Introduction to Quantum Mechanics
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Lecture – 04
Representing a moving particle by a wave packet

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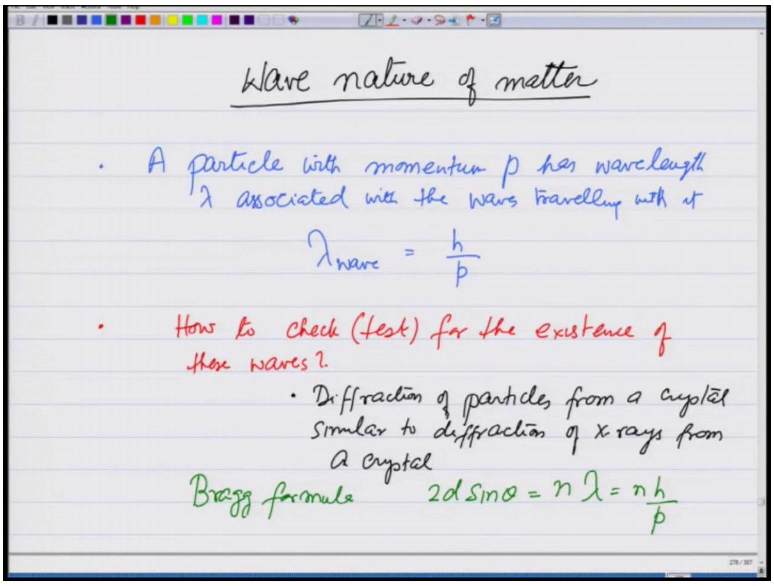

We have been discussing the wave nature of matter. And what I have presented the last couple of lecturers is that a particle with momentum p has wavelength lambda associated with the waves travelling with it; that means, lambda wave is h over p. What we have not said what kind of waves these are. The only thing we know is that there is some wave travelling with the particle when it moves and it has a wavelength h over p. And we have also discussed how to check or test for the existence of these waves.

And 3 experiments that I had discussed was diffraction of particles from a crystal, which is similar to diffraction of x rays from a crystal. Both satisfy the Bragg relationship, that is has 2 d sin theta equals n lambda. Where for x rays it will be the wavelength of x rays and for electrons or the particles that are diffracted from crystal it will be lambda of those particles which is n h over p of the particles.

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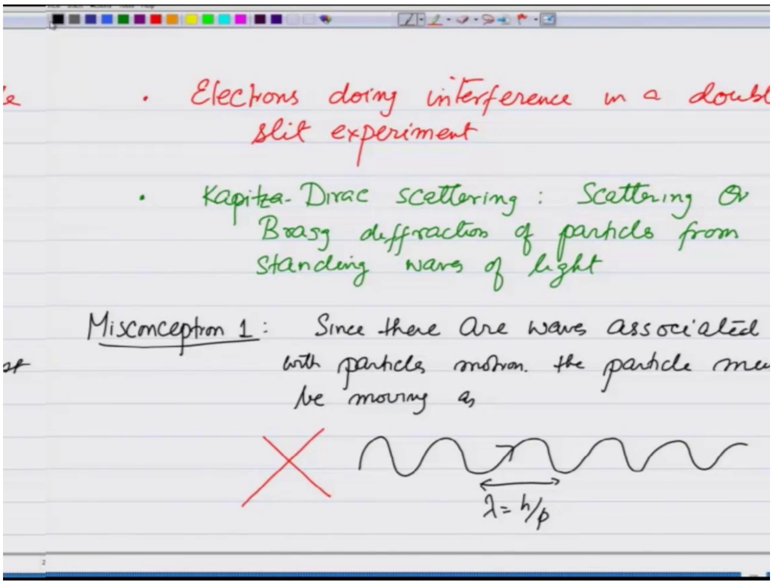

I also presented to you of particles from standing waves of light. Just dwelling on this concept little more I want to emphasize that their some misconceptions that should be cleared right away. Misconception one, and that is that since there are waves associated with particles motion the particle must be moving as has a wave itself.

That means sometimes there is a misconception that the particle itself is performing a motion which is wave light, with a wavelength which is lambda given by h by p. This is planed wrong.

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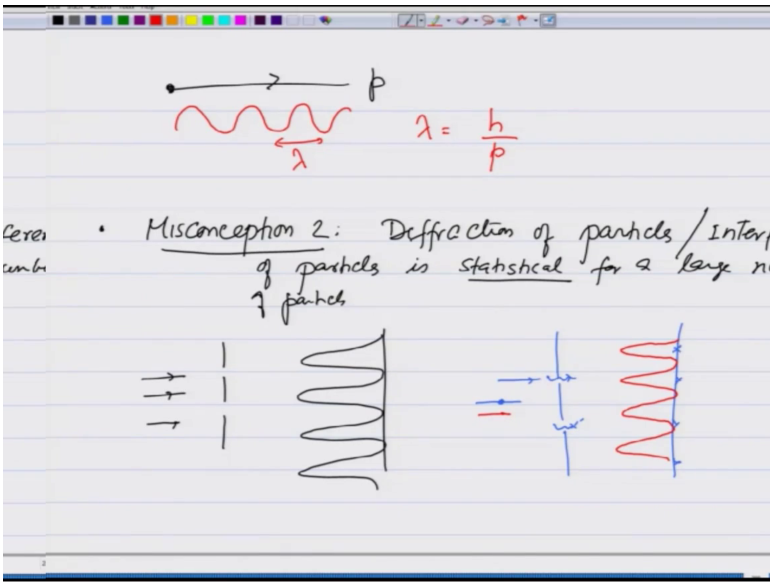

What this wave business associated with particle says is that If there is a particle which is moving with momentum p, with the particle there is a wave associated. And we do not yet know what this wave is until we write equation in start interpreting, but experimentally it is established that there is a wave associated which has a wavelength lambda which is h over p.

So, this should be very clear. There is a particle moving and there is a wave associated with it. Number 2 diffraction of particles or this is same as interference of particles is statistical for a large number of particles. What I mean to say in this is suppose I am doing a double slit experiment, with electrons coming from this side and finally, I see an interference pattern. The misconception is that when I let these electrons go through one electron comes and the second electron interferes with this electron and then they interfere and go somewhere that is a misconception. What is happening is each electron that comes somehow gets divided it does not get divided it is wave gets divided. And it is the wave associated with each electron that form a interface pattern is just that one electron can go either here or here or here or here. So, we do not really see the interference pattern, but it is being formed.

So, what happens when slowly, if you let one electron come at a time and collect a lot of data you even when only one electron is allow to pass through the slits at one time after we collect a lot of data, you see a pattern emerging like this. And what that means, is that each electron is interfering with itself. More precisely the wave associated each electron goes through both the plates and when it comes out just like light wave interference in the double Young's double slit experiment, these waves also interfere and then accordingly particle go somewhere, particle cannot be divided.

So, never think that it is only when large number particles come they interfere with each other and then the form in interference pattern. No, it is the wave associated with each particle single particle that interference with itself and creates a pattern same thing in the crystal diffraction experiment also. Wave associated with each electron each particle interferes with itself and then when you collect a lot of data you see the pattern emerging the particles going somewhere.

So, I hope I have cleared these 2 misconceptions which are quite prevalent. Now having done that I also want to give some time to Dirac-Kapitza effect in which what we said is that if I take a standing wave of light.

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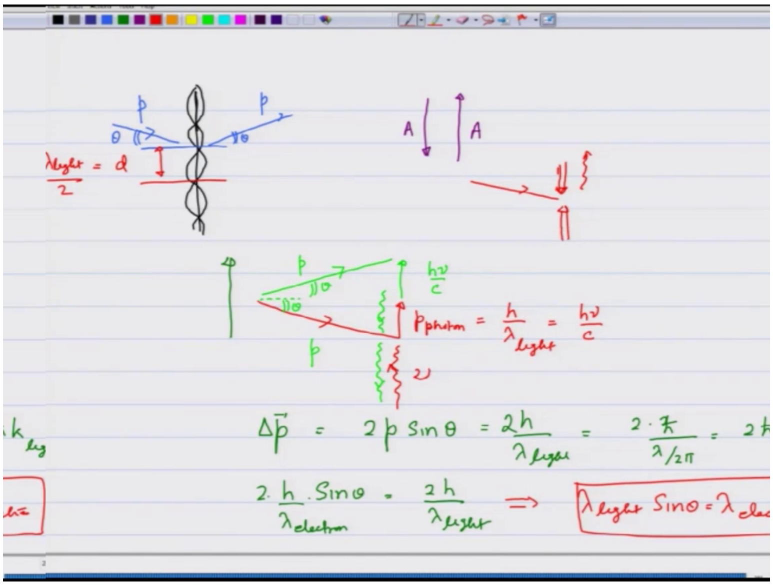

And let electrons or some particles come from this side at particular angle theta you see a large number of electron diffracting at exactly, the same angle with the same momentum p, how does this happen? So, what happens here; is recall from our discussion on waves that a standing wave is a wave travelling this way and a wave travelling the other way with equal amplitude. So, what happens when electron comes in? It may absorb a photon from wave coming down and therefore, it gets a kick in certain direction. And then the wave going the other wave may stimulated to give out that photon. So, it gets another kick, let us do it more systematically.

So, when this momentum p is coming in and suppose the electron absorbs a photon of frequency nu from this side. It gets a kick of momentum of photon, which can also be written as h over lambda light, which is h nu over c. Then since this is the standing wave there is a light coming this way also. It can stimulate this electron to give out the photon going the opposite direction. And this adds another kick to the photon which is again h nu by c.

And therefore, finally, the photon momentum makes it go this way. So, it was coming at an angle theta and went out at this angle theta, keeping the p the same. How much is delta p? Delta p is this big arrow. So, delta p is in the direction of photon, which is equal to 2 p electron sin of theta which you can see from this diagram that I have made. And this should be equal to h over lambda light times 2 because the 2 photons that are involved which is 2 I can write this 2 times h cross over lambda over 2 pi which is same as 2 h cross k light. You can write in many different ways, but this also tells you h cross k is the momentum of light.

And therefore, sin theta times p of electron I can write as 2 times h over lambda electron, which is same as 2 times h over lambda light and that immediately gives you, that lambda light times sin theta equals lambda electron. Which is the same as Braggs condition as derived in the previous lecture with this difference between 2 planes same planes is being d which is lambda light divided by 2.

So, this is a way of understanding Dirac-Kapitza or Kapitza-Dirac effect, where electron absorbs a photon and then it is stimulated to emit that photon in the opposite directions we gets twice the kick. So, having done that now we want to understand about this wave when the particle little more.

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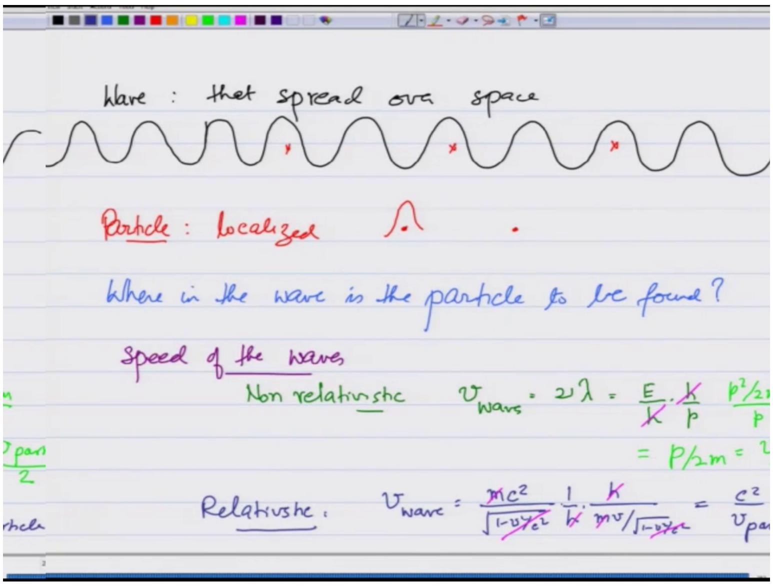

Now, a wave is something that is spread over space. So, if you take a plane wave it may be spread all over space, some drawing it as much as possible. So, this is a wave which is spread over space. On the other hand if you look at a particle, it is look like somewhere. It may be here, it may be here, at one position. So, this is localized at one point. If there is a wave associated with a particle, particle is localized, but the wave is spread all over the place, where is the particle? Is it here? Is it here? Is it here? Where is it in the wave?

So, question arises where in the wave is the particle to be found. That is one question. The other question is let us also calculate the speed of these waves. If I do a calculation in non relativistic term non relativistic v waves is going to be the frequency, times lambda frequency is nothing but energy E over h times lambda which is h over p, h cancels and I get E over p which is nothing but p square over 2 m over p, which is p over 2 m which is v particle over 2.

On the other hand if I do a relativistic calculation right. So, let me do a relativistic calculation, then I have again v wave is equal to energy, which is now going to be m c square over square root of 1 minus v square over c square times 1 over h times the momentum which is h over momentum is m v over square root of 1 minus v square over c square. Then I cancel this term I cancel m, I cancel h and I get c square over v particle. In both the cases I see that the wave speed is not the same as v particle. So, what is going on? So, 2 questions that have a reason out of this consideration of a wave associated with the particle. Number one is where in the wave Is the particle located.

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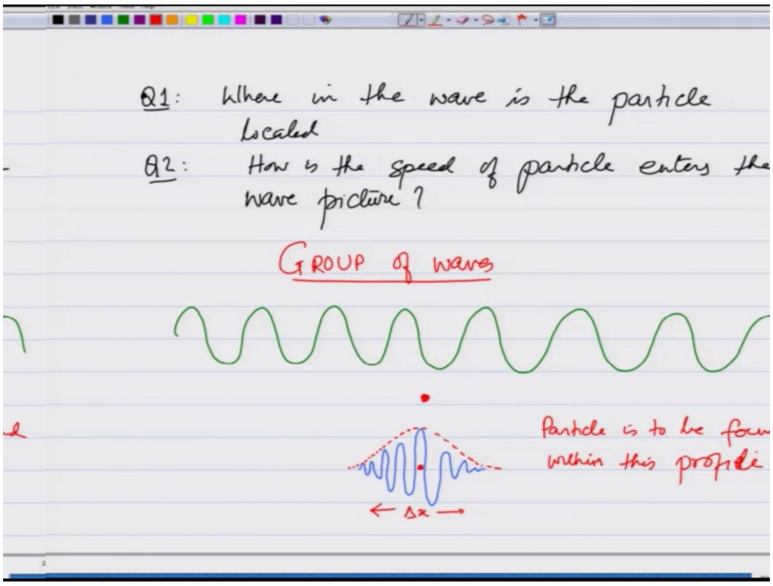

And question number 2 is how is the speed of particle enters the wave picture. And both of these are answered by something call the group of waves. And that is what I am going discuss next. So, when we say that there is a wave all over the place, and I do not know where the particle is located which should have been somewhere here where are show it by the dot, let dot put out says the particle as actually when is moving it is not represented by infinite train of wave.

Rather, this wave could be localized around the particle position. So, particle is somewhere there is a large amplitude. So, it has a profile the wave has the profile. And particle is to be found within this profile; so let say if this is delta x, the extent of this wave spreading then particle is found to be found within this profile. Notice by making that statement I am actually saying much more than what I just state it. What I am saying is if I associate how wave with the particle, it can best be located within some delta x it cannot be located precisely, and that is a quantum mechanical true statement.

I cannot locate the particle precisely with delta x being 0, because there is a wave associated could be anywhere within that wave.

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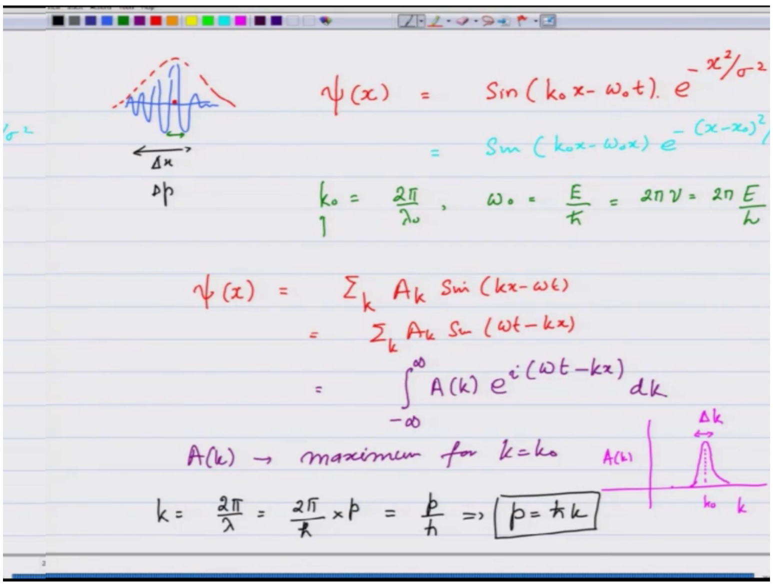

So, for example, when I take this particle and a wave associated with it. With a profile that I made earlier, it could for example, be that the waveform with now I will introduce this Greek symbols psi x because, I am going to use it later is going to be of the form for example, it could be sin of sum k 0 x minus omega 0 t times let say e raise to minus x square over some constant sigma square. So, that as you go far away from that position of the particle it decay.

So, if the position of the particle is at x naught I could write this as sin k naught x minus omega naught x e raise to minus x minus x naught square over sigma square. Where k naught is related to this lambda of this undulation. So, I can write this is as 2 pi over lambda 0, omega 0 is related to this energy of the particle over h cross, which is same as 2 pi nu which is same as 2 pi E over h.

So, there is some wavelength associated, but there is also profile. So, not only k naught is present in such case other case also present, and how are they present? Again I will go back to what I told you about how to represent a general wave in terms of eigen functions. So, this I can write as a Fourier transform or sum over different case time some constant A k sin of k x minus omega t I could equally well write this as summation A k sign of omega t minus k x.

Depending on which way do you want to saying so I can just change the sin and I would like this. K and in continuous case this is written as a Fourier transform which is integral of some a as a function of k e raise to i omega t minus k x d k, k wearing from minus infinity to infinity. And A k is maximum for k equals k 0 because you see that is a predominant waveform. And it decays away from this So for example, it could be of the form if I go to plot A k verses k it could be of the form this is suppose k naught it is largest there and then the decays.

So, there are some waves being mixed, this is delta k within some delta k. What is k related to? By de Broglie’s hypothesis k is nothing but 2 pi over lambda which is nothing but 2 pi over h times p. This gives p over h across which implies that p equals h cross k. So, momentum of the particle is related to k as h cross k. And when I say that their sum is spread delta k; that means, momentum of the particle is not specified very precisely a large chunk of it is k 0, but there is also a spread in it. So, notely the position is not specified exactly. Even the momentum has a spread. Momentum would have been precise if the. So, the plane wave of constant wave length spreading from minus infinity to plus infinity. So, for a localized particle delta x it is position is not known exactly is momentum is not know exactly.

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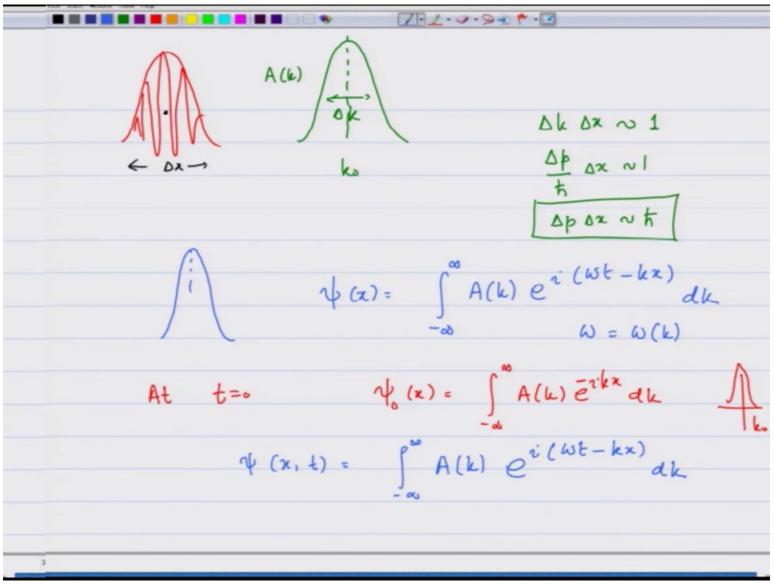

And In fact, you can show that if there is this particle which is located within this amplitude it is located somewhere here, is the spread is delta x. And the corresponding Fourier transform has a spread around k 0 this is A k and speed is delta k then delta k delta x is of the order of 1, and delta k is nothing but delta p over h cross delta x is of the order of 1, or delta p and delta x is of the order of h cross. This is another consequence of particle having a wave nature. If I want to reconcile the 2 I cannot give the momentum and the associated position, the same direction with the same precision as I do classically and they product actually is h cross. This is a statement of uncertainty principle which will come back to in later lectures.

But right now just I wanted to convey that if you look at the wave and particle nature together, you cannot specify the momentum and position to as much as accuracy as you like. They have a reciprocal relationship delta p delta x is of the order of h cross. So now, back to this representing this wave packet which in the previous slide I wrote as a Fourier transform, and I am going to write this psi x as equal to minus infinity to infinity A k e raise to i omega t minus k x d k. You may wonder why I am integrating only over k, it is because omega is also a function of k. Now let us say at equal to 0 the profile was given by psi 0 x which is integration minus infinity to infinity A k e raise to i minus i k x d x and this is psi 0 x. What happens sometimes later? And this k a is peak around k 0.

So, I am not right now psi x comma t and recall from earlier lectures, that when I want to write the time dependence I will just put in the time dependence as happens for a wave. So, it will be e raise to i omega t minus k x d k. With time this is how the psi function or the wave function is going to look like. Recall when I did the string and I said if I give it a shape of triangular string and displays it, the with time the displacement was given as a sum of all the eigen frequencies and eigen functions, in exactly the same manner this would I am going to write.

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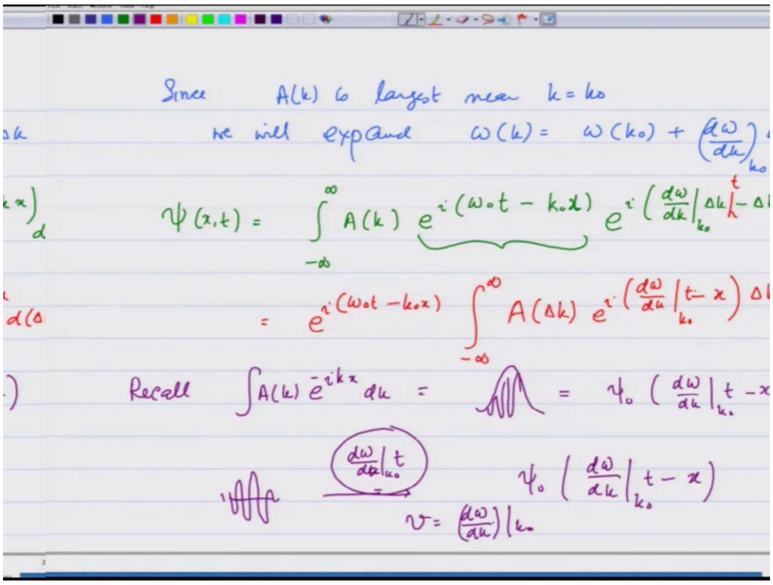

And now since A k is largest near k equals k 0, we will expand omega k as omega of k 0 plus d omega d k at k 0 delta k. So, I am going to write now psi x t is equal to integration A k e raise to i omega 0 which is corresponding to omega t t e which is corresponding k 0, minus k 0 x times e raise to i d omega over d k at k 0 delta k minus delta k x d k, minus infinity infinity.

Since the proportion shown in the curly brackets does not depend on k I can pull this out and write this as e raise to i omega 0 t minus k 0 x integration, minus infinity to infinity. Instead of writing A k I can write this as a function of a delta k, where delta k is difference from k 0. E raise to i d omega over d k A k 0 minus x there is a t here sorry, there was a t here delta k and integration over delta k. And what is this? Recall, A k e raise to minus i k x d k was this function which was giving the profile of this wave right. So, this is what I had shown you earlier that this was So that psi 0 x. So, I am going to have the same profile come in and this is going to be psi 0 except now I am going to have d omega over d k at k 0 t minus x.

So, what does happened? This profile that was there in time t it has shifted by distance d omega by d k at k 0 times t. And therefore, it is now given as psi 0 of d omega over d k at k 0 t minus x. If it has shifted by this much d omega by d k k 0 t; that means, it has moved with a speed v equals d omega by d k at k 0.

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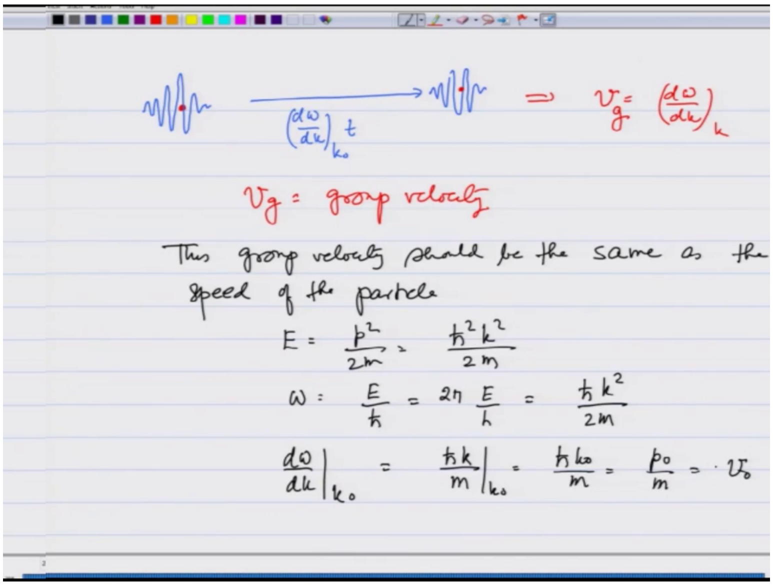

So, there was this particle which is being given by this profile of wave. And in time t this fellow moves by a distance d omega by d k calculated at k 0 t and moves same shape the particle has reached here. And therefore, this implies the speed with which it has moved is d omega by d k at k 0. And this is known as the group velocity, v g is the group velocity. Because the group of waves that has moved and this group velocity should be the same should be the same has the speed of particle and let us check that. So, E is given as p square over 2 m which by now your familiar is s cross over k square over 2 m.

So, omega which is E over h cross is same as 2 pi E over h is h cross k square over 2 m and therefore, d omega over d k at k 0 is going to be h cross k over m at k 0 which is h cross k 0 over m, which is same as p 0 over m, which is same as the particle velocity.

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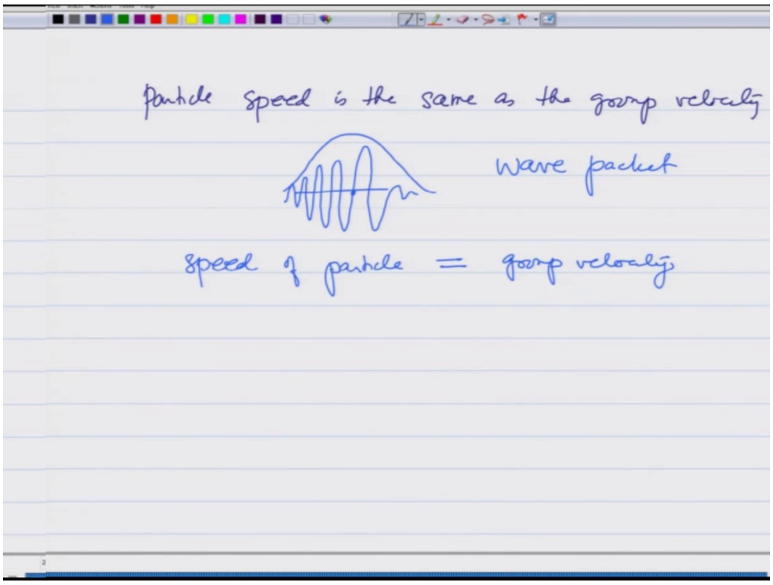

So, we say that the particle velocity of particle speed is the same as the group velocity. So, to conclude when I associate a wave with a particle: the particle, particle is described by what I call a wave packet. This is a wave packet. And the speed of particle is same as group velocity. In the next lecture we will develop the wave equation for this wave, and start solving problems using the Schrodinger equation.