Introduction to Quantum Mechanics
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Lecture – 02
Wilson-Sommerfeld quantum condition II - Particle moving in a Columb potential in a plane and related quantum numbers

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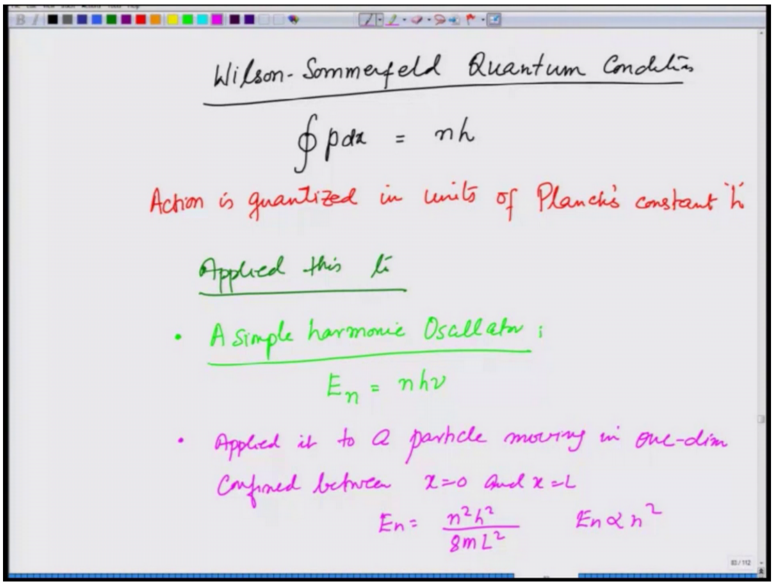

So, the previous lecture I introduced the Wilson Sommerfeld quantum condition that said that for a periodic motion p d x was equal to n h or the actions is quantized. So, let me write this inwards action is quantized in units of Planck's constant h and e applied this to applied this 2 number; one the simple harmonic oscillator that is the particle of mass m performing simple harmonic motion and we found that the energy n th level is given as n h nu which matched with whatever explain the black body spectrum earlier and we also applied a 2; a particle moving in one dimension say along the x axis confined between x equals 0 and x equals L and we found that E n was given as n square h square over it m L square.

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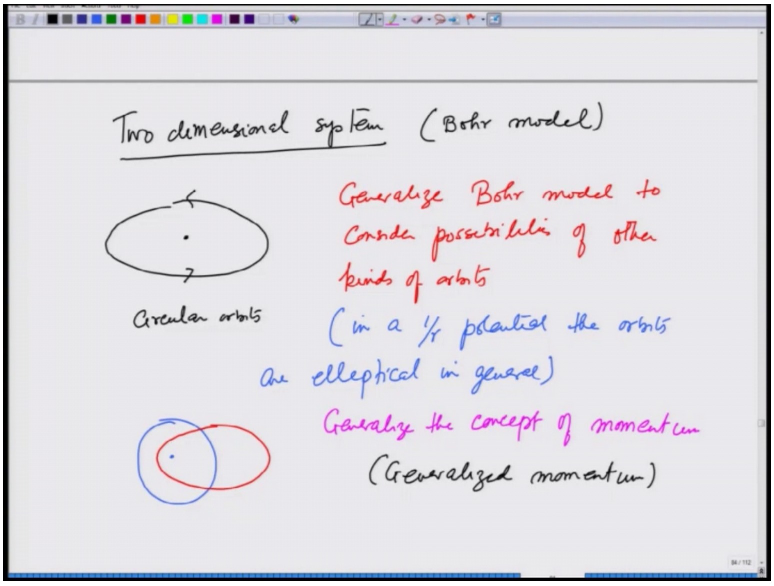

So, E n was proportional to n square; let us now apply it to a 2 dimensional system which will correspond to Bohr model. So, recall what the Bohr model did Bohr model did was that it considered electrons to move be moving in circular orbits around the nucleus and because angular momentum is conserved the plane of the orbit is the same. We are going to now generalized this. So, we are going to generalize Bohr model to considered possibilities of other kinds of orbits and what are the other kinds of orbits from classical mechanics I know that in a one over r potential the orbits or elliptical in general; that means, what I can have is I can have an electron moving in a circle I can also have an electron moving in an elliptical orbit and how are these described and you will see that they come out naturally from Bohr Sommerfeld quantization condition.

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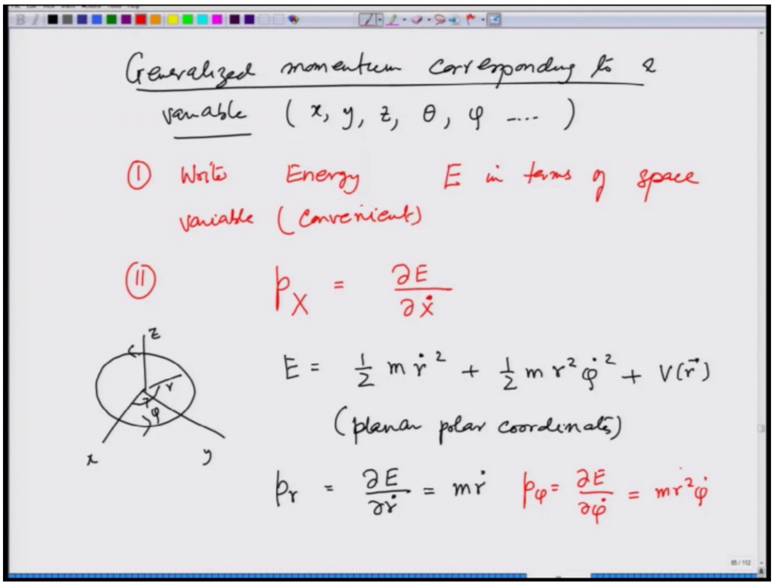

So, in this again, now I am going to now first generalize the concept of momentum. Now I am going to call this generalized momentum and for our purposes for our purposes what I am going to do is define it in the following manner corresponding to a variable that variable could be x, y, z theta phi. So, I can take any variable to describe the position and define a generalize momentum. So, what we do for this is right step number one write energy E in terms of space variables whichever ask convenient and 2 p corresponding that variable let me for the time being; call it capital X is given as partial derivative of e divided by partial x dot.

So, let me give you an example, suppose a particle is moving in a 2 dimensional world then the energy, E is given as 1 half m r dot square where r is the distance, plus one half m r square phi dot square where phi is the angle from the x axis. So, what I am writing in the planner polar coordinates and in addition the potential energy which may depend on r in general a vector, but right now let it be r.

Then p r will be defined as partial of E over partial r dot which will come out to be m r dot similarly p phi will be defined as partial e by partial phi dot which will come out to be m r square phi dot. So, this is the concept of generalized momentum and let me show you why it is called generalized momentum.

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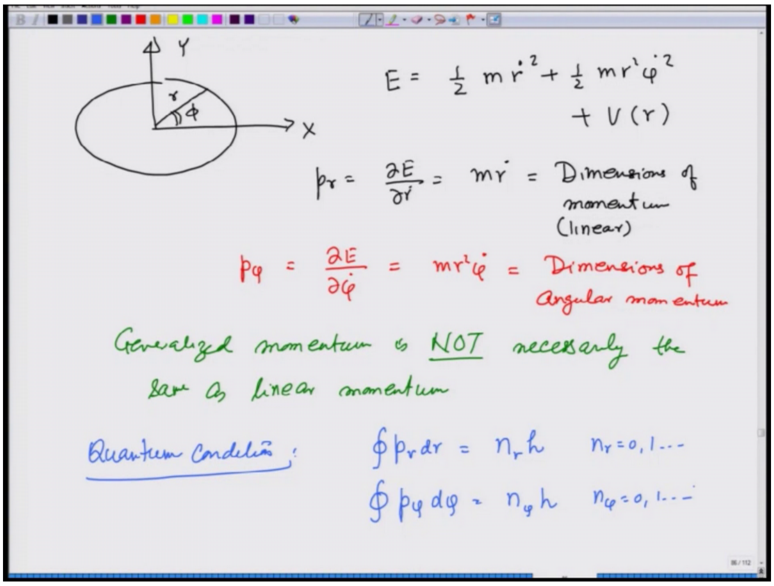

So, we considered a particle moving in 2 dimensional space; let us just confined our say as to 2 dimension x and y and we considered the variables that describes phi and r. So, we have energy which is one half m r dot square plus 1 half m r square phi dot square plus V r which may depend on the vector itself, but I will take v r only. So, that its central potential then p r which is d E by d r dot is m r dot and has dimensions of momentum that is linear momentum and p phi corresponding to variable phi is partial e over partial phi dot which is m r square phi dot and this has dimensions of angular momentum.

So, notice that generalized momentum is not necessarily same as linear momentum it could be angular momentum and the quantum condition now would be. So, the quantum condition in terms of generalized momentum would be integral p r; d r is equal to an interior n let me call it n r h an integral p phi d phi is going to be equal to sum n phi h where n r would be 0 1 and so on and phi could be 0 1 and so on; this is the quantum conditions now notice that now there are 2 conditions 2 quantum conditions one on r and 1 on phi and these 2 quantum conditions determine the orbit right. So, let us just spend some more time on this.

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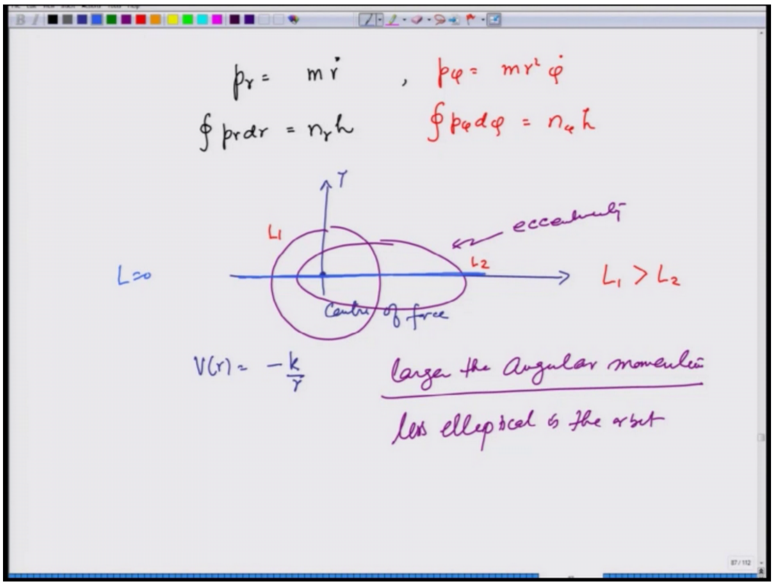

So, I have p r which is m r dot and p phi which is m r square phi dot and the corresponding quantum conditions integral p r d r and is equal to n r h and integral p phi d phi and against this period is equal to m phi h input a could be.

What is happening is that if you consider the motion in the x y plane with centre of force, we know from classical mechanics that in this case if V r is equal to minus k over r that is it is an attractive coulomb or gravitation kind of potential, then the orbits are like this either they could be circular or they could be elliptical and this has some eccentricity and larger the angular momentum less elliptical is the orbit. So, if I have considered suppose the circular orbit has a angular moment rum L 1 and the elliptical one is L 2 and L 1 could be greater than L 2 this makes sense because if the angular momentum is 0 suppose angular momentum is 0 L equal to 0, then the motion will be linear passing through the origin. So, in that case the motion will be linear passing through the origin.

So, smaller the angular momentum larger the ellipticity or extensity of the orbit right now let us supply quantum condition, now in this case is notice that for a given orbit.

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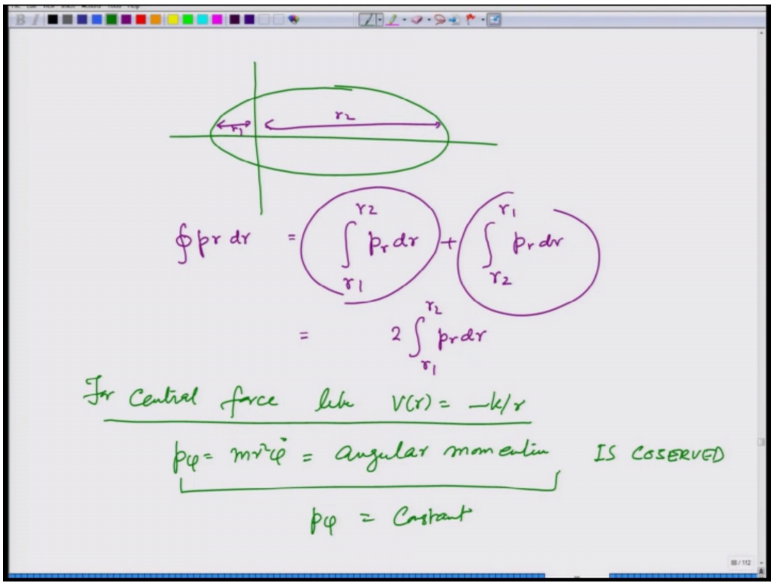

So, if there is a given orbit like this, there is a distance r 1 smaller distance r 2 and when I calculate for example, p r d r, I will be going from r 1 to r 2 p r d r plus to complete the full period I will be coming from r 2 to r 1 p r d r and that would give me both integrals give me the same answer because the symmetry and therefore, this is nothing, but 2 times r 1 to r 2 p r d r.

On the other hand for central force like V r equals minus k over r, p phi which is m r square phi dot which is nothing, but the angular momentum is conserved this quantity is conserved and therefore, p phi is going to be constant for a central force. So, now, we are now ready to apply all this that we have done so far to calculate energy levels of the system doing a motion in a plane or in a planar orbit in columbic fees and that should give us the same answer as Bohr model.

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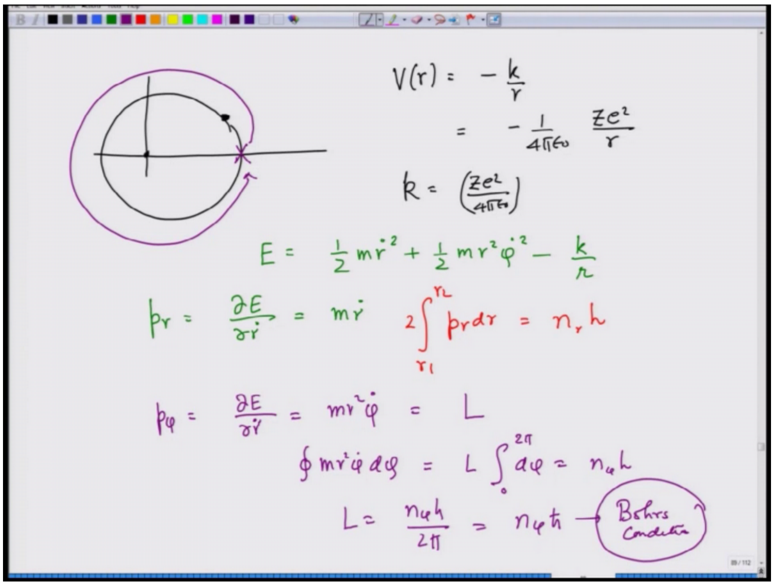

What we are now doing is taking the motion of a particle in a plane around a potential centre which is minus k over r and the case of atoms it is nothing, but minus 1 over 4 pi epsilon 0, z e square over r where e is the charge of the electron going around, alright, but for the time being just for convenience I will keep this as k. So, k equals z e square over 4 pi epsilon 0 and the energy of the system is given as 1 half m r dot square plus one half m r square phi dot square minus k over r, p r is equal to partial e over partial r dot which is m r dot and the corresponding quantum condition is r 1 to r 2 times 2, p r d r is equal to n r h the corresponding p phi equation is equal to till partial e over partial r dot which is m r square r dot.

Phi dot which is a constant let us call this L and the corresponding condition is m r square phi dot d phi over the whole period; that means, if I start from one point from this cross and go all over that is v do d phi integral from 0 to phi this is equal to L time 0 to 2 pi d phi.

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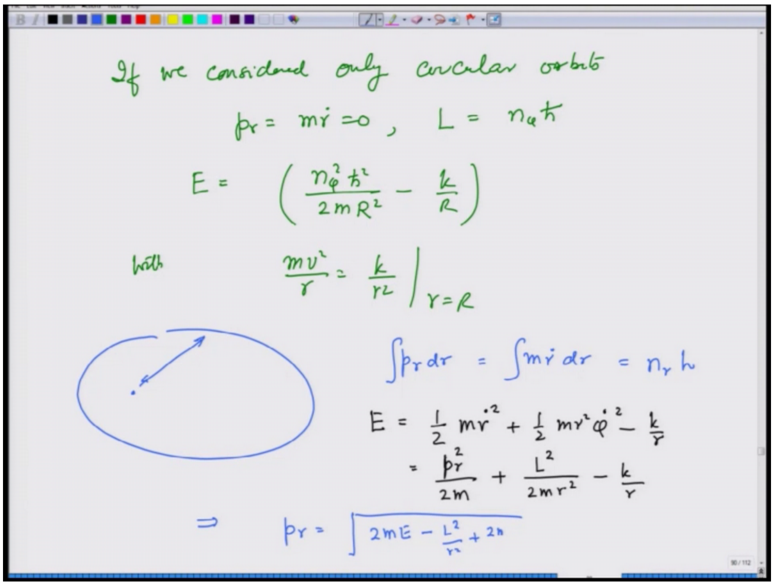

This is equal to n phi h which gives me L equals n phi h over 2 phi or n phi h cross which is the same as Bohr's condition. If we considered only orbits then we have p r is equal to m r dot is equal to 0, L is equal to n phi h cross and the energy would be equal to one half m r dot square which will be 0 and phi square h cross square over 2 m radius square minus k over r, and with m v square over r equals k over r square with r equals R this will give me the same answer as Bohr model.

Now, we are considering the possibility that the particle can also perform motion in r direction. So, there is an r dot involved. So, this gives me the elliptical orbits and for that I have p r d r which is non 0 which is m r dot d r equals n r h. So, now, I have 2 quantum numbers n r and n phi. So, let us calculate this. From the energy condition e equals one half m r dot square plus one half m r square phi dot square minus k over r, which is same as p r square over 2 m plus L square over 2 m r square minus k over r, I get p r is equal to square root of 2 m E minus L square over r square plus 2 m k over r, right.

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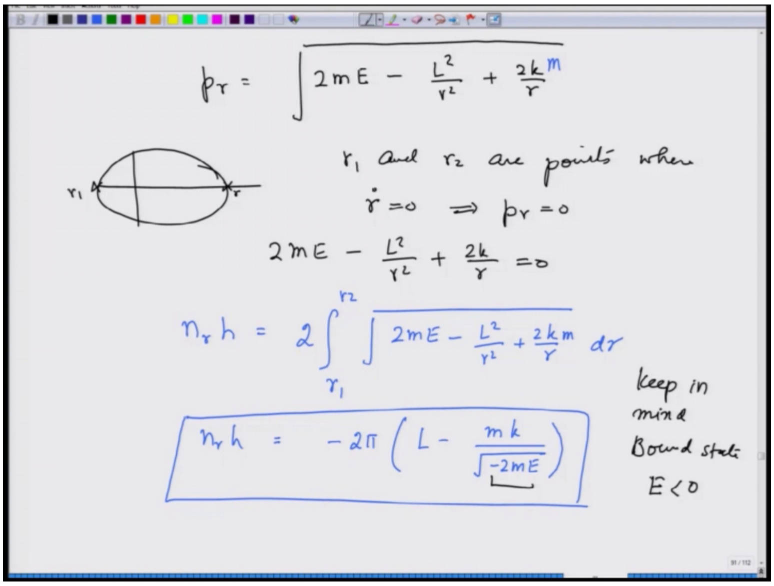

So, I have p r is equal to 2 m E minus L square over r square plus 2 k over r square root.

The solve it is like this I have a r 1, I have a r 2, r 1 and r 2 points where r dot is equal to 0 which implies p r is also be equal to 0. So, those can be calculated from the expression while writing 2 m e minus L square over r square plus 2 k over r is equal to 0. So, what one I have is therefore, n r h would be equal to integral r 1 to r 2 integral of 2 m e minus L square over r square plus 2 k over r t r times 2 this integral slightly complicated. So, I will just give with the answer, the answer for this whole thing comes out to be minus 2 pi L minus m k sorry this is a m here, there is a m here m k over square root of minus 2 m e that is it. So, this is n r h this giving you the final answer for the integral and this is what is going to determine the energy.

Now, you may wonder why these minus 2 m E? Keep in mind that this is a bound state and therefore, E is less than 0. So, what we get is.

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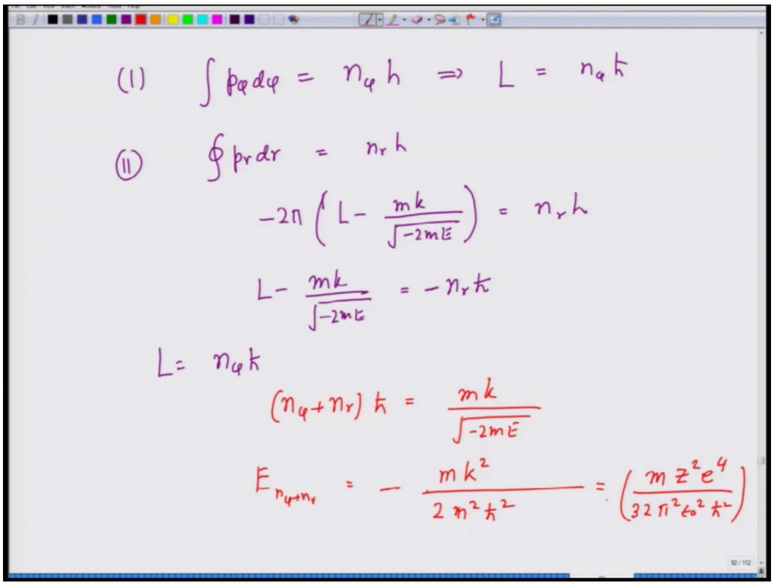

So, from the first condition integral p phi d phi is equal to n phi h, I get L equals n phi h cross and from the second condition that integral p r d r is equal to n r h, I get minus 2 pi L minus m k over square root of minus 2 m E equals n r h and therefore, L minus m k over square root of minus 2 m e is equal to n r h cross with the minus sign or L which is m phi h cross, if I substitute that I get n phi plus n r h cross is equal to m k over square root of minus 2 m e or e equals m phi plus n r equals k square over 2, there will be an amount of n square h cross square with the minus sign which is same as m z square e raise to 4 over 32 pi square epsilon 0 square h cross square to the minus sign 1 over times n over m square.

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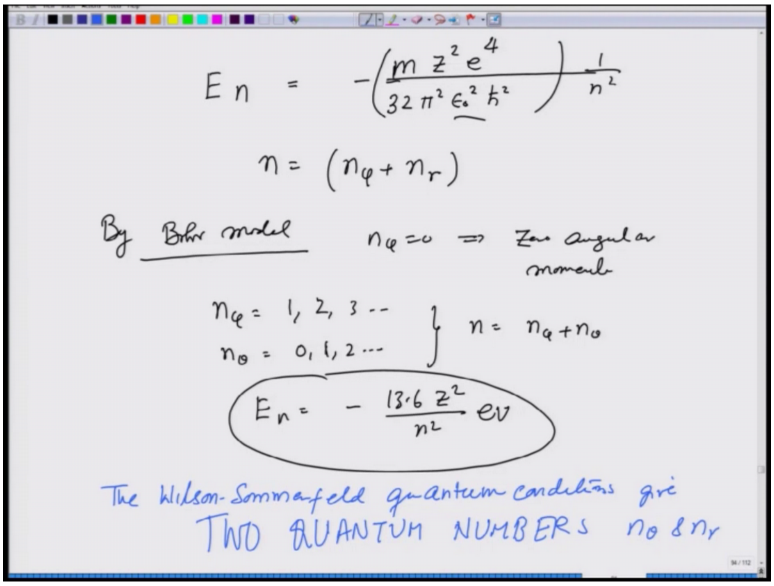

So, we getting an energy e which is equal to minus m z square e raise to 4 over 32 pi square epsilon 0 square h cross square 1 over n square, let me call this n where n is n phi plus n r. Now by Bohr model or also because n phi equal 0 would imply 0 angular momentum how to exclude that and call n phi equals 1, 2, 3 and so on and n theta equal 0 1, 2 and so on and n equals n phi plus n theta and. So, energy E n is given by this number which is same as the Bohr answer minus 13.6 z square over n square electron volts you can calculate this number in that comes out to be, which is exactly the same answer as the Bohr model; what it gives you; however, are now let us just analyze this the Wilson quantum conditions give 2 quantum numbers and phi and n r.

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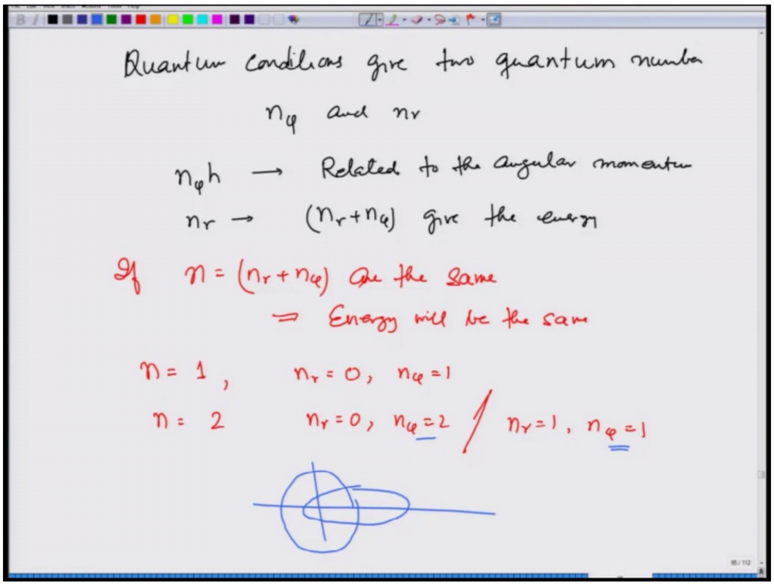

So, quantum conditions give 2 quantum numbers n phi and n r and let us see; what they mean. N phi h is related to the angular momentum and n r the such that n r plus n phi give the energy. So, if n equals n r plus n phi are the same energy will be the same; however, these are 2 different orbits for example, suppose n is equal to one I could have n r equals 0 and n phi equals 1. Suppose n equals 2, I could have n r equals 0 n phi equals 2 or n r equals 1 and n phi equals 1 they both give the same energy.

Now, what kind of orbits would they be if n phi is 2, I would expect this to have less eccentricities and that could be a circular orbit; n phi is one that is less of an angular momentum. So, it could be an elliptic orbit. So, what we have now introduce generalize the idea of these orbits of electrons moving in a plane when they are moving under the influence of coulomb potential through Wilson Sommerfield quantization conditions now will a more orbits then one that have the same energy; however, different quantum numbers.

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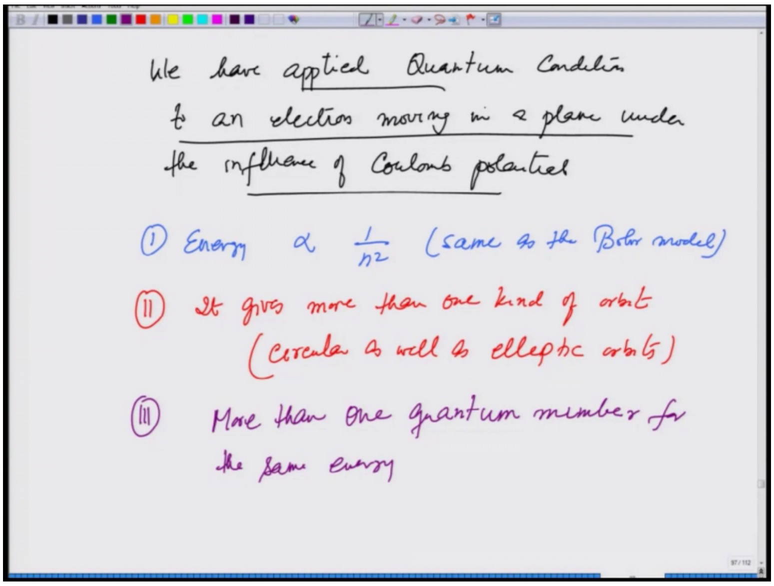

So, to conclude this lecture, we have applied quantum conditions to an electron moving in a plane under the influence of coulomb potential and what do we find one energy comes out to be proportional to 1 over n square same as the Bohr model. More important however, is it gives more than one kind of orbit and circular as well as elliptic orbits and it introduces the idea of more than one quantum number for the same energy more than one quantum number means they could be more orbits than one orbit which has the same energy and in this case they happened to be circular elliptic orbits of different eccentricities.