Introduction to Quantum Mechanics

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Lecture – 01
Wilson-Sommerfeld quantum condition I - Harmonic Oscillator and particle in a box

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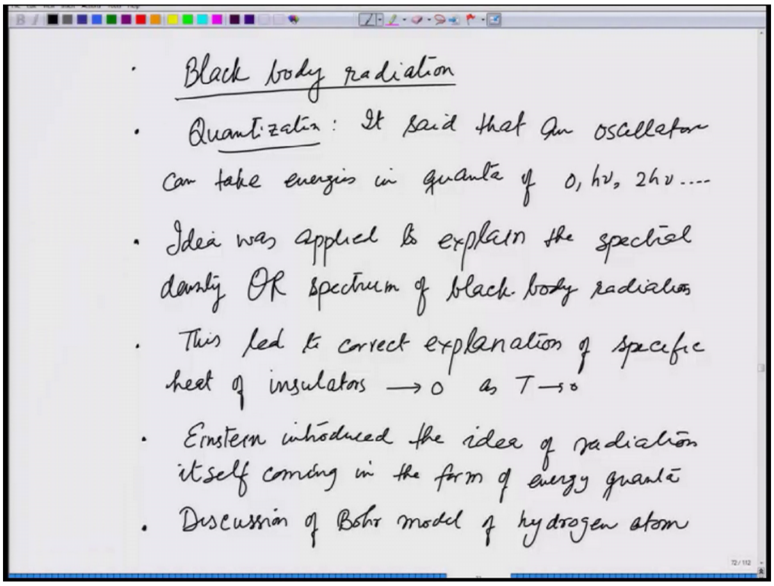

Last week we started our discussion on quantum theory with black body radiation. And introduced the idea of quantization it said that an oscillator and I am not distinguishing between an oscillator giving out radiation or a mechanical oscillator. So, it said that an oscillator can take energies in quanta of 0, h nu, 2 h nu and so on so integer number of h nu. Then this idea was applied idea was applied to explain the spectral density or spectrum of black body radiation.

Also this led to correct explanation of specific heat of insulators going to 0 as temperature goes to 0. Then Einstein introduced the idea of the radiation itself coming in the form of energy quanta, which we now call photons. And finally, we ended the week with the discussion of Bohr model of hydrogen atom, where by applying quantum foundations, he obtained the correct explanation of hydrogen spectrum.

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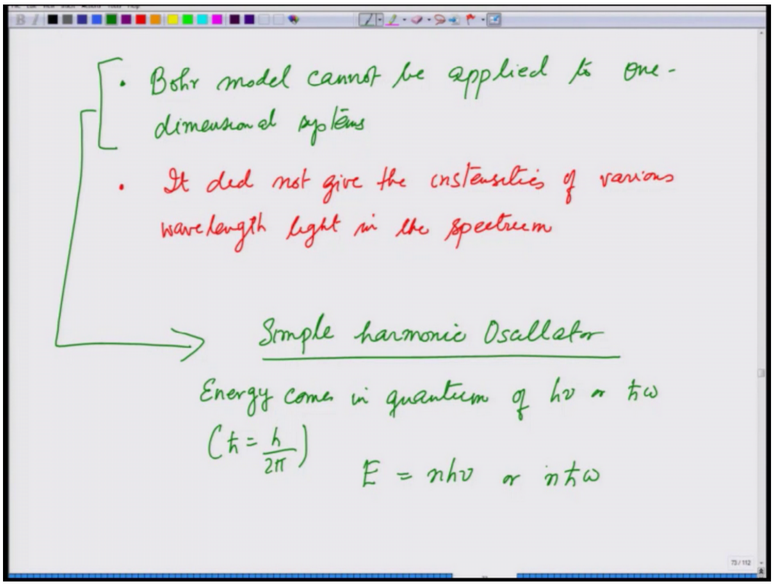

At the end of that, the final lecture I had said that Bohr model cannot be applied to one-dimensional systems and also I had said that it did not give the intensities of various wavelengths light in the spectrum. So, in that sense, all though quantum ideas were applied, they gave the right answer, but certainly it was not complete, it was just the beginning. And one had to find answers one had to develop a theory as to how to get the answers of say one-dimensional systems or other system systems other than hydrogen atom, and all also how to calculate intensities of various wavelengths that exist in a spectrum.

So, what we are going to focus today in this lecture on is the first part. And let me start again with a simple harmonic oscillator for which I know that the energy comes in quantum of h nu or h cross omega, where h cross is h upon 2 pi. And because of that explained the black body spectrum and also explained the going to 0 of the specific heat of solids, how to get an answer for this. The correct answer that is E equals n h nu or n h cross omega correctly what kind of theory would give it.

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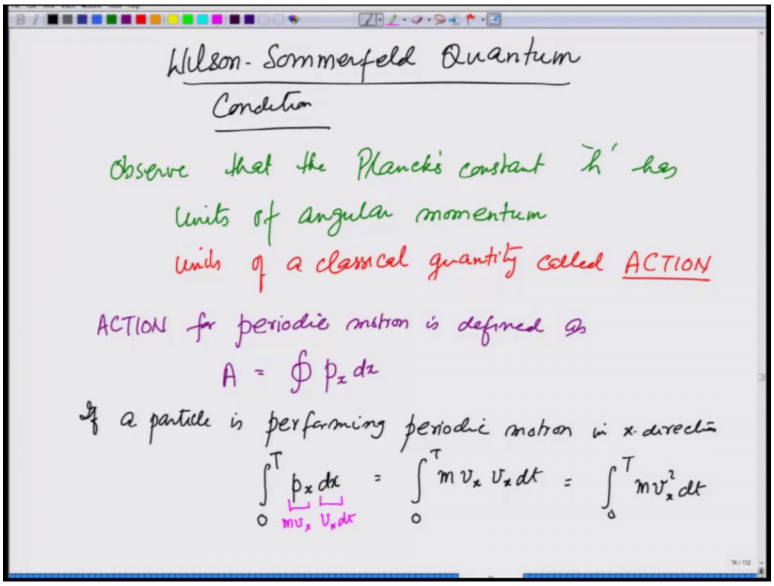

And that is where Wilson Sommerfeld quantum conditions come into the picture. This condition is such that it correctly it produces Bohr model at the same time makes quantum mechanics applicable to other systems. So, for these observe that the Planck's constant h has units of angular momentum, and I am actually going to introduce one more word and that is it has units of a classical quantity called action. And actions for periodic motion is defined as I will explain what this circle means p, let us call it p x d x this is reaction.

So, what it means if a particle is performing periodic motion then integrate over the entire period from 0 to T p is momentum suppose p performing periodic motion n x directions they take p x dx which will also be the same as m v x v x dt, because this quantity dx is nothing but v x dt. p is the momentum. So, this is m v x and this is integrated from 0 to T. And this is therefore, 0 to T m p h square dt this is the quantity called action.

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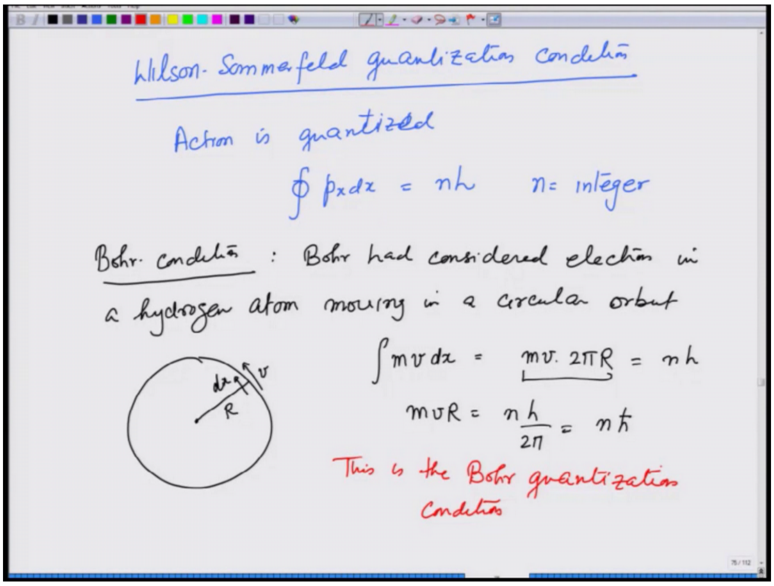

And what Sommerfeld quantization condition cells is that Wilson-Feld quantization condition it says that action is quantized that means, integral over a period of p x dx comes in units of h, they have the same units. So, n h, where n is an integer. Let us first see if it reproduces Bohr condition. The Bohr had considered electron in a hydrogen atom moving in a circular orbit. So, if it moving in a circular orbit of radius r, its velocity is in this direction, then m v times dx will give me. so v is in the same direction has dx this is the displacement dx right all though I am take considering circular motion v is a constant m is a constant we will give me 2 pi r. And according to Wilson-Sommerfeld condition this must equal n h.

And therefore, m v r would be equal to n h over 2 pi which I will write as n h cross, and this is precisely the Bohr condition. This is the Bohr quantization condition. So, Wilson-Sommerfeld quantization condition is consistent with Bohr's quantization conditions. So, it will explain hydrogen spectrum. In addition, now I am going to show you that will give me the energy levels of a harmonic oscillator also.

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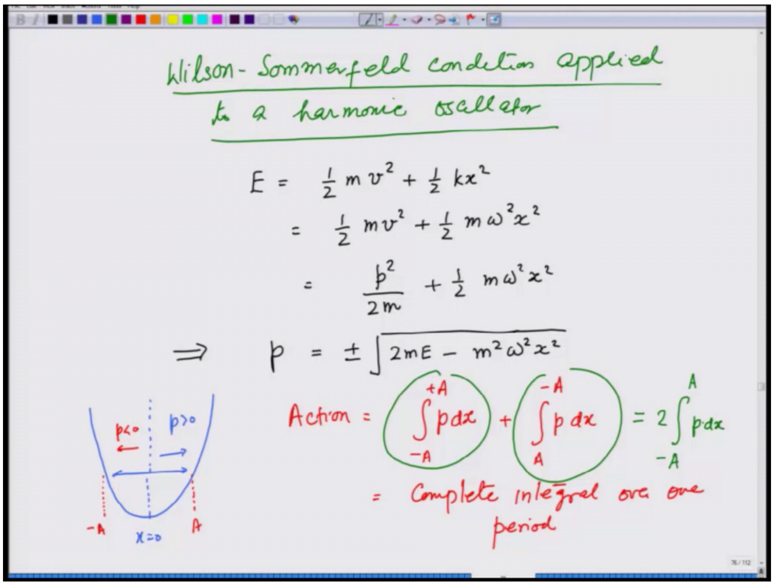

So, let us do that. So, Wilson-Sommerfeld condition applied to a harmonic oscillator for which recall that we already know the answer. So, for a harmonic oscillator, the energy E is given as one half m v square plus one half k x square which is also written as one half m v square plus one half m omega square x square. I can write this as p square over 2 m plus one half m omega square x square. This immediately gives me that the momentum p is equal to square root of 2 m E minus m square omega square x square does the momentum, it could be plus or minus. So, in a harmonic oscillator when the particle is going back and forth, and let us say that I take the origin to be at the minimum, when the particle is going to the right p is positive and when the particle is going to the left p is negative.

And what I want to calculate is p action is going to be integral p dx from starts form let us say the minus the amplitude to plus the amplitude minus A to plus A and then back plus integral p dx A to minus A that would be complete integral over one period. So, that will be the integral now while going from right to left p is negative and d x is also negative while going from right to left p as positive and dx is also positive. So, both these quantities will give me exactly the same answer. So, I can easily write this as two times x equals minus A A p dx that will be the action.

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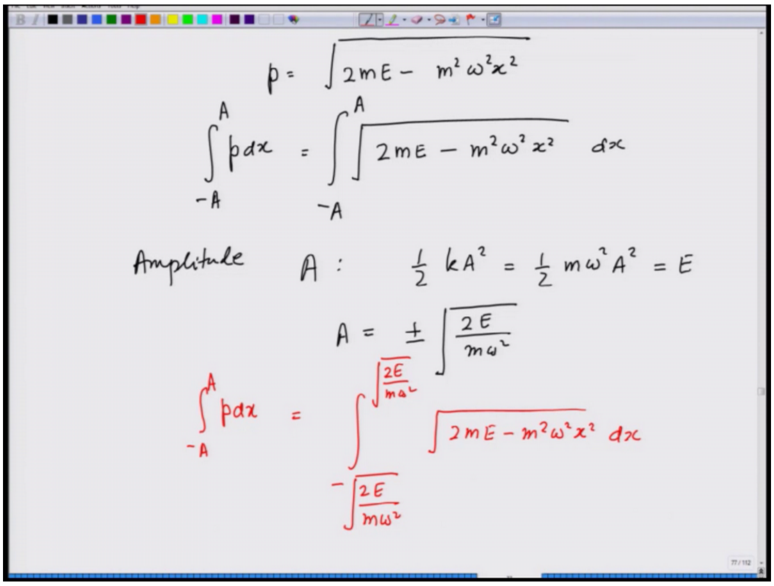

So, let us evaluate this. So, when I evaluate this, I have p equals 2 m E minus m square omega square x square square root. So, p dx from minus A to A will be equal to square root of 2 m E minus m square omega square x square dx from minus A to A. So, now, the amplitude A such that one half k A square which is one half m omega square A square is equal to the total energy. And therefore, amplitude A is plus or minus square root of 2 E over m omega square. So, I have minus A to A p dx is equal to minus square root of 2 E over m omega square to plus 2 E over m omega square square root of 2 m E minus m square omega square x square dx, and this is an easy integral to calculate.

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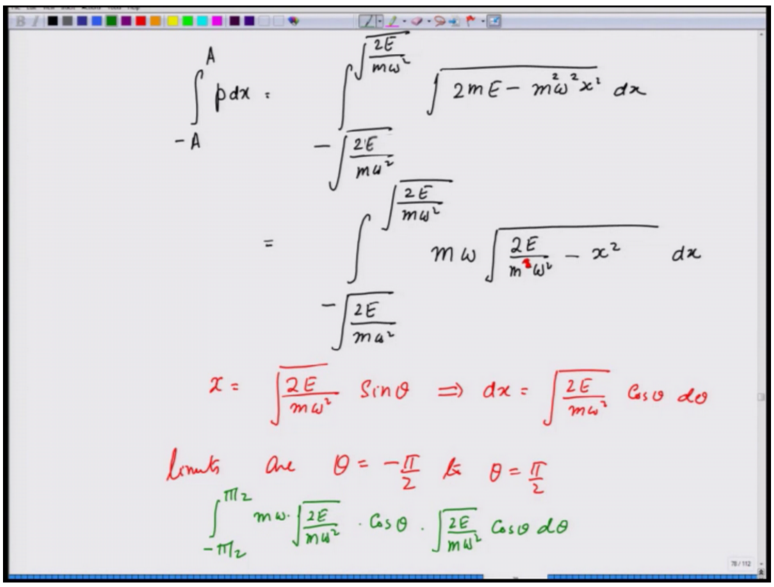

So, on calculating integral minus A to A p dx which is equal to integral minus square root of 2 E over m omega square to square root of 2 E over m omega square square root sin theta, so that of 2 m E minus m square omega square x square d x which I can write as interval minus square root of 2 E over m omega square to square root of 2 E over m omega square. I will take out m square omega square, so that comes out as m omega and I am left with the square root of 2 E over m square omega square minus x square dx. Maximum and minimum values of x r 2 E by m omega square, there is no two here. So, I will take this out this is m omega square.

Calculate this integral I will take x equals 2 E over m omega square square root sin theta, so that dx is equal to a square root of 2 E over m omega square cosine of theta d theta and the limits are theta equals minus pi by 2 to theta equals pi by 2. And those thetas to a corresponding value of a or x becomes 2 E by m omega square minus or plus. And therefore, this integral becomes minus pi by 2 to pi by 2 m omega times square root of 2 E over m omega square cosine of theta times square root of 2 E over m omega square cosine of theta d theta does the integral which I can write as minus pi by 2 to pi by 2, I had m omega 2 E over m omega square cosine square theta d theta.

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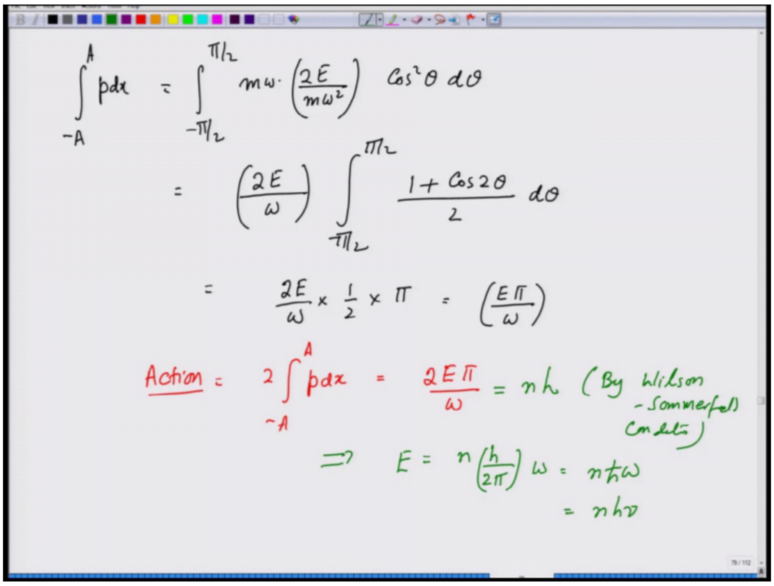

So, this is this integral minus A to A p dx which is equal to 2 E over omega integral minus pi by 2 pi by 2 1 plus cos 2 theta divided by 2 d theta. And that gives me an answer 2 E by omega times 1 by 2 times pi and cosine 2 theta integrated over minus pi by 2 pi by 2 is 0. So, this is E pi over omega. So, action now which is 2 times minus A to A p dx becomes equal to 2 E pi by omega and it should be equal to n h by Wilson-Sommerfeld condition and that gives immediately gives E equals n h over 2 pi times omega which is n h cross omega which is same as n h nu. So, with this condition, I also get the answer for the harmonic oscillator and the correct answer. So, this seems to be the right condition for applying quantum mechanics.

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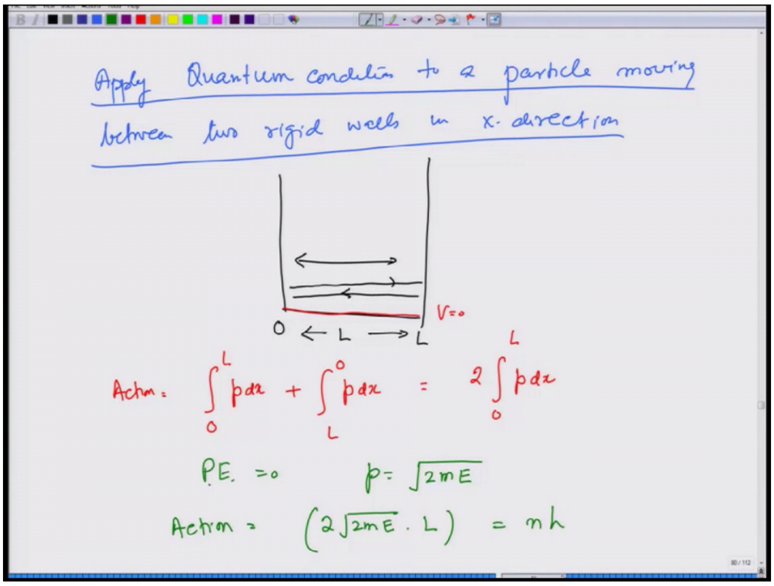

Let us now apply to another systems apply and I am not going to write Wilson-Sommerfeld again and again, I will just say quantum condition to a particle moving between two rigid walls in x direction. So, this is still one-dimensional motion. So, problem I am considering is that the particle of mass m that is moving back and forth between two rigid walls, where the distance between them is L. So, what will this particle do? It will go this way, hit the wall and come back and that will be one complete motion.

So, if I what do calculate the action for this, action will be 0 to L p dx plus L to 0 p dx and both will contribute the same. And therefore, I can write this as two times integral p d x zero to L. What is p, now sense this is particle moving freely. So, v is 0 the potential energy is zero potential energy of the particle is 0; and therefore, p is nothing but square root of 2 m E. So, what we have is action is going to be 2 is square root of 2 m E times L and that by the quantum condition should be equal to n h.

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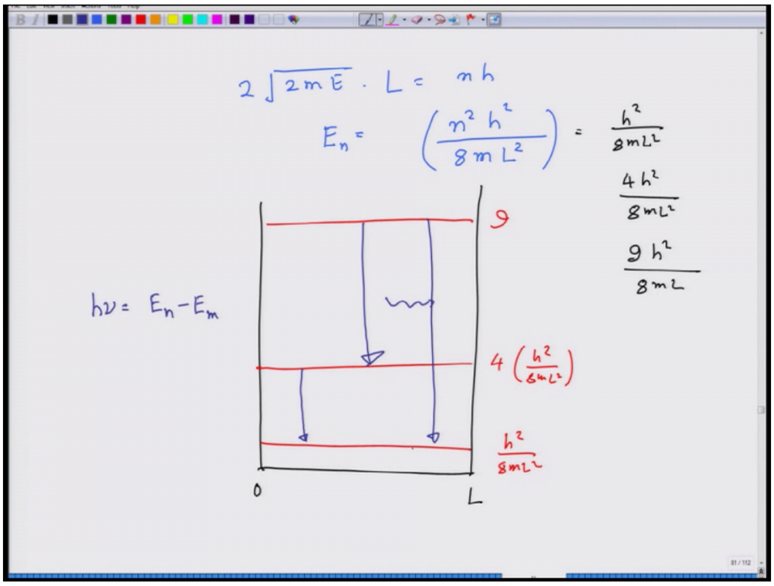

And what is this lead to this leads to the square root of two m E times l is equal to n h. And therefore, E equals n square and h square over 8 m L square, this is E n. So, what this tells you is that if I have this particle in a box confined between of size L, the energy is equal to either h square over 8 m L square or 4 h square over 8 m L square or 9 h square over eight m L square and so on. So, the energy levels would be here this is let us say I just square over 8 m L square then it increases becomes four times h square over eight ml square and then it increases even more this becomes 9 times and next would be 16 times and so on and these are the energy levels.

When an electron makes a jump from one level to the other it gives out radiation by Bohrs rule. So, h nu the frequency of light coming out would be given by E n minus E m when electronic makes jump from the energy level to mth level and that will be the corresponding color can be calculated. So, this sort of concludes the introduction to Wilson-Sommerfeld quantization condition which has been applied to one dimension.

Now, in the next two lectures, I will generalize it to two dimensions and three dimensions and three dimensions very important when I consider it for a central potential like coulomb potential because it gives you the answer for atomic levels, and it introduces an idea called quantum numbers right. So, we will do that in coming lectures.