Introduction to Quantum Mechanics
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Lecture – 07
Heisenbergs formulation of quantum mechanics: Application to harmonic oscillator

So, what we have learnt in the previous 2 lectures is the Heisenberg's formulation of quantum mechanics and 2 things that we have learned so far our number one that quantities like xt or vt or other derived quantities are to be expressed as a collection of numbers, which we now know are matrices solid as matrices.

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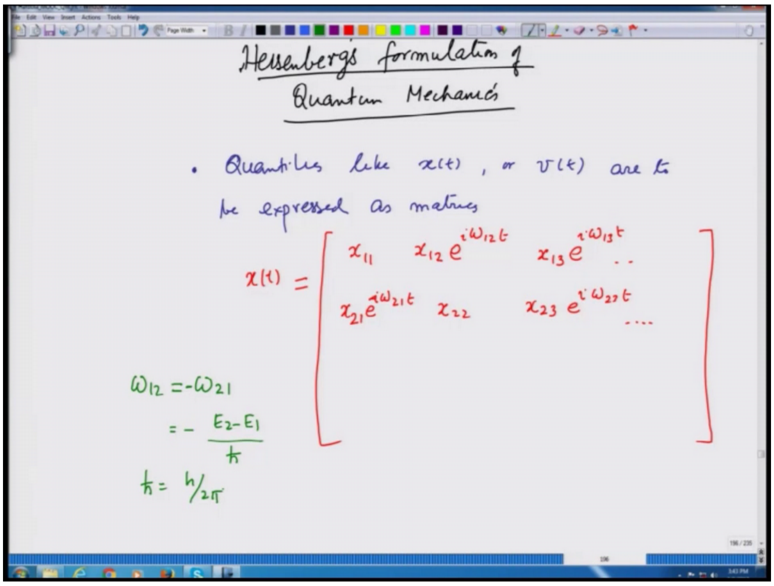

And I like this for example, if I were to express xt I would write this as x 11, x 12 and x 12 has a time dependence i omega 1 to t, x 13 e raise to i omega 13 and so on; x 21 e raise to i omega 21 t, x 22 x 23 e raise to I omega 23 t, t here and so on.

Notice that if a quantity is time independent; that means, it does not depend on time all the off diagonal terms anything containing e raise to i omega 12 t, omega i 12 or 13 t would be 0 and the matrix would be diagonal. Let me also point out that omega 12 is equal to omega 21 with a minus sign in front which is minus E2 minus E1 divided by h cross; h cross is h over 2 pi. So, that much is something that we have learned about the kinematic part of the theory, and then we obtain the quantum condition.

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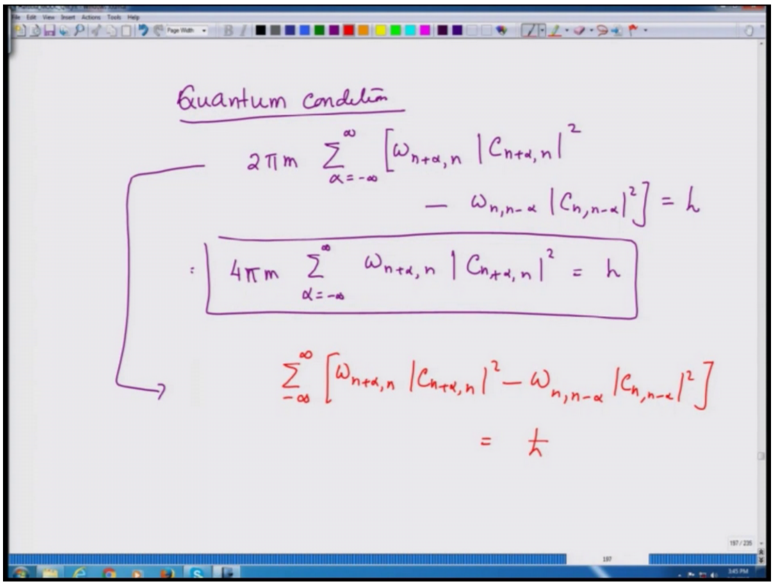

Which said that 2 pi m summation of alpha equals minus infinity to infinity, omega n plus alpha to n modulus C n plus alpha n square minus omega n, n minus alpha mod of C n, n minus alpha square is equal to h and then we also express this as 4 pi m summation of alpha equals minus infinity to infinity, omega n plus alpha n mod C n plus alpha n square equals h. This can also be written as I will take the first condition, summation minus infinity to infinity omega n plus alpha n mod C n plus alpha n square minus omega n, n minus alpha mod C n, n minus alpha square equals h cross that is also equivalent condition.

So, these are all equivalent, I just wanted to clarify something here before I proceed further and that is along the way while deriving this condition we use the C.

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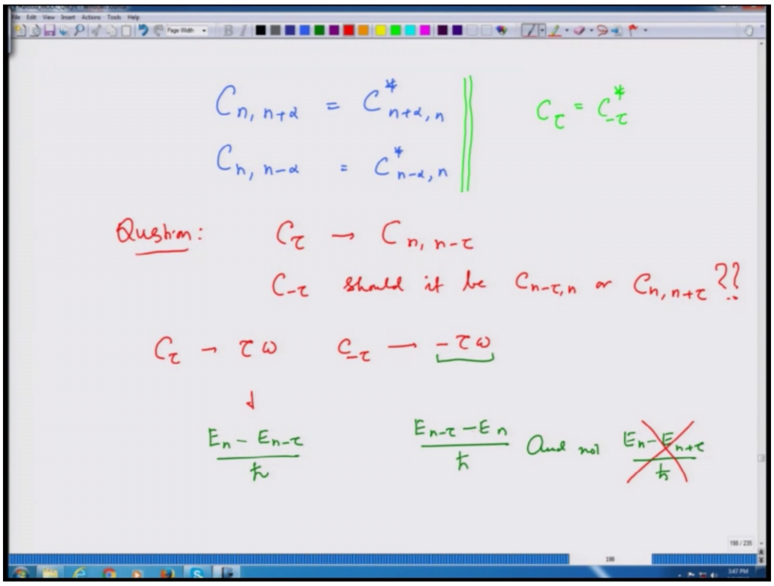

N n plus alpha was equal to C star n plus alpha n or equivalently C n, n minus alpha is equal to C star n minus alpha n, and sometimes a question is raised particularly light off that we base all this in correspondence principle, where we have said that C tau was equal to C minus tau star, and the question that arises; question is that C tau suppose it corresponds to in quantum mechanically C n to n minus tau, then C minus tau should it be C n minus tau n or C n, n plus tau that is the question. And what I have claimed is what I have shown in blue on this screen and the reason for this is as follows.

We call that C tau corresponds to the frequency tau omega, and C minus tau corresponds to the frequency minus tau omega. And tau omega quantum mechanically goes to E n minus E n minus tau divided by h cross. Correspondingly minus t tau omega would go to E n minus tau minus E n divided by h cross, and not En minus E n plus tau divided by h cross it does not go to this and therefore, the correct coefficient for the negative tau quantum mechanically would be En minus tau minus En, and the corresponding coefficient C would be C n minus tau n and not C n 2 n plus tau. So, this has been done very carefully and everything is consistent. What we are going to do now is write what the condition on the dynamical condition should be.

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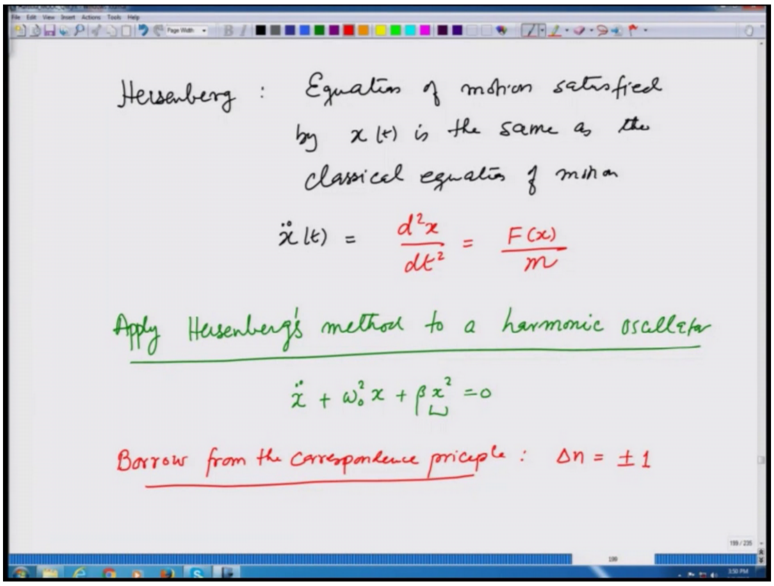

So, what Heisenberg proposed that the equation of motion satisfied by x t is the same as the classical equation of motion; that means, x double dot t which is same as d 2 x over dt square is going to be equal to Fx divided by m, except that now each x and each f x has to be calculated using these matrices and then you solve the problem. What we are going to do now is apply Heisenberg's method to a harmonic oscillator which also heated in his paper. He took an approach whereby he considered the anharmonic oscillator.

So, the equation for that is x double dot plus omega 0 square, x plus some coefficient beta x square equals 0. He took an harmonic oscillator so that he could use his product conditions. We are going to take a slightly different route because I do not want you to get walked down on how to solve for an harmonic oscillator. So, we are going to borrow something from correspondence principle. So, I am going to borrow to solve the problem from the principle, and what I borrow here is that delta n remember we derive can be maximum plus or minus 1. Heisenberg in is paper considered the n harmonic oscillator equation and could obtain delta n equals plus minus 1 in the lowest order approximation for omega.

We are going to assume that and our basis of that assumption is going to be borrowing this thing from the correspondence principle right. So, this is what we are going to do. So, let us write it step by step number one.

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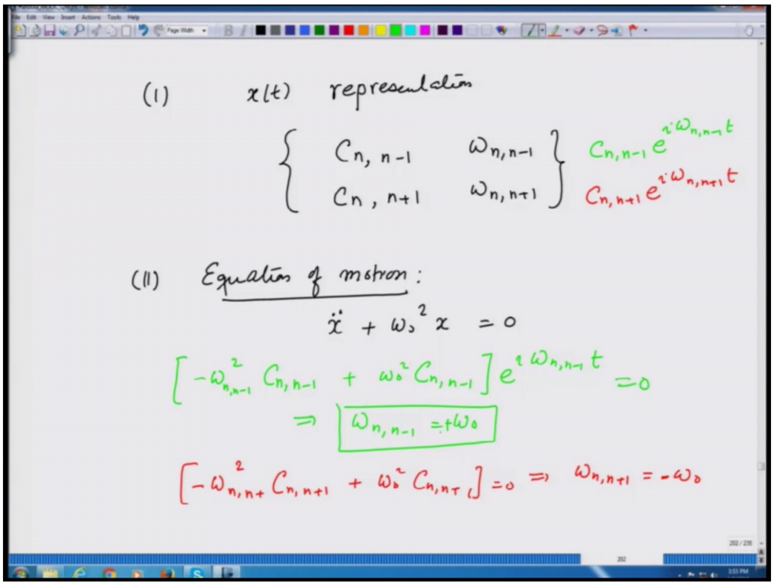

Xt representation is going to be a collection of numbers of C n, n minus 1 or C n, n plus 1 because maximum n change could be 1 and the corresponding frequency here let it be omega n, n minus 1 and omega n, n plus 1. So, that the time dependence can be written as let me write that in a different color C n, n minus 1 e raise to i omega n, n minus 1 t and the other one as C n, n plus 1 e raise to i omega n, n plus 1 t that is the time dependence.

Number 2 equation of motion is the same as the classical equation of motion, which says that x double dot plus omega naught square x is equal to 0. Let us substitute for one of the components. So, I am going to get for example, for the C n, n minus 1 I am going to get minus omega n n minus 1 square plus omega 0 square C n, n minus 1 outside I am going to have e raise to i omega n, n minus 1 t is equal to 0, and this immediately gives me that omega n, n minus 1 is equal to omega 0. In the same manner I am going to get minus omega n, n plus 1 square C n n plus 1 plus omega 0 square C n, n plus 1 is equal to 0 and this imply omega n, n plus 1 is equal to omega 0, but with a minus sign right keep this in mind because one is the frequency when the electron is making a jump from the upper to the lower level and that we have been taking a positive frequency, the other one is going to be a negative frequency.

So, once that is clear from the classical equation of motion we are ready to apply our quantum conditions.

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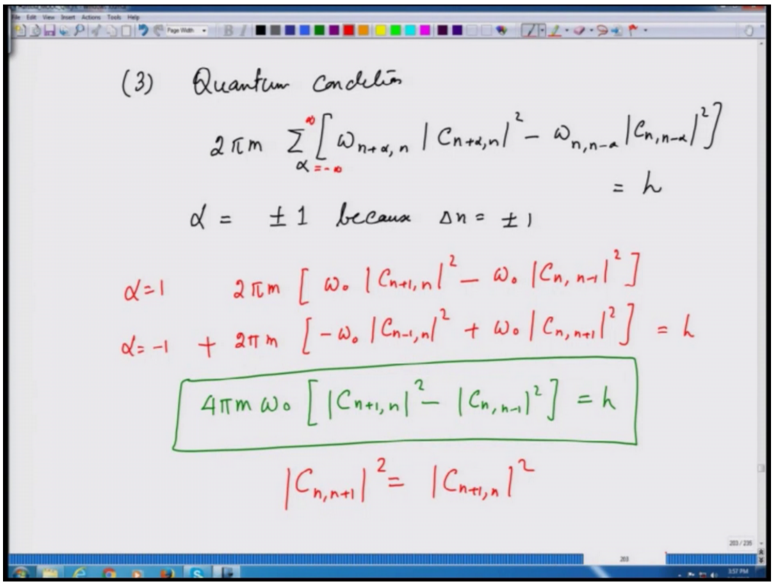

And quantum condition says the 2 pi n m summation over alpha omega n plus alpha n mod C n plus alpha, n square minus omega n, n minus alpha mod C n, n minus alpha mod square is equal to h. Now alpha in this case is plus or minus 1 because delta n is plus or minus 1. Now alpha equal to one gives you 2 pi m omega 0 that is n plus 1 to n mod C n plus 1 n square minus omega 0 mod C n n minus 1 mod square, and alpha equals minus 1 will give me 2 pi m minus omega 0 mod C n minus 1 n square minus, minus minus plus omega 0 mod C n n plus 1 mod square and some of the 2 would give me h. And you add the 2 terms you are going to get 4 pi m omega 0, C n plus 1 n mod square minus C n n minus 1 mod squared is equal to h that is what quantum condition gives me, alright.

So, this is the quantum condition and this can help me determine C n, C n plus 1 and what I have used here I have also used that C n n plus 1 mod square is same as mod C n plus 1 n mod square and same thing about C n n minus 1. So, what quantum condition has given me now is that 4 pi m omega 0.

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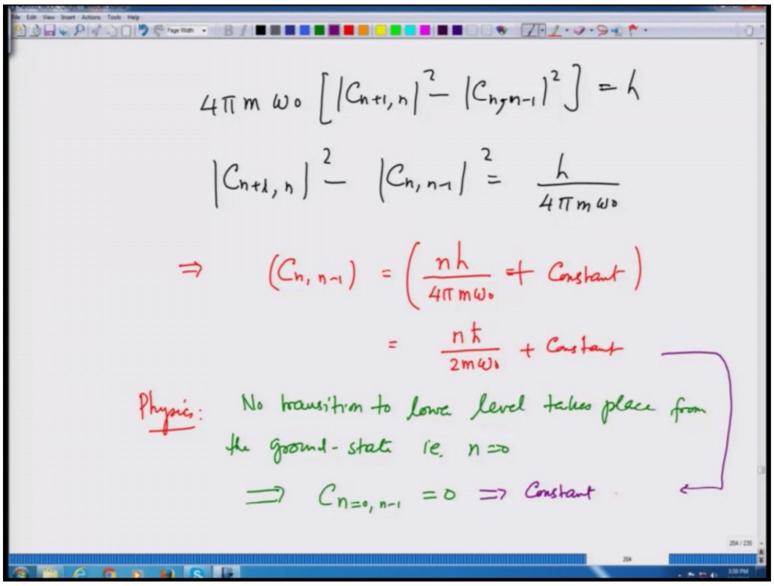

C n plus 1 n mod square minus C n my n comma n minus 1 mod square is equal to h or C n plus 1 n mod square minus C n, n minus 1 mod square equals h over 4 phi m omega 0 and this immediately implies the solution is very straightforward that C n, n minus 1 is equal to n h over 4 pi m omega 0 which I can also the plus some constant remember Heisenberg had said that that constant should be determined from quantum conditions this I can also write as n h cross over 2 m omega 0 plus some constant.

Now, we are going to use some physics. Physics is no transition to lower level takes place from the ground state that is n equal to 0 and this implies that C n equal to 0 n minus 1 should be equal to 0, which immediately tells you from this equation that constant is equal to 0.

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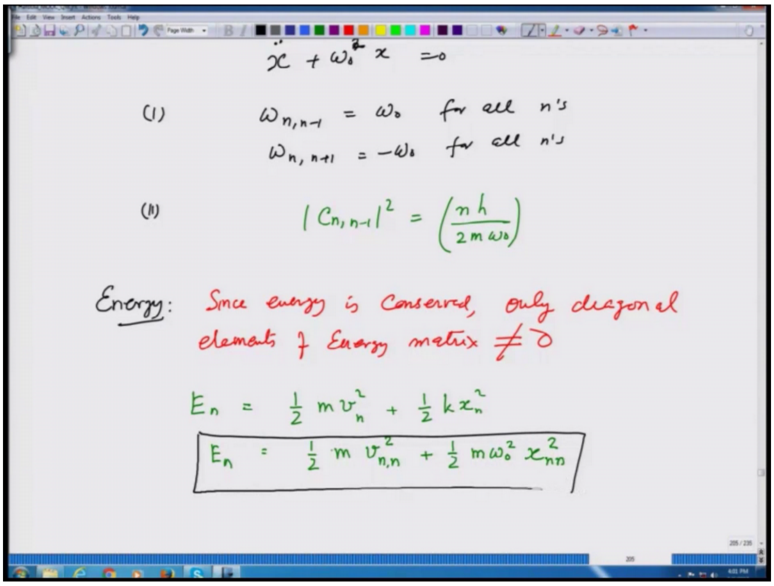

So, taking the harmonic oscillator equation the first step the classical equation gave me that omega and n minus 1 is equal to omega 0 for all ns and omega n, n plus 1 is minus omega 0 for all n's.

Number 2 we applied the quantum condition that is what we did we applied the quantum condition and got that mod C n, n minus 1 square is equal to n h over 2, m omega 0 what about the energy of the system. So, we have got on these transition matrix element now energy. Since energy is conserved only diagonal elements of energy matrix are not equal to 0, because if there were diagonal elements they will necessarily involve e raise to i omega t and that would make a time dependent.

So, if I want to calculate the energy for the nth level, this would be equal to one half mv square for the nth level plus 1 half k x square for the nth level, and this is one half m in quantum mechanical notation it will be v n n square because that is the diagonal term plus 1 half k is m omega 0 square x n n square, and that will be the energy for the nth level. This is what we need to calculate now for the system to get the nth level energy and let us do that next.

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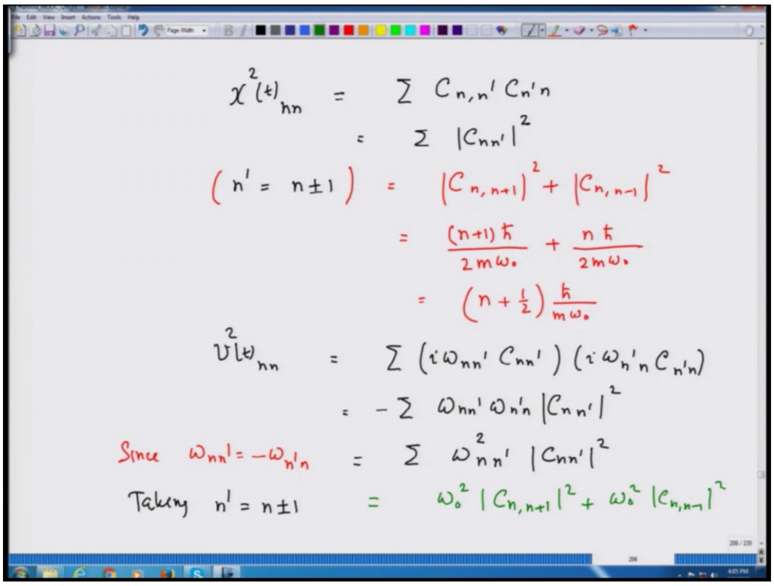

What is x this is represented by the C n numbers? So, x t square n n would be nothing but summation according to overrule C n n prime C n prime n right that is what it is and this is nothing but summation mod C n n prime square.

Now, we have seen that n prime could be n plus or minus 1 and therefore, because of this I am going to have C n n plus 1 mod square, plus C n n minus 1 mod square and these have been calculated this is nothing but n plus 1 h cross over 2 m omega 0 plus n h cross over 2 m omega 0 which comes out to be therefore, n plus a half h cross over m omega 0 what about vt prime? Vt square n n is going to be nothing but summation i omega n n prime C nn prime times i omega n prime n, C n prime n this is v square t t prime which is nothing but minus omega n n prime, omega n prime n C n n prime mod square

Since omega n n prime is minus omega n prime n, we get this equal to summation omega n, n prime square mod C n n prime square and therefore, taking n prime equals n plus minus 1 and all omega n n minus 1 to be omega 0, we get this whole thing to be equal to omega 0 square mod C n, n plus 1 square plus omega 0 square mod C n n minus 1 square, which is the same as for x square and therefore, this value is also going to be equal to omega 0 square n plus half h cross over m omega 0.

So, you have gotten v, we have gotten omega 0 energy e.

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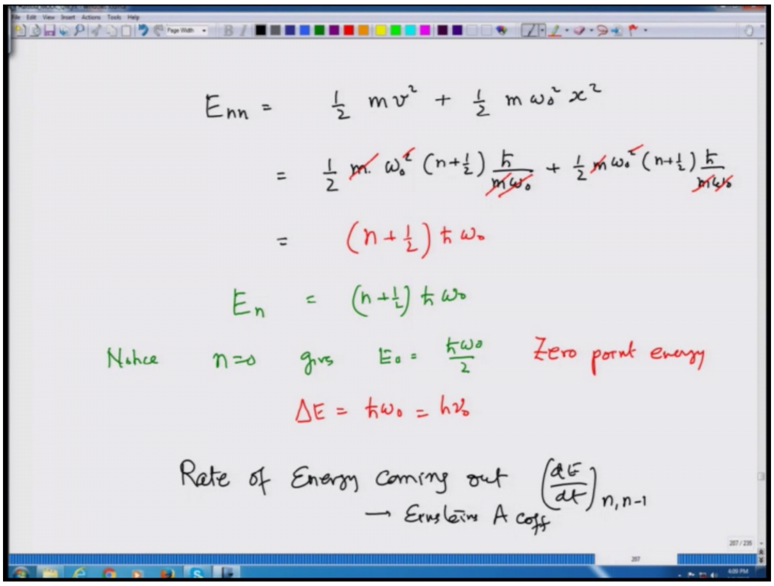

N n is one half mv square plus 1 half m omega 0 square, x square which then comes out to be one half mv square we have just calculated is nothing but omega 0 square, n plus a half h cross over m omega 0, and the second term is one half m omega 0 square this is also n plus a half h cross over m omega 0.

So, this comes out to be m cancels. So, does one of the omega zeros same thing here and you get n plus a half h cross omega 0 that is the energy.

So, according to Heisenberg's quantum mechanics then we get e for the nth level to be equal to n plus a half h cross omega 0 notice n equal to 0 gives e 0 to the h cross omega 0 x 2 it is not 0 energy as in the classical sense or even in Wilsons Somerfield field quantum conditions the lowest energy the ground state energy is h cross omega 0 by 2. This is known as 0 point energy, and delta E is a still h cross omega 0 or h nu 0 which it should be if we were to reduce the blackbody radiation formula. So, that comes out correctly and the energy has 0 point energy what Heisenberg has been able to do in all this is calculate the energy purely from quantum mechanical considerations, where the quantum mechanical, quantities are expressed as matrices quantum conditions and equation of motion help you determine these coefficients that the proposed.

What about the rate of energy coming out or dE dt when transition is made from n to n minus 1 and that gives you Einstein's a coefficient also, remember we had earlier calculated it using correspondence principle. So, let us do that.

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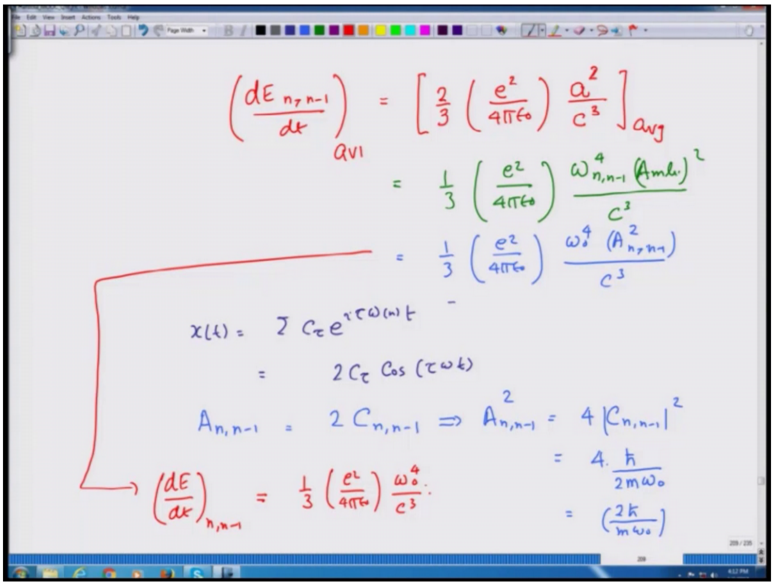

DE n to n minus 1 by dt, I am going to use the classical formula is going to be given as time averaged of 2 thirds e square over 4 pi epsilon 0, acceleration raise 2 acceleration square divided by C cube this is the formula we had used and when I take the average I take the average of this, this comes out to be one third e square over 4 pi epsilon 0 omega alright n to n minus 1 raise to 4 amplitude square divided by C cube.

Now, we have already seen that omega n n minus 1 and all these things are the same as omega 0. So, this is going to be one third e square over 4 pi epsilon 0 omega 0 raise to 4 amplitude square n to n minus 1 transition amplitude divided by C cube. What about this amplitude? Again go back to the classical C n n and I will say that the xt was written as summation, C tau e raise to i tau omega for the nth level t, which could be written as taking C to be real as 2 C tau cosine of tau omega t. In the quantum mechanical sense then I am going to write that the amplitude n to n minus 1 is going to be 2 C n, n minus 1 making that correspondence and this implies that A n, n minus 1 square is going to be 4 times C n, n minus 1 square which we have already calculated to be 4 h cross over 2 m omega 0.

So, this comes out to be 2 h cross over m omega 0, I will substitute this in the formula here and get dEby dt n to n minus 1 to be equal to one third e square over four pi epsilon 0 omega 0 raise to 4 over C cubed and 4 a n square I get 2 h cross over m omega 0. Let us cancel a few terms this omega 0 gives me omega 0 cubed and therefore, I get this 2 I can bring in front and therefore, I can write dE dt transition from n to n minus 1 is equal to 2 thirds, e square over 4 pi epsilon 0 omega 0 cubed at h bar over m c cubed.

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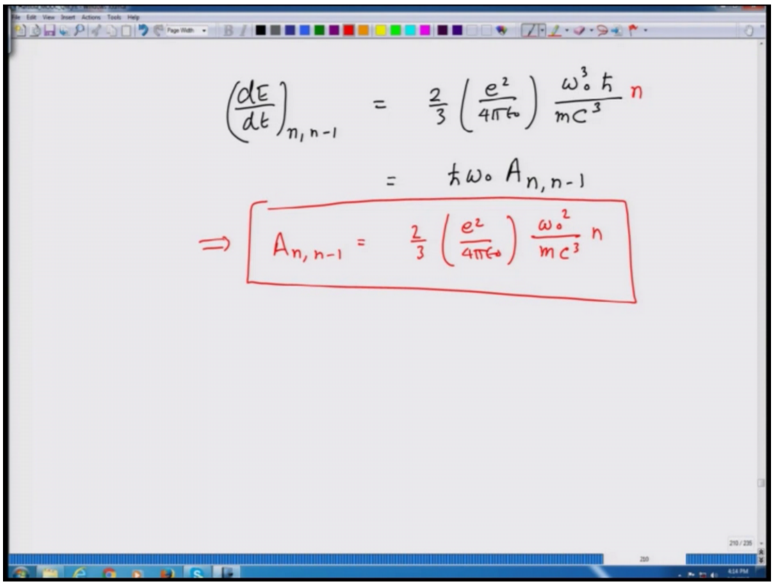

I recall that this is also equal to h cross omega 0 times the Einstein coefficient for a to n, n minus 1 and this immediately gives me the answer that the Einstein coefficient n minus 1 is going to be 2 thirds e square over 4 pi epsilon 0 omega 0 square over mc cubed. There is an n also because the C n factor has n and this is precisely the answer we had obtained earlier. So, quantum mechanically we have also calculated this through a purely quantum mechanical calculation.

So, let me now conclude this lecture on Heisenberg's application of his equation to harmonic oscillator and write.

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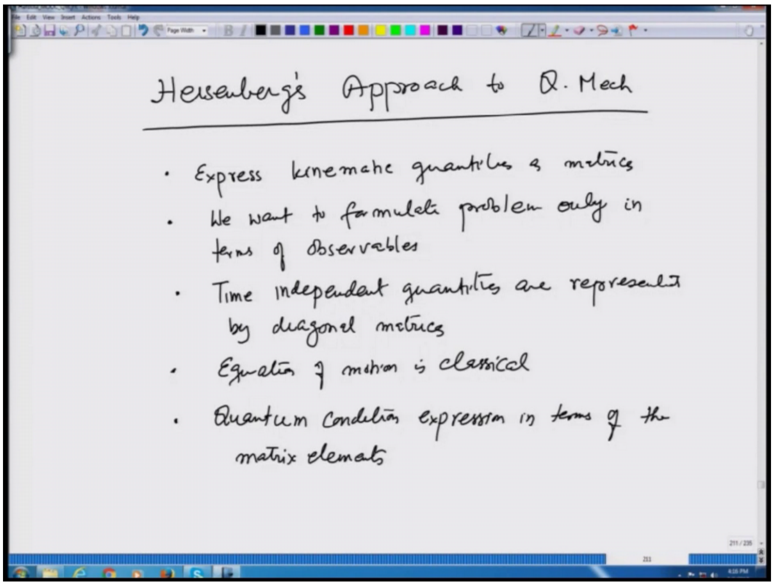

So, this is basically Heisenberg's approach to quantum mechanics is conclude number one express kinematic quantities as matrices. And that is done because we want to formulate problem only in terms of observables time independent quantities are represented by diagonal matrices, then equation of motion is classical and quantum condition expressed in terms of the matrix elements.

So, this is the conclusion of Heisenberg's approach, this was made more general and put on a stronger footing and something called the matrix mechanics, to which I will just give you a brief introduction in the next lecture.