Introduction to Quantum Mechanics
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Lecture – 02
Stationary waves, eigen values and eigen functions

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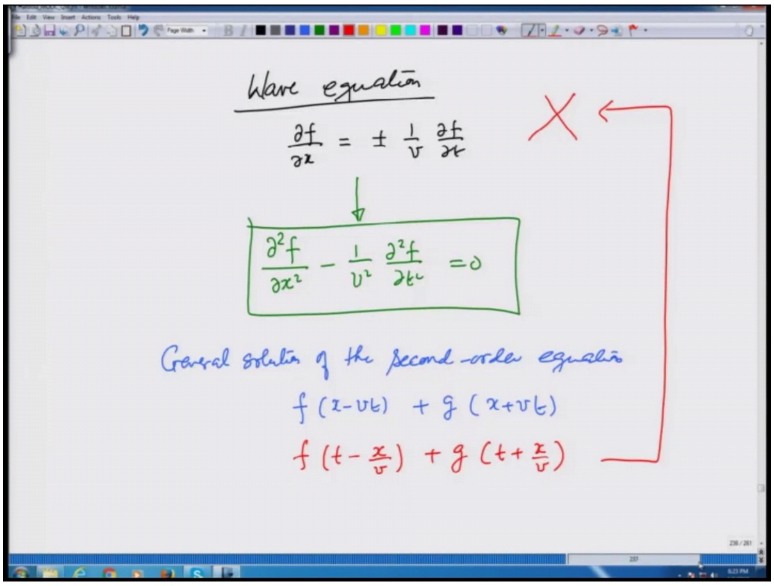

In the previous lecture, I described the wave equation starting from waves for the equation for waves travelling to the left or the right which was given by d f by d x equals plus or minus 1 over v d f by d t which was combined together to give a wave equation d to f by d x square minus over v square d to f by d t square is equal to 0 and we said the general solution is it could be the wave travelling to the right wave travelling to the left and the linear combinations.

So, a general solution of this of the second order equation is some function f x minus v t plus some other function g x plus v t, I could also write this as some function f of t minus x over v plus some other function g of t plus x over v. Notice that this combination cannot be the solution of this equation, it cannot be for them it is either t minus x over v or t plus x over v.

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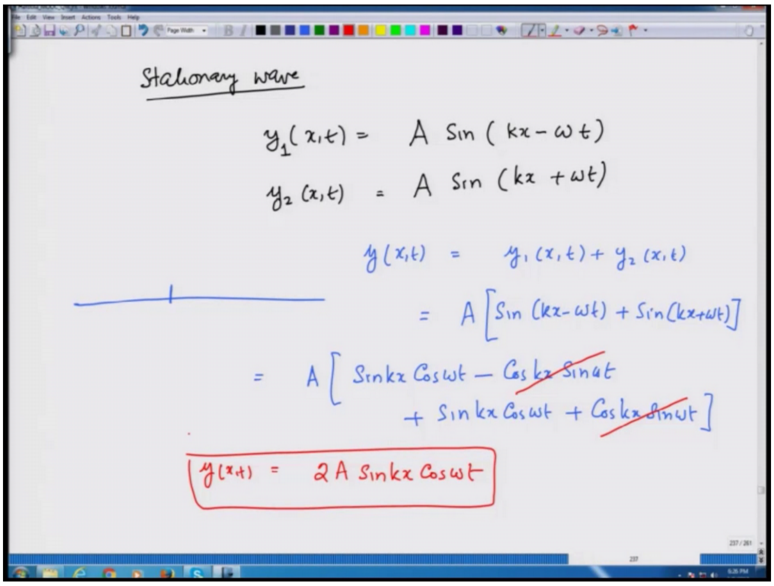

So, only when you combine them, it gives you possibility of combination of waves travelling both to the left to the right now that gives rise to the concept called stationary wave and let me describe that suppose I have the solution y x t given as some amplitude sin k x minus omega t sinusoidal wave. This is a wave travelling to the right, I also have another way. So, let me call this y 1, I have also have another wave y 2 x t which has exactly the same amplitude except that it travels to the left. So, it is going to be k x plus omega t, I have 2 waves of equal to amplitude travel into the left and travel into the right and at any point.

So, suppose they are travelling on a strain at any point, the net y x t is going to be a combination of the 2, right, the point there does not know whether it should be it y 1 or y 2 displacement in that displacement and this is going to be A sin of k x minus omega t plus sin of k x plus omega t which is A sin of k x cosine of omega t minus cosine of k x sin of omega t plus sin of k x cosine of omega t plus cosine of k x sin of omega t and immediately notice that term cosine omega t cancel, and the answer I get is 2 A sin of k x cosine of mega t thus in that displacement.

Let us look at it carefully and see what does it mean?

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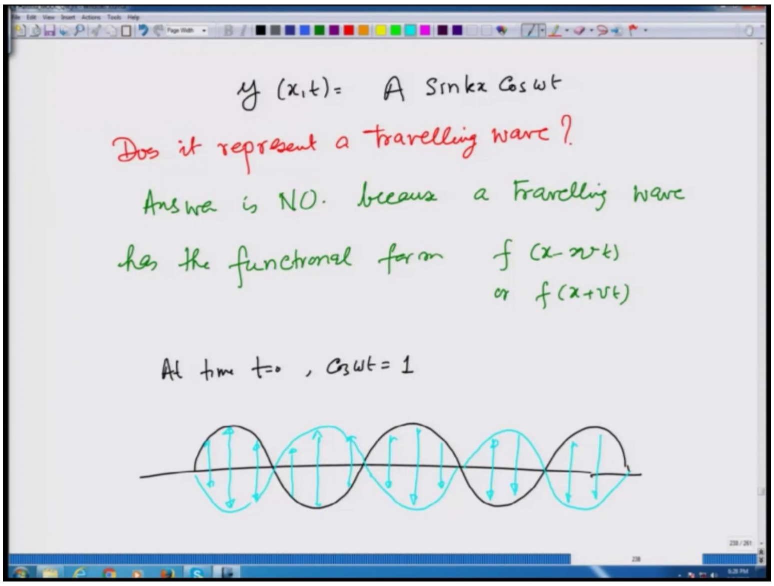

So, I have y x t is equal to 2 A which I am going to call again some sort of an amplitude A; what is more important is I have sin k x cosine of omega t, let us ask the question; does it represent a travelling wave and the answer is no, why because a travelling wave has the functional form of f x minus v t or f x plus v t x and t come in that combination. So, this is not a travelling wave.

So, what does it represent? Let us try to plot it. So, suppose at time t equal to 0, let see at time t equal to 0 cosine of omega t is going to be 1 and therefore, this whole thing is going to look like A sin function and what happens is the time progresses all that happens is that each point goes back and forth oscillating back and forth with the frequency omega.

So, all the points oscillate between these 2 points; all right. So, all the points any point on this strain is oscillating back and forth and the disturbance is not traveling either to the left or to the right; you see in this; suppose, I take a strain tie it between 2 ends and give it a displacement and just goes back and forth. So, nothing travels this is a stationary wave or standing wave as you may have learnt your twelfth grade.

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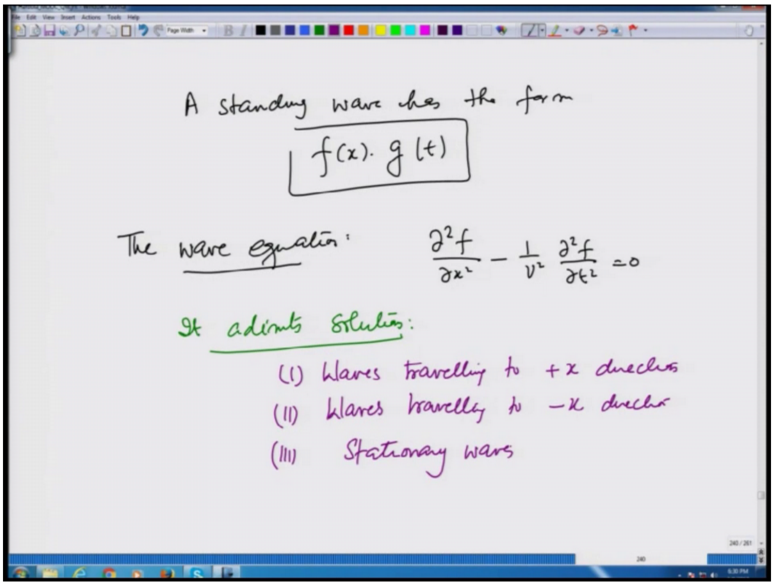

So, necessarily a standing wave has the form some function of x times some other function of t, it is separable, it does not come in a combination x minus v t or x plus v t and it is as if each point is just oscillating back and forth its performing periodic motion that is what a standing wave is.

So, if I look at the wave equation which is d 2 f by d x square minus 1 over v square d 2 f by d t square is equal to 0, it admits solutions which are number 1 waves travelling to plus x direction 2 waves to minus x direction and 3 stationary waves; these are waves that just step out at one place in the points oscillating back and forth.

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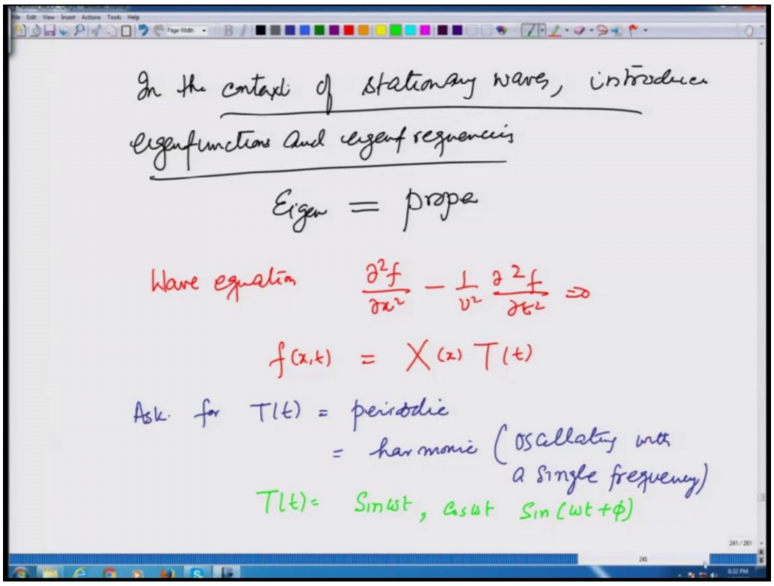

Now let us look at stationary waves a little more carefully and what I am going to do now is that in the context of stationary waves I am going to introduce Eigen functions and Eigen frequencies; Eigen is a German word which means proper.

So, let me introduce that idea now. So, let us look at this equation the wave equation which is d 2 f by d x square minus 1 over v square d 2 f by d t square is equal to 0 and I said it admits solution of this kind f x t is let us now call x x T t where x is the function of capital C is the function of x alone and capital T. The function of t alone and this is stationary wave, it is not a travelling wave because x and t do not come in combination of x plus v t or x minus v t and now ask for T t to be periodic and more than periodic harmonic.

That means, oscillating with a pure frequency with a pure means with a single frequency. That means, I want t of the form some sin omega t or cosine omega t which is same as sin omega t which in general I can combine the 2 and write this is sin omega t plus some phase phi using complex notation this is also written sometimes as exponential.

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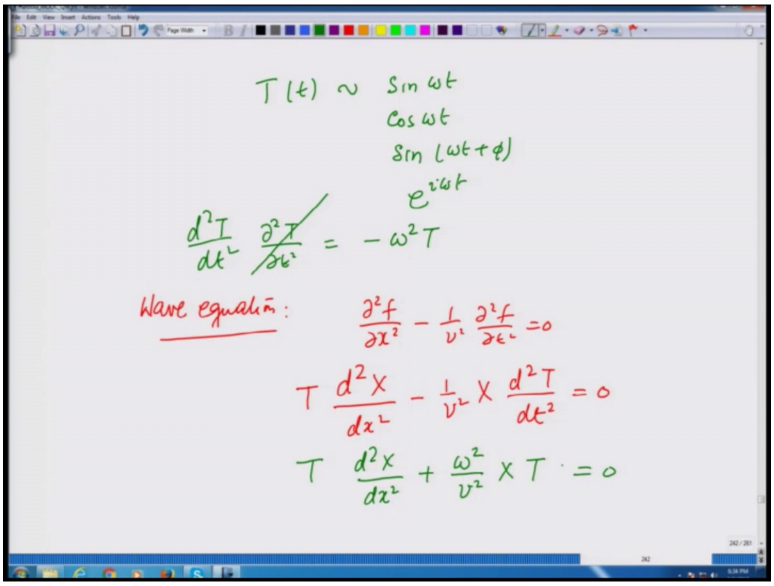

So, I can have sin omega t or cosine omega t or sin omega t plus phi or some e raise to i omega t. They are all the same thing; their periodic function pure a single frequency omega, but what is important in this is that d 2 t by d t square in this case is going to be equal to minus omega square T that you can easily check you have learnt this in the context of simple harmonic motion that a pure frequency omega. We take the second derivative is respect to time it gives you omega x square t and therefore, the wave equation becomes d 2 f by d t.

Let me write this completely first 1 over v square d 2 f by d t square is equal to 0 becomes d 2 x by d x square T remains here minus 1 over v square x comes out side d 2 t over d t square is equal to 0; oh, I made a mistake here, I should have written this green thing as a complete derivative because t depends only on time and this gives me t d 2 x over d x square minus minus plus omega square over v square x t is equal to 0.

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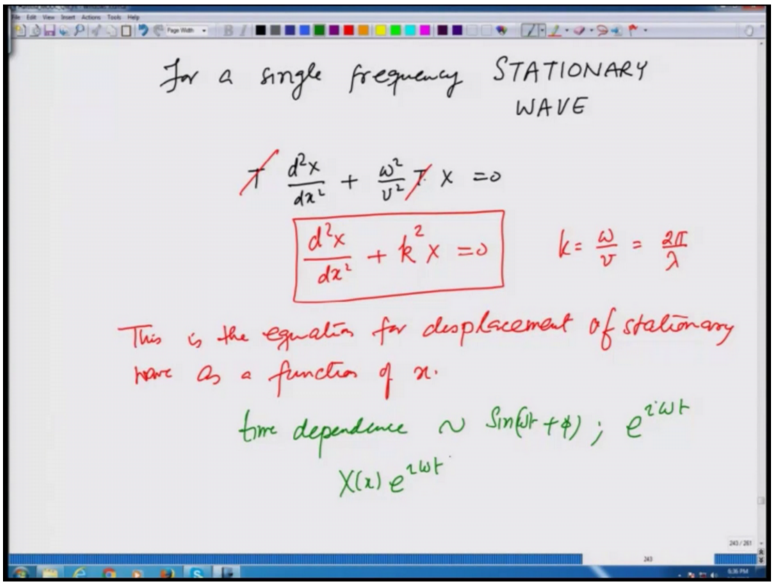

So, for a single frequency and I am going to emphasize is stationary wave; stationary because I have separated the x and d parts; I have the equation t d 2 x by d x square plus omega square over v square t x is equal to 0 or I can cancel t out throughout and I have d 2 x by d x square plus k square x is equal to 0 and we recall this k is which is known as a wave number is given as omega over v which is same thing as 2 pi over lambda.

So, this is the equation for displacement of stationary wave as a function of x time dependence of course, is like sin omega t plus phi or using complex notation i t e raise to i omega t that is the time dependence. So, the final solution is x; x times e raise to i omega t that is the final answer now does this give me re displacement.

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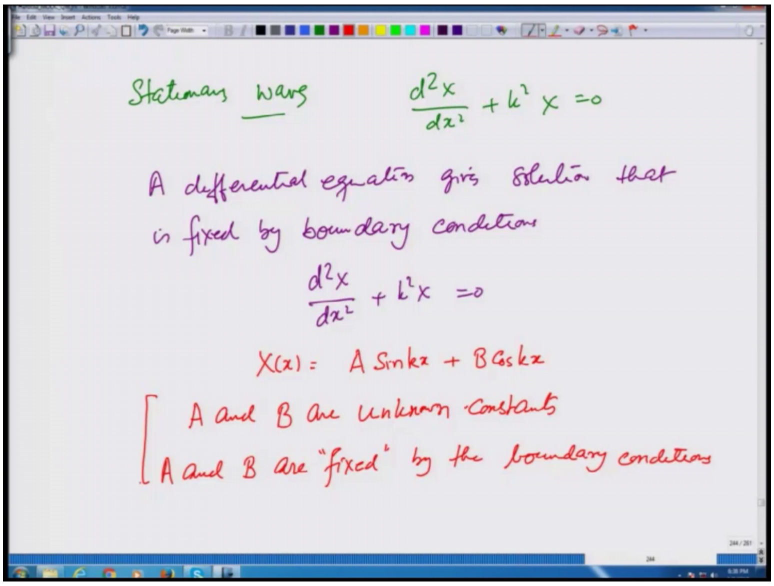

So, stationary waves is d 2 x over d x square plus k square x is equal to 0.

Now, I just want to remind you that a differential equation gives solution that is fixed by boundary conditions let me explain that you see this equation d 2 x over d x square plus k square x equals 0, the very simple equation its solution is you can write X x is equal to A sin of k x plus b cosine of k x where a and b are unknown constants.

So, the solution is not really known until I know A and B; to know A and B, I have to know the boundary condition. So, A and B are fixed by the boundary conditions. Once you set the boundary condition, then A and B are fixed; so that is number 1, so without a boundary condition, solution is not really fixed it is some arbitrary solution number 2.

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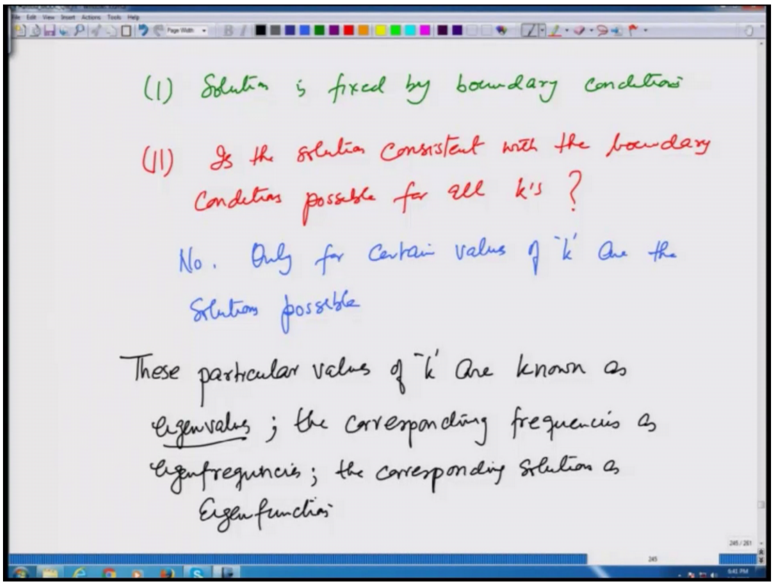

So, one I have seen that solution is fixed by boundary conditions.

Let me make another point and this is very important, I will show this by the example that is the solution consistent with the boundary conditions possible for all k s for any value k s is it possible and the answer is no only for certain values of k are the solutions possible what does; that means, the fix solution is possible only for certain values k for other values of k boundary conditions are not satisfied. And B cannot be fixed and this case these particular values of k are known as Eigen values these are the proper values the corresponding frequencies as Eigen frequencies and the corresponding solution as Eigen function.

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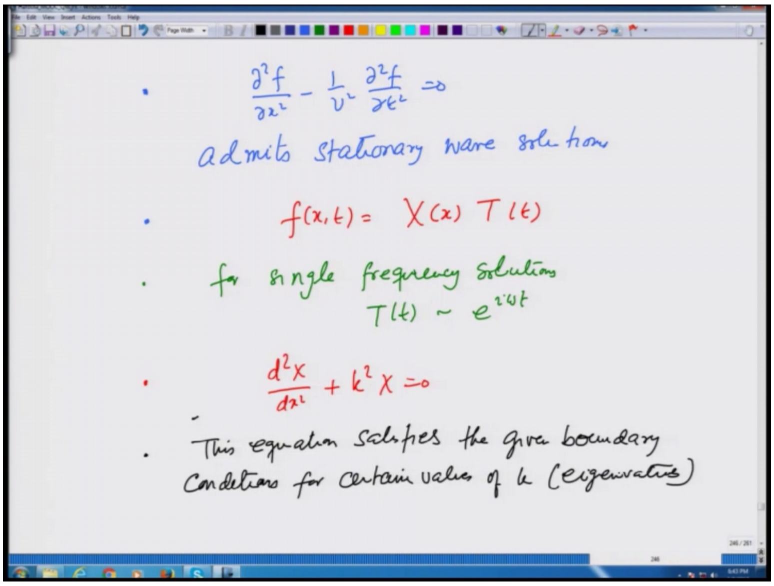

So, let me just summarize this. So, d 2 f over d x square minus 1 over v square d 2 f by d t square is equal to 0 admits stationary wave solutions that is number 1; what does that mean; that means, I can have f x t as some function of x alone and some function of t alone and third for single frequency solutions T t is of the form e raise to i omega t that include sin cosine and everything and the equation becomes d 2 x by d x square plus k square x equals 0 and finally, we wrote this equation satisfies given boundary conditions for certain values of k.

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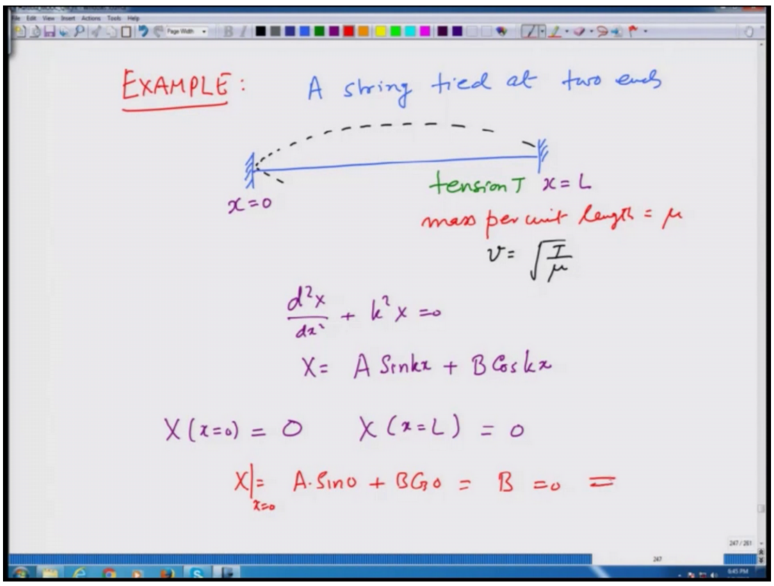

So, and these are known as Eigen values and now an example I will take a string tied at 2 ends. So, I have a string which is tied at 2 ends it has tension T and mass per unit length equals mu. So, this speed of wave is square root of t over mu this you know from twelfth grade that is not our concern, if I want to have to stationary waves on this and what would stationary waves 2, they will basically oscillate back and forth this you have done in the twelfth, but I want to see this from the perspective of these stationary wave equation that we have written.

So, that equation is going to be d 2 x by d x square plus k square x equals 0 which gives me at x equals sum A sin k x plus b cosine of k x and what are the boundary conditions the boundary conditions are that x at x equal to 0 should be 0 and x at x equals L, there is a length of a string should be 0 where this is point on the left x equals 0 point on the right is x equals L. Let us substitute this boundary condition in the solution. So, x is equal to A sin 0 plus b cosine of 0 which is equal to b this is at x equal to 0 and they should be 0 implies b is 0 and therefore, my solution for this string of length L tied between 2 ends is X x equals A sin k x that is consistent with the solutions being 0 at 2 ends, but I am still not done.

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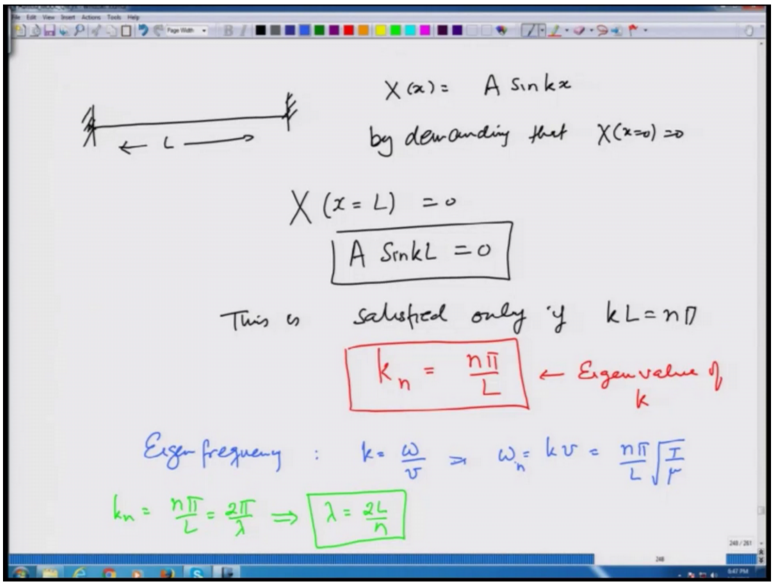

So, we have just got on this by demanding that x at x equal to 0 is 0 what about the second boundary condition which says that capital X x at x equals capital L 0 and; that means, A sin k L must be 0 no matter what a is this will not be satisfied for arbitrary k this is satisfied only if k L is n pi or k; I will label it as n k n is n pi by L.

So, you see this boundary conditions satisfied only for the certain values of k and this is it; Eigen value of k the corresponding Eigen frequencies we have seen that k is nothing but omega over v and therefore, omega is k v and these will be k is n pi by L square root of t over mu let me connect it with what you already know from your twelfth grade k n equals n pi by L is also 2 pi by lambda which immediately tells you that lambda is 2 L by n; this you know from the twelfth grade and the corresponding frequency these are the Eigen frequency let me also label these frequencies as omega n.

So, what I have told you is that when I am looking for stationary state solutions, these are not for pure frequency. These are not satisfied for any frequency, but certain frequencies which are known as Eigen frequencies and the corresponding solution. Now X x is going to be that some amplitude A sin of k n x which is the Eigen function you can try the similar solution for you know these open organ.

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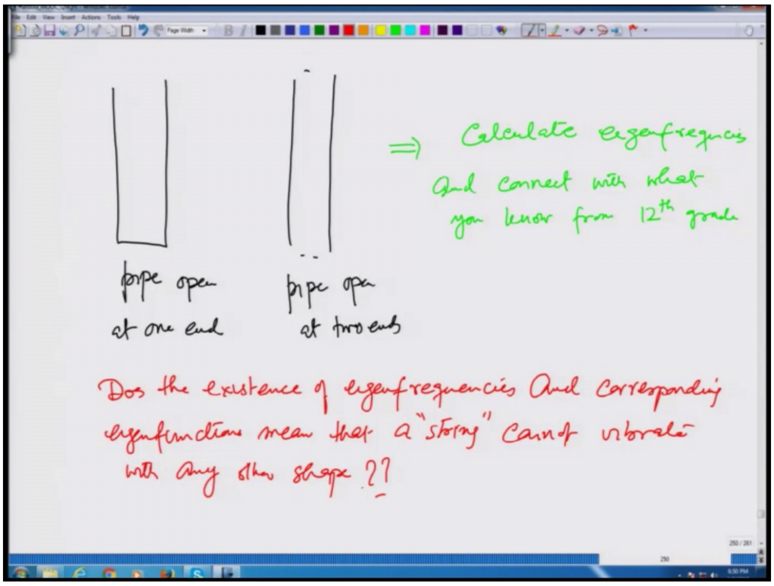

Pipes one open at one end or organ pipes which are open at 2 ends this you have seen in your twelfth grade and you can try to connect and get the Eigen frequencies for these and so this is pipe open at 2 ends this is pipe open at one end try to write these boundary conditions in the solutions for a stationary state for stationary waves in these and connected with what you know from twelfth. So, I will just say calculate Eigen frequencies and connect with what you know from twelfth grade.

Now, this is the question. So, does it mean let me raise this question does the existence of Eigen frequencies and corresponding Eigen functions mean that a string since i am taking the examples of string I will just stick to string cannot vibrate with any other shape and the answer is no it can vibrate with any other shape.

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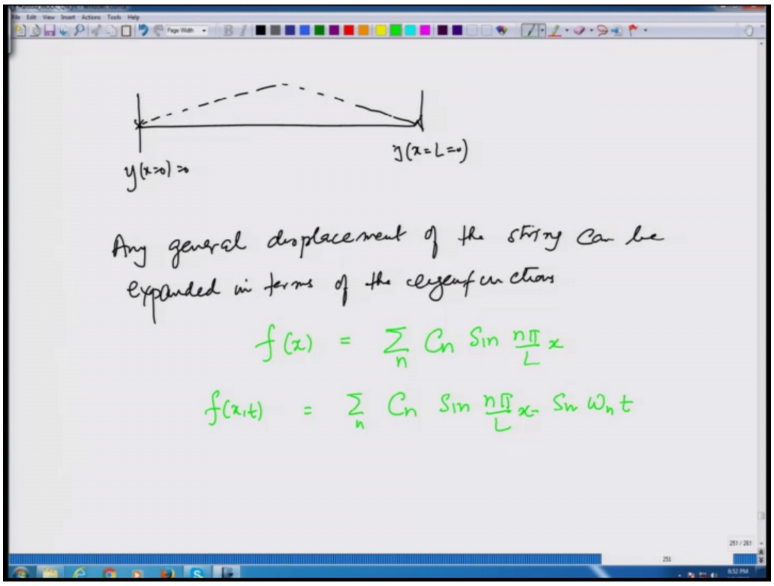

For example if I have this string and I initially disturb it like this I am make a triangle and leave it, right. So, Eigen the boundary condition satisfied that y at x equals to 0 is 0 y at x equals L it is still 0 and I just make a triangle and leave it and it vibrates; how do I describe that. So, now, let me just state this any general displacement of the string can be expanded in terms of the Eigen functions this is like the Fourier expansion we did earlier.

So, if I had the triangular shape or any other shape, let us say f x; I can write this ad summation n c n sin of n pi over L x where sin of n pi L x, we saw are the Eigen functions and therefore, f x t is going to be summation over n c n sin of n pi over L x and the corresponding time dependence is sin of omega n t. So, general time dependence will be given by all these combinations of the basic frequencies. So, it can have general time dependence and it can vibrate any other shape with time dependence being given like this.

So, what I have introduce in this lecture is the idea of Eigen functions, Eigen frequencies, Eigen values in connection with stationary states. It should be useful when we describe stationary sates solutions of the Schrodinger equation.