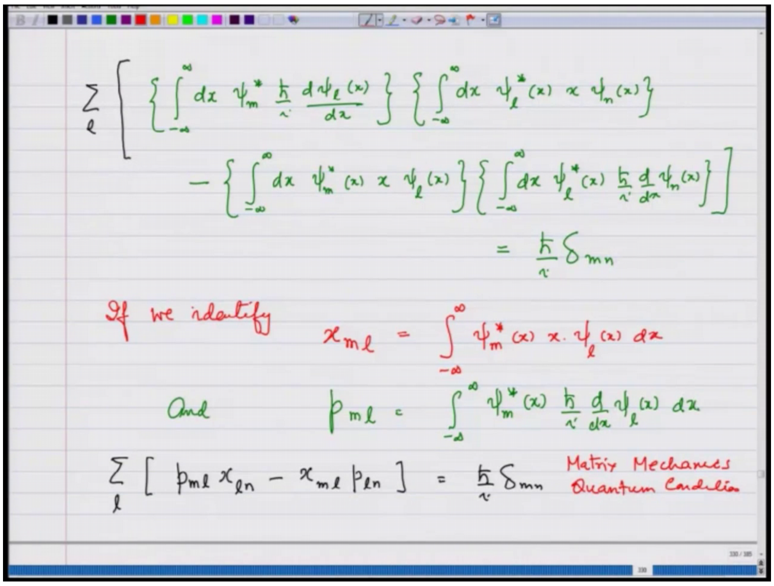

This is a matrix element with m and l indices. This is a matrix element with l and n indices, I have summation over l and I will put this whole thing in a big bracket and I have the first integral minus infinity to infinity d x psi n star h cross over i d psi l x over d x, second one is integration minus infinity to infinity d x primes, but since these a dummy variable, I can still write this as d x psi l star x, x psi n x minus minus infinity to infinity d x psi n star x x psi l x and integration minus infinity to infinity d x psi l star x h cross over i d by d x of psi n x big bracket and recall all this is equal to h cross over i delta m n.

Now, let me assert if we identify x m l as integration psi m star x x psi l x d x and p m l as minus infinity to infinity psi m star x h cross over i d by d x of psi l x d x, we identify it like this, what does this reduce to it reduces to the summation over l inside the bracket i have p m l second term is x l n minus i get x m l p l n is equal to h cross over i delta m n and this is matrix mechanics quantum condition. So, if you want to have the 2 formulism equivalent then there is a particular way of defining the matrix elements of Heisenberg's approach to quantum mechanics using the wave function in a very particular way and that way is now called through operators.

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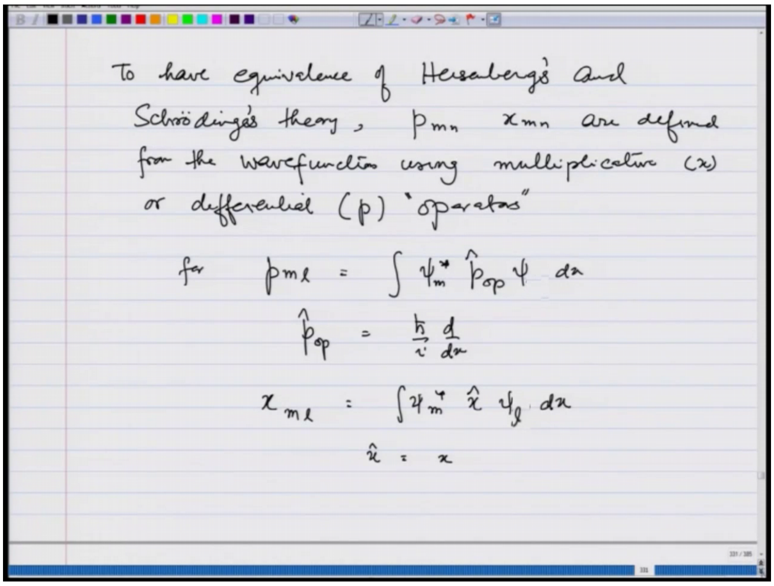

So, to have equivalence of Heisenberg's and Schrodinger's theory p m n and x m n are defined from the wave functions using multiplicative that is four x or differential for p operators.

So, I am introducing a term called operator these are known as operators. So, for p m l and is this equal to integration psi m star then I apply p operator, sometimes it is written with a hat on top psi l d x where p operator is equal to h cross over i d by d x for x m l x m l i have psi m star x operator i put a hat psi l d x where x operator is nothing, but just x and multiply by that what about p square and x square.

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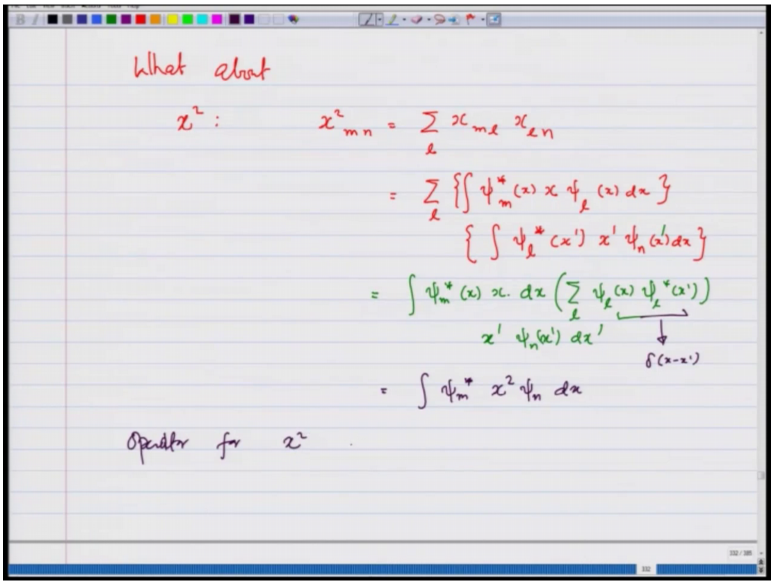

So, let us now check what about x square remember in Heisenberg's mechanics x square m n would be equal to x m l x l n summation over l which would be same as summation over l integration psi star m x x psi l x d x multiplied by integration psi n star let me write this as x prime x prime psi n x d x and again using completeness i can write this as integration psi n star x times x d x i will use summation over l psi l x psi l star x prime then there is an x prime psi n this is an x prime x prime d x prime.

This is nothing, but delta x minus x prime this is delta x minus x prime. So, this becomes integration psi m star x square psi n d x.

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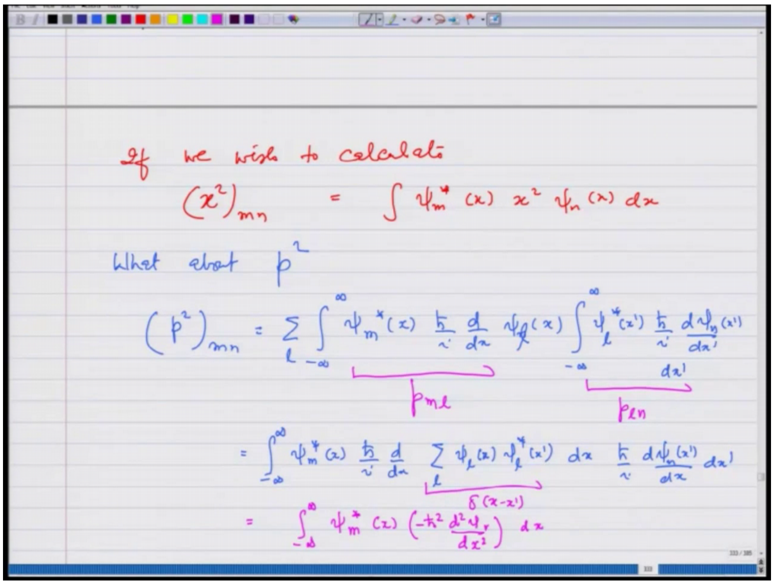

So, operator for x square is x square. So, if we wish to calculate x square m n this is going to be nothing, but square of psi m star x x square psi n x d x. Let us see what about p square p square m n is going to be integration minus infinity to infinity psi m star x h cross over i d by d x on psi n or l x times minus infinity to infinity psi l star x prime h cross over i d psi m by d x prime this is all prime d x prime summed over l because the first term is nothing, but this is p m l and this is nothing, but p l n.

So, by the rule of matrix multiplication this is what it is and this again I can write as minus infinity to infinity psi m star x h cross over i d by d x operating on summation over l psi l x psi l star x prime, I can write it like this because d by d x does not act on x prime, there is an integration over x and then i have h cross over i d psi n x prime over d x d x prime.

But again this term is delta x minus x prime. And therefore, this whole thing becomes minus infinity to infinity psi m star x h cross square d 2 psi by d x square d x.

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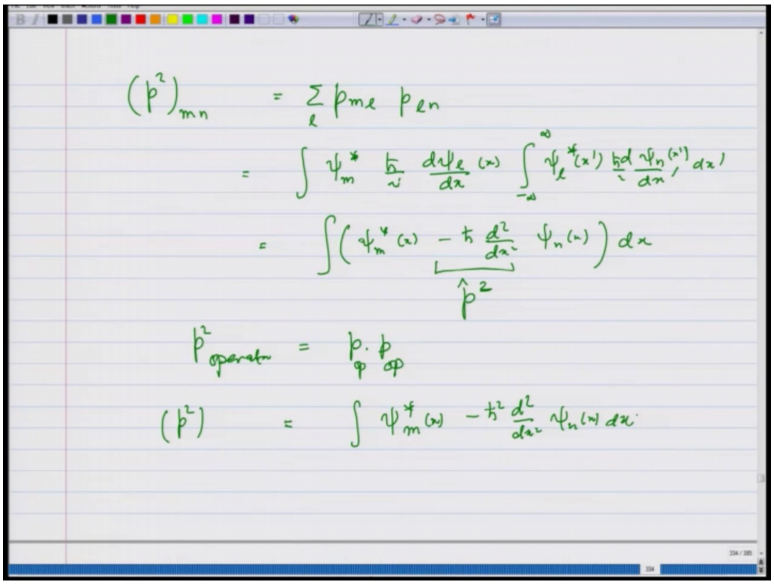

So, for p square m n i did p m l p l n summed over l and this i wrote as integration psi m star h cross over i d psi l over d x x and this is minus infinity to infinity psi l x prime star h cross over i d psi n over d x prime this is all prime d x prime then, I use the orthogonality condition to show that this is equal to psi m star x minus h cross square d 2 by d x square psi n x d x integration this therefore, is the operator for p square and you can see that p square operator is nothing, but p times p operator. So, if I want to calculate p square I do not really have to go through the matrix suit, I can straightaway write this is equal to psi star m x p square operator which is minus h cross square d 2 by d x square psi n x d x.

So, you see how things become very easy I do not really have to do matrix multiplication all the time and I can perform these operations, I can make operators for anything, they just multiply the operators and you get the answer. So, this now makes a equivalence of the Heisenberg picture with Schrodinger picture where now form the wave function I extract the matrix elements by applying the corresponding operators does it make sense.

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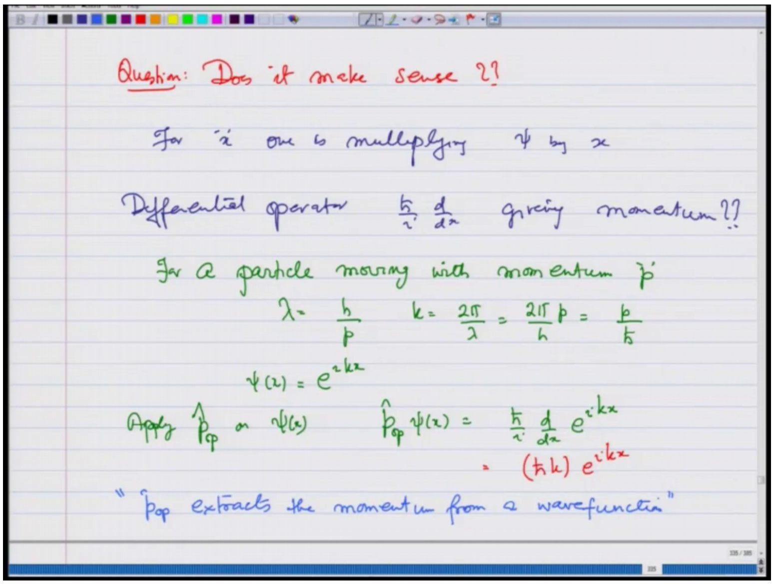

Let us see that; so the question; now I have question, does it make sense in particular for x 1 is multiplying psi by x. So, there it seems to be that if I want to calculate x, I just multiply by x that will be clear in the next lecture when I give the born interpretation for the wave function, but differential operator h cross over i d by d x giving momentum that looks a little odd, but let us see does it make sense.

Recall for a particle moving with momentum p wave length is h over p k which is 2 pi over lambda is 2 pi over h p which is p over h cross and the static part of the wave function is e raise to i k x psi x it is a plane way for a fixed momentum and let us now apply p operator on psi x and see what does it give. So, when I do p operator psi x, it gives me a crossover i d by d x operating on e raise to i k x and that gives me h cross k e raised to i k x. So, in this case when the particle has a pure momentum p applying p operator on psi gives me that momentum. So, it does make sense that p operator gives me momentum; that means; this is an operator that extracts right the momentum out of a wave function.

So, in a very loose statement I will just write this p operator extracts the momentum from a wave function and that is the formalism in Schrodinger that whenever you want to extract a quantity it has to have a corresponding operator that then acts on the wave function and gives you that quantity incidentally here since you get exact momentum. So, this is a momentum Eigen state with the Eigen value of the operator being h cross k let us look at it slightly differently what about the energy the Schrodinger equation is minus h cross over 2 m d 2 by d x square operating on psi n plus v x psi n is equal to E n psi n.

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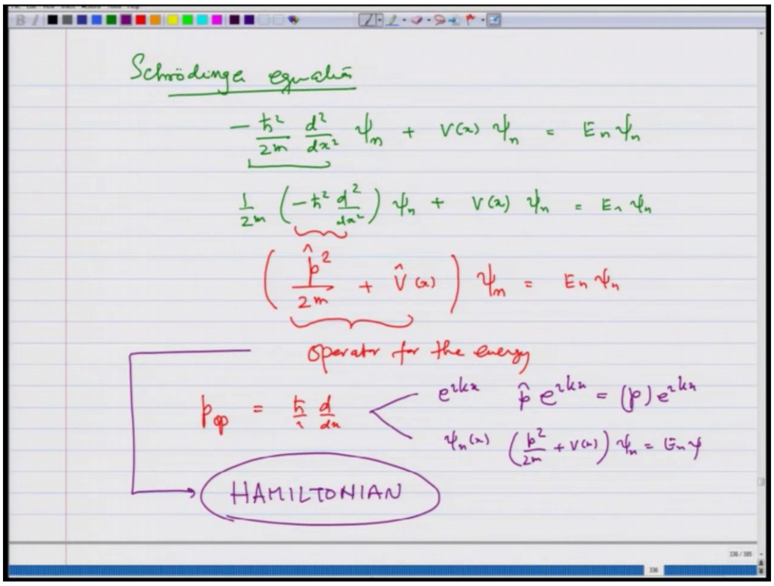

What did we say this operator was this is 1 over 2 m minus h cross square d 2 by d x square psi n plus v x. Now v x you can write as a polynomial or whatever and we saw that that also is like an operator operating on psi n is equal to E n psi n in operator language this will become this as p square. So, it will become p square operator divided by 2 m plus v operator which is a multiplicative operator acting on psi n gives me E n psi n and what is this p squared over 2 m plus v x it is like an operator for the energy.

So, operator for the energy acting on the wave function gives me the energy. So, this suddenly starts making sense that p operator being written as h cross over i d by d x in 2 instances, it has given me the right answer for e raise to i k x p acting on e raised y k s gave me the true momentum p times e raise to i k s and for psi n x p square over 2 m plus three x on psi n gave me the energy. So, this operator for the energy is known as the Hamiltonian and in this course this is an introductory course, I am not going to go more into details of why this is called Hamiltonian, but this is the Hamiltonian operator. So, Hamiltonian is the operator for energy it acts on psi n and Eigen function and gives you E n.

So, to conclude this lecture let me just make a comparative table for Heisenberg and

Schrodinger picture.

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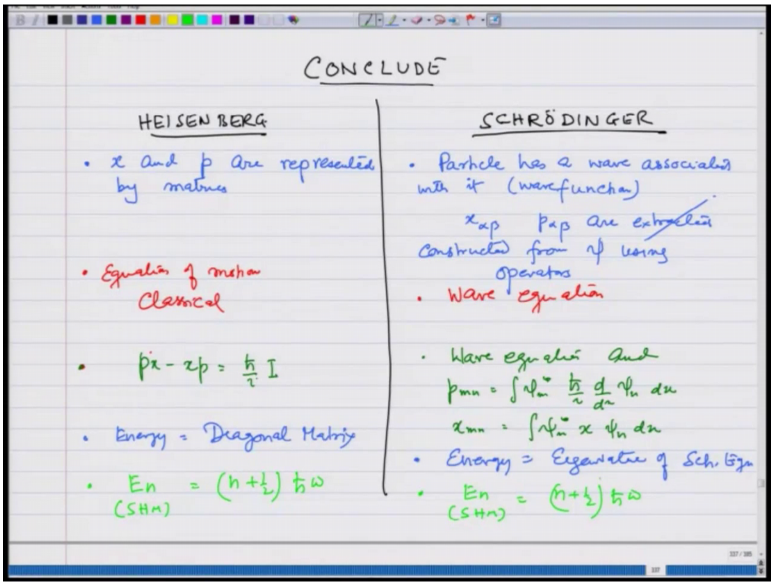

So, I will conclude this lecture by making a table of Heisenberg Schrodinger in Heisenberg's picture the x and p are represented by matrices in the Schrodinger picture particle has a wave associated with it which is known as the wave function x alpha beta or x m n and p alpha beta are extracted or I will use constructed from psi using operators. So, if there are matrices in here you have to apply an operator on the function, now here the equation of motion is classical except that now this is written in terms of the matrices here is the wave equation here the quantum condition is p x minus x p equals h cross over i, the identity matrix again here I will write wave equation and p m n equals psi m star h cross over i d by d x psi n d x x m n being given as psi m star x psi n d x.

Here the energy for stationary systems or conservative systems is diagonal matrix and you have done that in the past for harmonic oscillator here the energy is Eigen value of Schrodinger equation; however, through all this finally, the energy for harmonic oscillator right s h m comes out to be n plus a half h cross omega and here also we have solved this energy for s h m comes out to be n plus a half h cross omega and the 2 are equivalent.

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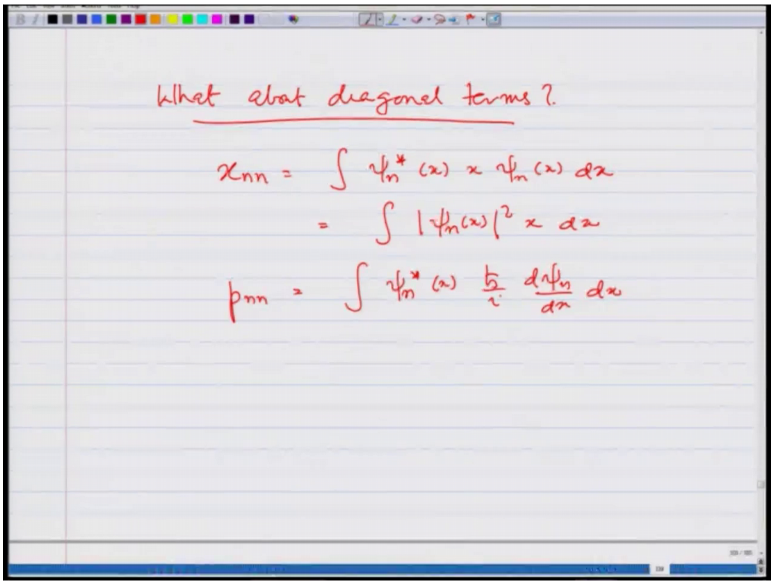

So, after this conclusion, I just want to leave you with a question what about diagonal terms for example, x and n which is integration psi n star x x psi n x d x which is integration mod psi n x square x d x and p n n which is integration psi n star x h cross over i d psi n over d x d x. These terms we have not yet addressed what do they mean and we will address this in the next lecture when we discuss the interpretation of psi n square.

So, I will leave you with that question in this lecture.