Computational Science and Engineering using Python

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Lecture - 10
Interpolation II: 2D, Spline

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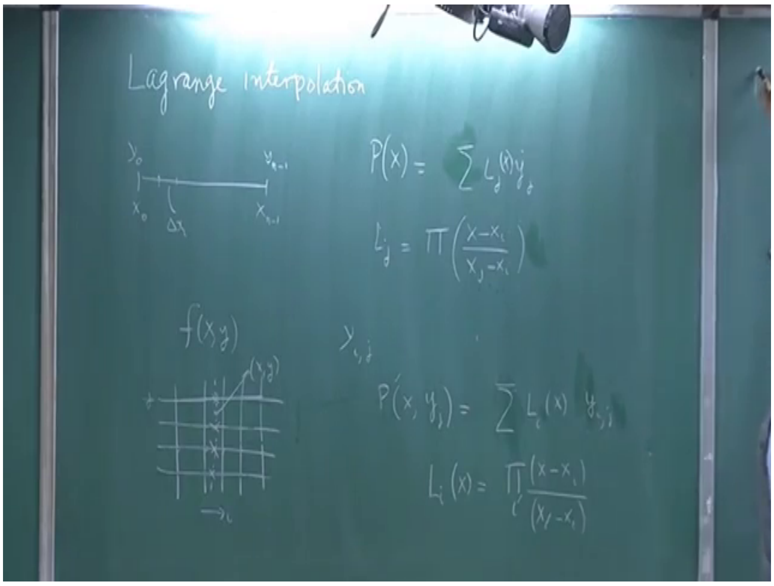

Lagrange interpolation, there is n points know here n points x 0 to x y (Refer Time: 00:34) So, Lagrange interpolation. So, we can get values of function approximate function basically you cannot get accurately because we really have only finite number of points, but you can get approximate function according to your x and that was product.

So, the this was sum of sum L j y j. So, L j is a polynomial and L j equals product x minus x I, x j minus x i that was (Refer Time: 01:30) So, with this I can get the value of the function at any place, but it is approximate value it is not the real value and if we use large n of a number of points then you get reasonably good approximation. So, we get first order polynomial second order third order depends how many points you choose. The easiest is 2 points which is linear interpolation it is clear to everyone.

Now, we can do this to 2 dimension. So, there are 2 dimensions are there function. So, for example, temperature ness on a surface of on a plate. So, that function will be f x comma y, but again when I make the measurement I will only make the measurement it is finite number of points, for this class let us assume that I have. So, I have did which is

(Refer Time: 02:30) with it need not be any form, here also it is x j minus x i it is not they are not legal for each neighboring points right they I do not did not assume uniform separation between the points.

Here is also, but I assume that this is not same as that this is not same as this, but I assume as square units a (Refer Time: 02:56) units. So, delta x i are not equals. So, this also can be called delta x. So, how do I interpolate how do I derive formula for this. So, there is (Refer Time: 03:16) formula is on similar lines. So, let us first derive along x for. So, I will say at I will fix y I y j. So, I am going to use this slightly change this one this one on this these (Refer Time: 03:38) will b j these (Refer Time: 03:39) will be y.

So, my value my function is y I comma j here j was the running well. Here j was the running index, but I am going to use j for y coordinate and I for x coordinate change that notation. So, for given j this relation I can do interpolation, so I formula. So, let us first fix j, so my polynomial P x comma. So, this is of x I am going to put that I here in in a minute, this function. So, this is. So, now I can use what (Refer Time: 04:40) I 1 j x, but now I am summing all is. So, you see remember this is i. So, I am recently fixing here I equal to 0, I equal to 1, I equal to 2 like this (Refer Time: 05:01) this now I comma j now this will depend on I prime. Now this is no right now there is sum this is sum. So, I am summing over I no no product, this errors. So, this sum over I now L I will depend on points here. So, L I of x is product.

I just write this x minus x i, but I do not want to use j because j is for y coordinate. So, x i prime I am use I prime minus x i i prime. So, I priming is the index varying index in the direction. So, this is going to give me interpolate values on a given line horizontal line. Now for every j I can write these functions. So, interpolate along x and my y will of course, are fixed they are basically those along the lines which are given to you.

Now, you can interpolate along that direction. So, suppose somebody wants the value function here which is neither on the horizontal line nor on the vertical line. So, what do I do? So, I draw a dash line and now I know the values of the function here by this formula because that case wants to this particular j values of the function here, here all these values now interpolate along y.

That is the idea. So, I want now. So, we can call it p prime the intermediate, it is not the full function I want to p will be interpolate at any point in between. So, this is a point x y.

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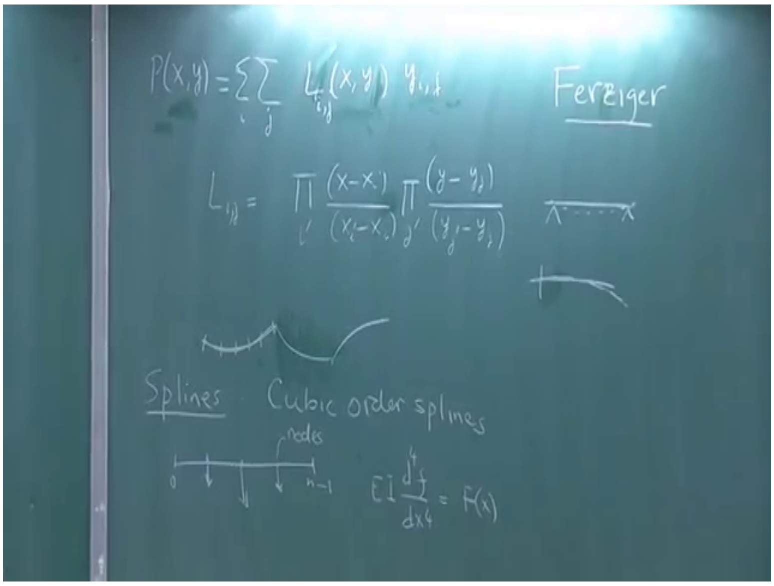

So, P x y it will be now. So, I will use this function to interpolate sum P prime x y j or I (Refer Time: 07:43) j j (Refer Time: 07:46) P x y xy P x y. So, this is my what I am doing L I j l I j x y y I j. So, this what is my interest.

Now, this L I j will be including both x and y variables. So, L I j (Refer Time: 08:21) the 2 sums product x minus (Refer Time: 08:30) pi and it is for the y, I have to do to for the y product, this I prime, j prime y minus y j. So, you can do 2 steps. So, this is a formula (Refer Time: 09:03) this will give you interpolate value at any x value in the domain.

You can generalize the 2 three dimension or any dimensional value. So, for three dimensional I will have this I j k (Refer Time: 09:24) three dimension. So, you will three products. So, we do it like this it is very useful many times we need to work with functions in 2 d 3 d now this sums up to an Lagrange polynomial interpolation. Now there are many more. In fact, there is a something called using Hermite's polynomial, the disadvantage of this scheme is that suppose I have set of points then I can do Lagrange interpolation here and then Lagrange interpolation in 2 segments I can do let say I have n minus 1 points and n minus 1 points here.

Now, in all this stuff I needed value only at the value of the function at y js I did not use the derivative. So, one may say well I will give you function as well as derivative, then I can do I can I need with (Refer Time: 10:31). So, I think this part is different idea this part is the different idea. So, here I will have points and or a I give you the derivative as well as the value of the function and with n points I can get 2 n minus 1 order polynomial. So, I will have 2 n coefficients, but that is bit a more mathematics. So, I will skip that one.

So, I will give you the notes which has some details with maybe there is some errors typographical errors, but that has some details the reference book for this is ferziger it is there in the website the full detail is also some copies are there in the library. So, a Hermite polynomial is described here if you are interested you can do that. Now I am going to work out another one called splines. So, let me just before I going to spline I will make another remark you can do piecewise there are second p, I can do interpolation of may be 5 points here.

Then another 4 points here 3 points here. So, it is possible to do things like this. So, let say I have 100 points. So, you know how to do with 100 output let us too clumsy and (Refer Time: 12:05). So, we work with piecewise and in fact, the idea such spline is goes in the same direction.

So, splines are objects which are smooth. In fact, if you done engineering drawing those plastic curves you know that is splines so. In fact, we can give any continuous function by those splines and if you if you drew some drawing packages, you can say move it around and the curves takes very smooth functions. So, there are of course, lots of splines many types of splines I will focus only in cubic orders splines, this is one of the very popular interpolation package used in graphics every in graphics.

So, graphics do not have (Refer Time: 13:05) all the points by saying interpolate (Refer Time: 13:06) into. So, g p s will be doing lots of this stuff form. So, if you are playing games g p s really working hard to you will have pixel and it will do some interpolation to give you a better image. So, the idea of spline is. So, I have again n minus 1 at 0 to n points, but I want to construct function f passing through this points which follows the equation of a beam I am I do not know how many of you know the equations beam hanging beam.

So, this is the beam right now it is stable the beam from one side on the wall to other side of the wall a rod just. So, beam is you can have a rod attached like this. So, it is has. So, this is called beam you could have beam where which it is it is attached to a side and (Refer Time: 14:16). So, this is called beam now. So, I am going to use the construct a function beam function which will go through this point and each point there is certain load. So, the idea is in fact, it comes a civil engineering.

So, I have a function f. So, equation of the beam is b 4 x, b x 4 and their position in front is f this is equation of b, it is a fourth order it is one of few equation which involves 4 fourth order now these are elasticity coefficient and moment of inertia. So, you got this now this function is reasonably smooth it is fourth order. So, it is just smooth.

Higher order function it is smoother, now I will assume that at each point given points if a this points are called nodes at each point is a weight and because of the weight I will a ovals it will look oblique. So, we construct f given this condition. So, if it is fourth it is have derivation is the equations are bit lengthy, but the idea is quite simple now I am going to demand something on f then. So, what is that (Refer Time: 15:57) demand on f.

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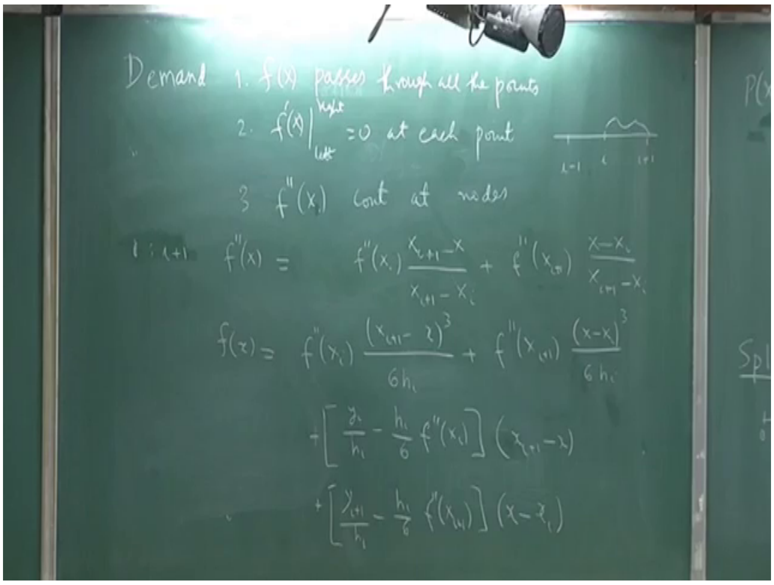

That this f should pass through all the points that should know. So, f where the derivative is continuous across. So, each point I just have the derivatives and derivative also is continuous. So, the slopes are equal from both the sides. So, f prime x left it is a differentiable function at each points.

Student: (Refer Time: 17:01) one more thing.

Yeah.

Student: (Refer Time: 17:02).

Sorry.

Student: (Refer Time: 17:06).

So, e is a elasticity coefficient and I is the moment of inertia of the (Refer Time: 17:13) we do not need to worry about it, this is the equation I will use it I am not going to specify this. So, we will come become clear in a minute we do not need now this is fitting I am borrowing the idea, but I am not using really the beam (Refer Time: 17:44) also demand the second derivative. So, second derivative also continuous right you can first order second derivative and a the curve is. So, these are things which are demanded for all this function.

So, now how do I construct the function? Now do you done quantum mechanics with delta function know is one wave function solution. So, what can we say about this function, so what. So, since fourth order, this is a delta function at each node is a point mass which hanging. So, f is of the form delta x minus x i f I is a number I will again not specify it you will be coming from what I am going demanding by construction.

So, what can you say about fourth order derivatives or delta function at those points. So, what can we say about the third order derivative.

Student: Step function.

Third order is step function. So, third order will be step function. So, it is constant to the left and cross to the right of the each node. So, second order derivative must be linear. So, second derivative is linear. So, I have this set of points. So, I am going to focus on three points, i minus 1, i and i plus 1. So, let us write it here.

I minus 1 (Refer Time: 19:34). So, if f double prime not f eight over prime is linear in x you are happy with it now use the Lagrange interpolation formula. So, get the values of the f double prime between any interpolate is linear. So, f double prime of x let us say the interval i minus 1 to i no I to i plus 1 i to i plus 1.

I say this is a linear function each p is linear function, because it is a random function it just normal. So, I will use Lagrange interpolation to right on the function f double prime. Assuming that I know the wave, wave at each function so in fact. So, step one is suppose you know f double prime at each point you can write down piecewise linear functions.

So, what is this is my Lagrange interpolation, the formula is linear interpolation x i, x i plus 1 minus x divided by ok.

So, this interval similarly we can write for this interval as well. So, for every interval we have f double prime is linear now if I have the function then I can get f x. So, integrate this function, but we demand one more point one more condition, that f x must pass through y i and y a (Refer Time: 21:37) or y you put that condition is bigger than delta. So, we have 2 unknown coefficients now 2 (Refer Time: 21:49) coefficient how are they fixed that if f I the value function is y i and here this y i plus 1.

So, you have this and then integrate twice say that f x i is y i and f i, f x plus f x i plus multiply. So, I will not do the algebra and I just write down the function f x i this one I will write fully because I will write I share this notes and then write this plus . So, these this is pretty lengthy expression, but this is a cubic order polynomial for x ok.

So, that is what I said my function f will be cubic order and that comes from this equation.

Student: (Refer Time: 23:28).

Oh great. So, h i is this x i plus 1 minus x i gap and this will be h i minus 1, now I will only sketch the next part. So, you have to write down. So, this is f of x, I will denote because you write f of x for this part now once we have f of x at both sides I demand that my derivative is continuous.

So, I can compute the derivative here from both the sides and I demand the derivative from the left this (Refer Time: 24:09) with the right and I will get a condition, and if you do the algebra correctly you are welcome to do it (Refer Time: 24:19) if you do the algebra correctly, you will get equation which looks like.

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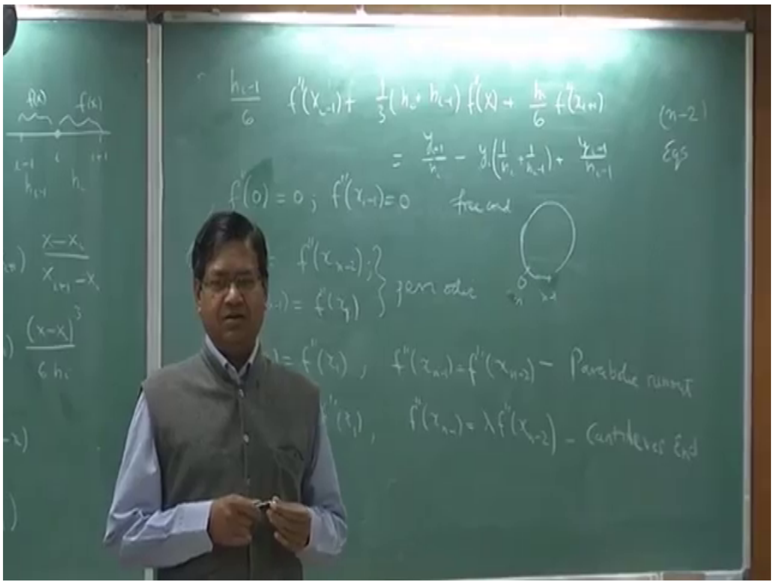

Which is not which is quite easy equation right h, I minus 6 by 6 plus one-third equals (Refer Time: 24:50) y. So, the equation ok.

So, this is by setting the derivatives from both the sides equally. So, how many equation will I get like this? I have considered a (Refer Time: 25:27) from a intermediate points I cannot setup at the end points right, so n minus. So, there will be n minus 2 equations these are known y i and h is are known what is not known is the second derivatives so. In fact, I assume that well I have said that suppose somebody gives me second derivative then I can set it up, but then now (Refer Time: 25:55) well I do not give you second derivative you have to figure it out.

So, I got n minus 2 equations and how many unknowns, I got n of the second derivatives unknown they are unknowns. So, I cannot figure out right now, but I can setup few extra conditions now. So, I setup some conditions one condition would be the derivative at the both ends are 0 double derivative. So, very simple condition is that f double prime f 0 0 is 0 f double prime, and this is a (Refer Time: 26:46) beam, if you have done this b equation. So, at the both end it is free.

So, you can hold it at the in the middle somewhere to you can put 2 two columns (Refer Time: 26:57), but the end there is no stress. So, this is for free end free condition. So, I got I give 2 conditions now I have n equations and I can say, they are all will already given to all, but n minus 2 you can figure it out by which there some other conditions called call one thing is periodic boundary condition. So, it goes this circle periodic means it is goes in circle. So, you demand that f double prime 0, well if I this one is exactly tricky. So, this for the double derivative it is not for the function. So, of course, we demand for the function that y 0 is 7 y n minus 1 that is continuity for periodic.

So, 0 is same as n minus 1. So, these are together. So, n minus 1 this simply lying on this is not n minus 1 that will be n that will be n this is n minus 1 0 is same as n we do not specify n aj no no periodic will what periodic will we do to the function should imply periodic should imply that n minus 1 should be equal to. So, I am not certain that I need this n minus n here or n here; I am not 100 percent sure. So, I can tell you what is for the double derivative.

So double derivative demand red or n x is here is the (Refer Time: 28:49) in next class. So, double derivative for the periodic boundary condition is f double prime let us n minus 1 and f double prime which is periodic and we can also have some other conditions. So, this is called parabolic run out. So, that is the condition third condition is f double prime x 0. So, since a double derivative is equal at 2 conservative points the functions if it is parabolic at (Refer Time: 30:04) function. So, it is parabolic near the end now these 2 can be combined to call cantilever n lambda ok.

Lambda lying between 0 and 1 and this is called cantilever condition cantilever end and these are 4 popular conditions. So, with this (Refer Time: 30:34) then you can find out f double prime once you know f double prime then you can find out actually piecewise unit I it demands and it is function is continues is first (Refer Time: 30:51) derivative is continuous and also second derivative secondary derivative also same is continuous. So, it is a very very good interpolating function and graphics is highly used.

Now, I will just show you how to do it in python. So, python has it is building function for doing spline, but before that maybe we can just workout the example by hand. So, this part I will leave it as homework, but I will just sketch what you do for the 4 points I discussed in the class, one over x the function is 1 by x. So, my function is 1 over x i, I give you this and they are at points 2 to 4 2 3 2 to 5.

So, now we worked out in the last class with the Lagrange interpolation. So, we can do the same thing with splines. So, how should we proceed? So, I need to write down this equation, I have just erased it these equation I should not erase that equation, but if you just do in a homework here h i is the same for all of them is equal 1 right h i is a difference between 2 consecutive x such points. So, now, this equation is condition equation for f double prime I get figure out for these 2 points intermediate points and these come out to be I just write down one of them.

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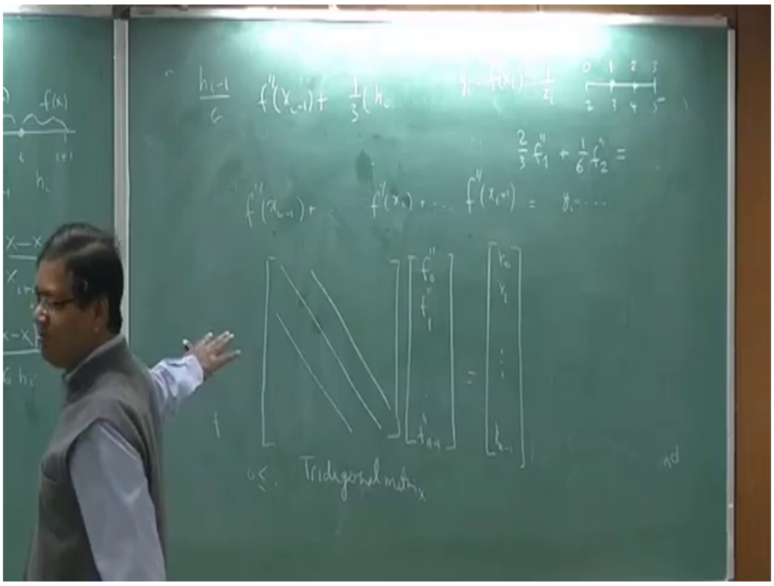

Two-third. So, here double derivative get 0.1. So, this is point 1 this is 0 equal to the right hand side that was y I plus 1 y i y i minus 1. So, you write it out there is a 2 equations because there is a 2 (Refer Time: 33:15) now for the end point, I will say that you use free monotone condition. So, I will get you get all the 4 f double prime and from that you construct this f x and plot it and how good it does it fit with one by x. So, that homework is clear ok.

Now, let us use python's spline function. So, now before going to python's spline let me also I have ten minutes I just want to make a remark. So, let us do a re look at this part. So, it has f double prime x n minus 1 plus dot dot dot f double prime x i (Refer Time: 34:01) equal to right hand side which was given to you.

Now, there is n minus 1 unknown. So, you can write this in terms a matrix correct these are unknowns. So, these become matrix inside the first entry, will be f double prime 0 here how will the matrix look like this part.

Student: (Refer Time: 34:46).

Three entries. So, this is quite right argument (Refer Time: 34:55) elements here (Refer Time: 35:00) here. So, we can write this equation in the terms of matrix. So, this is tridiagonal matrix and we can solve it, now this part I will do towards later product rules how to solve tri diagonal. So, we find that mathematic the computation physics they are all connected do the interpolation I need to know what is how to solve matrix. So, this part we do later now.

Student: Ok

So, full matrix algebra, will be done later part of the course.

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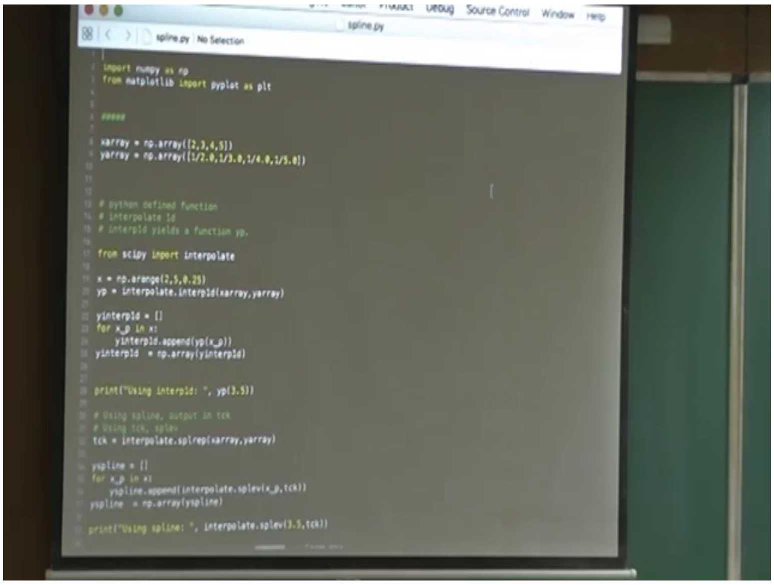

So, let us quickly look at, you know this is again looks tiny. So, interpolate will be one interpolation. So, this one interpolation, this is part of scipy scientific python and the full package is. So, within scipy there is package full interpolation, within interpolate we have (Refer Time: 36:20) functions. So, I can also interpolate dot inter inter key one d. So, interpolate one d. So, this interpolates one d. So, I have doing. So, this is I arrange it is a arrange function, but it is called real function. So, it starts from 2 it goes up to 5 with interval 0.25.

And, this is my x. So, interpolate y p will be result coming from this function it will write y, I want a point. So, there will be any function. So, y interpolate one d and basically I put in a loop when I will just copy everything in that function there will be a shortcut, but I just did that it is a (Refer Time: 37:04). So, the answer is here in my. So, this is go to do interpolation 1 d interpolation, at any point you give it you do give to the function and what where are the function I given this x actually yeah from i now I (Refer Time: 37:30). So, y p is a function y p is some kind of things generated by python.

Now, you have to give x p is the argument to y (Refer Time: 37:46) why these not an array y p let polynomial. In fact, we construct the polynomial is that part clear to you. So, y is not an array, it is it is some kind of it has all those parameters which you no need to worry about it. If it is a (Refer Time: 37:59) now why it is a will act like a function and st is in argument. So, this will give you a value which is appended to interpolate y.

So, my final answer interpolate 1 d is a array which are the values of the function the values at x equal to this array. So, I need to plot x versus y interpolate function, now this is for interpolate 1 d which is not a very good function it is a linear interpolation we will just see in a minute now for a spline. So, you have to focus on this part pretty similar to this. So, we give the x array and y array. So, x array and y array are yes 2 3 4 5 and y array is this you give it here. So, it again mix it gives some parameters it makes the function pck set of parameter.

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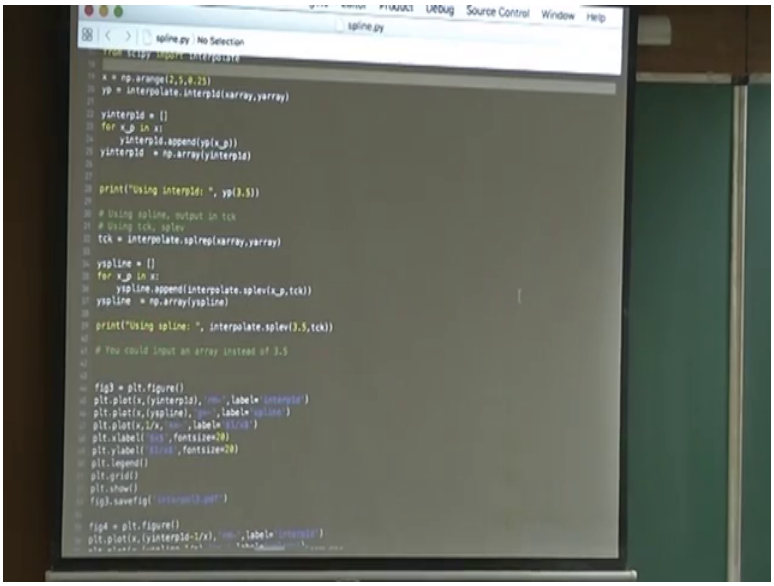

Now, p c k will become an argument for spline (Refer Time: 39:06). So, whichever x you want to compute is the value of the function interpolate function, we just pass on this x p. So, x, x p this is low for all s t lying in x is formed. So, I give a argument xp and set a parameters obtained by this function spline representative. So, p c k which is argument of this and we will compute and this is the total this spline of x. So, I am just adding this 2 by 1 and so, my result will be in y spline which is treated by me y splines is not system (Refer Time: 39:52). So, you have to do this. So, let us done it quickly. So, you make plots.

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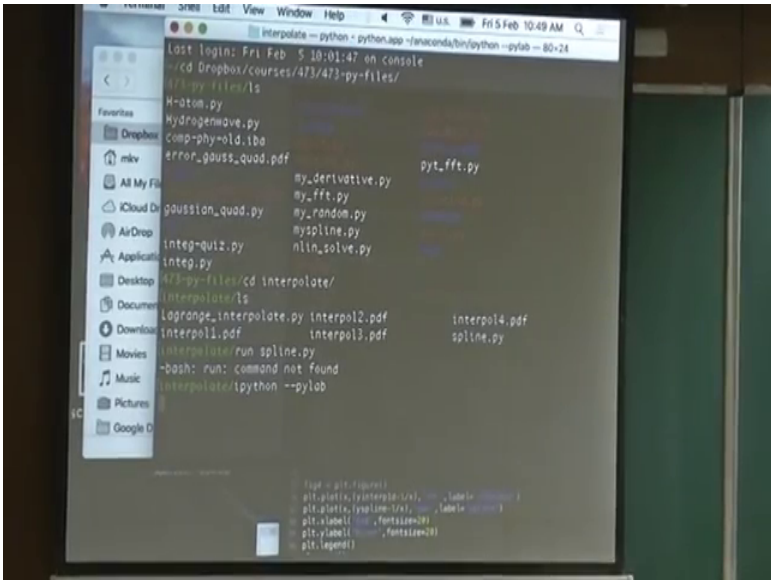

Some of you still struggling with this c d e stuff. So, run spline (Refer Time: 40:39).

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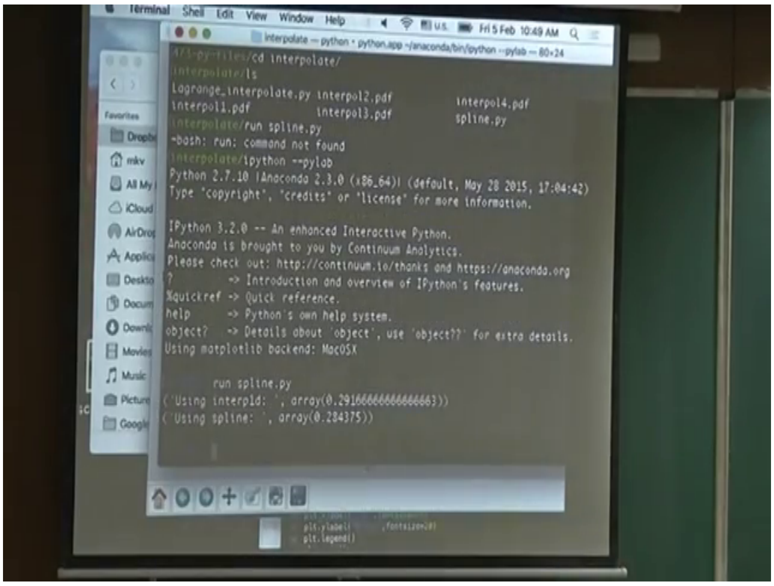

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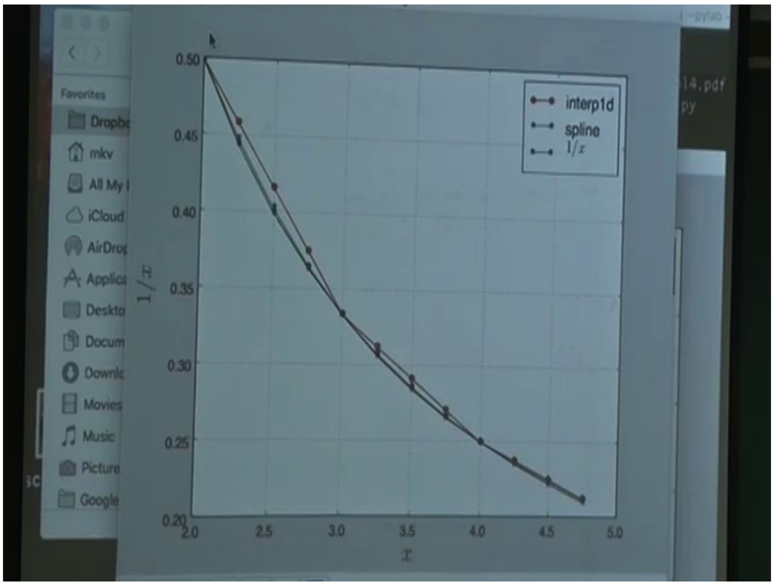

So, this is a plot. So, my function is that better than one by x spline is doing very good job this cubic. So, cubic is you know this is the interpolate 1 d this is linear (Refer Time: 40:59) good job. So, spline is easy to use. So, if you just pay attention which you are going to do it.

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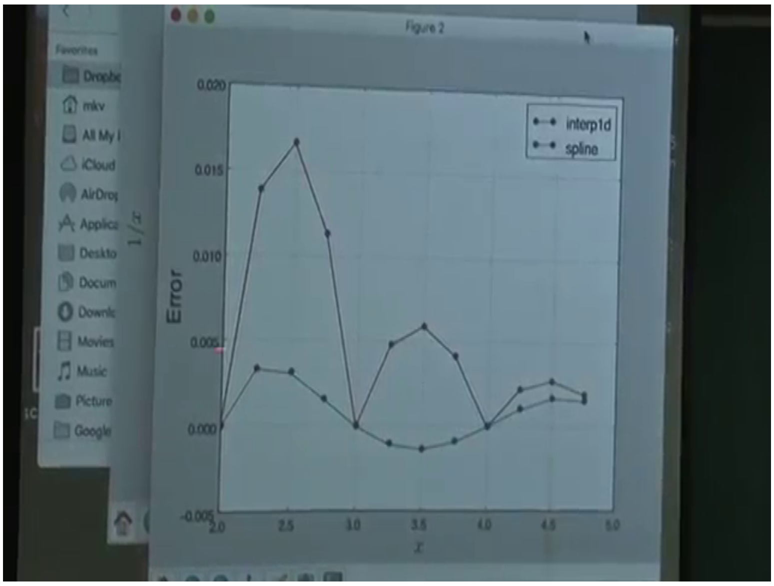

And the errors are got it here. So, error interpolate 1 d quite it is large just like error is (Refer Time: 41:18) there is a point f double prime a double prime x i plus h i by 6 f dot prime x i plus 1. So, this was right hand side. So, y is right.

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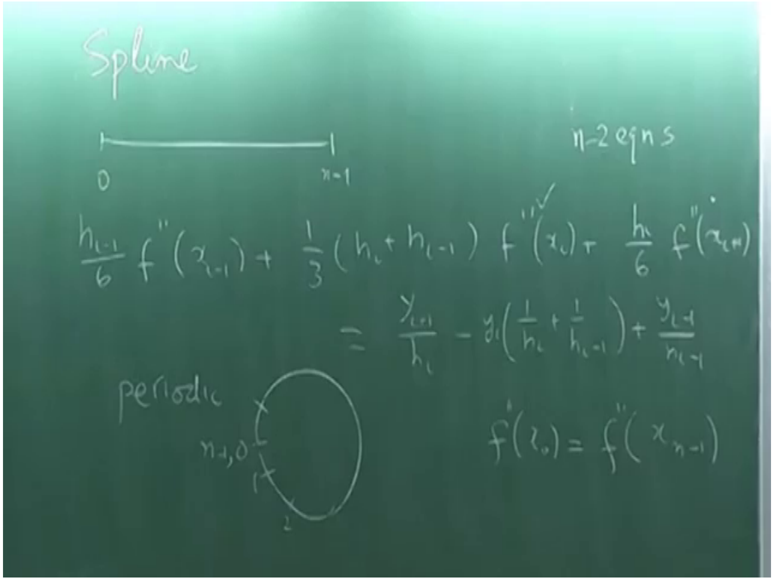

So, that was y I plus 1 by h i minus yi h i minus y. So, there were n minus 2 equations we do it for we skip this point and this point. So, there are n minus 2 equations, but I have n unknowns f double primes. So, we need to assume something. So, if we assume periodic then these 2 points are the same. So, it is like in a circle. So, 0 1 2 like this and this becomes n minus 1.

So, in effectibely we have one less point. So, we have one less point then we need one less condition. So, it is already n minus 2. So, I just need to match the double derivative here. So, what we want is f double prime x 0, x n minus 1 well they are same point, but you have to take double derivative from the left and from the right and that will that is what we are assuming in this.

So, that will give you n minus 1 conditions and we have n minus 1 unknowns and you can solve it the other conditions parabolic runout parabolic runout has. So, let me just finish it off.

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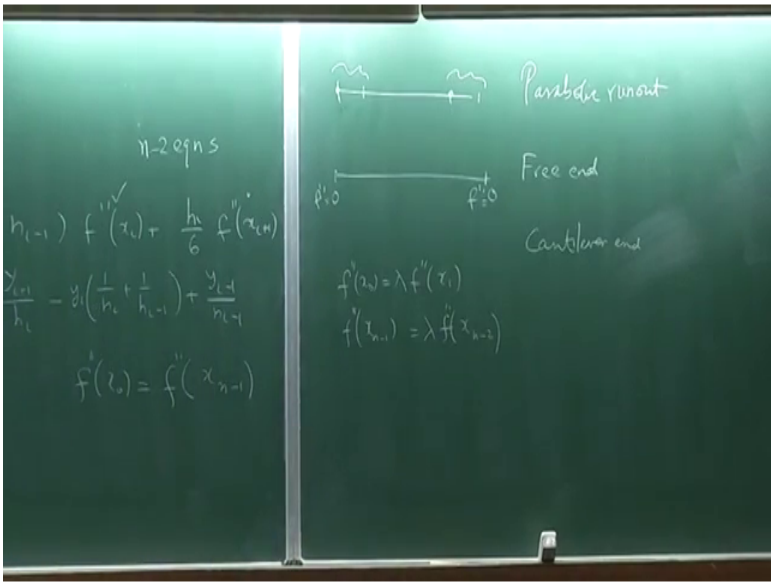

So, these 2 points have same double derivative and these 2 points are same double derivatives. So, we assume that the double derivative here is same as here double derivative here is same as here.

So, it is parabolic in this regime double derivative is constant. So, this is say periodic, this is parabolic runout and then free end these 2 points have 0 double derivative f double prime is 0 here (Refer Time: 43:51) and there is something called cantilever end.

So, there I assume that f double prime x 0 is lambda f double prime x 1. So, except the periodic case we need 2 conditions for periodic case we need only one condition. So, and they solved using solving the tri diogonal matrix which will cover later, later part of the course for you right now use 4 points and do the homework these 4 points we need to solve 2 equations. So, please do the homework.

So, piecewise cubic and it is first derivative and second derivatives are continuous there is a condition mean post.

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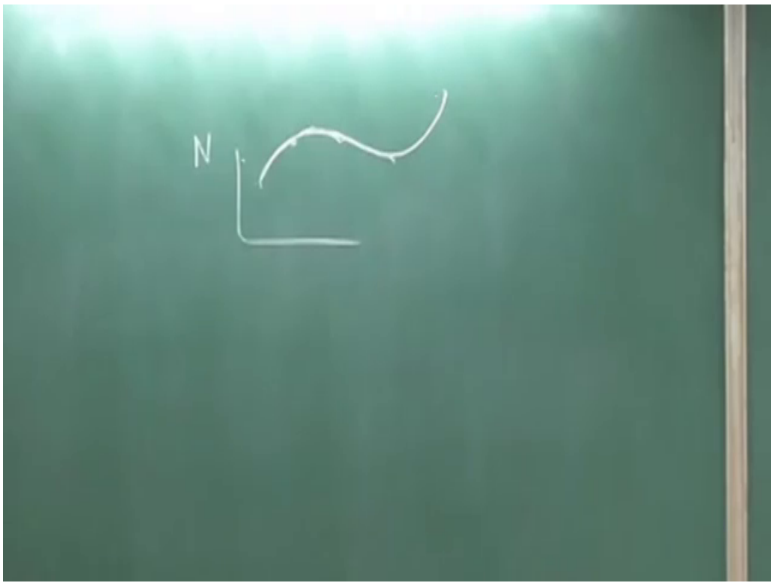

So, curves are quite smooth. Advantage of spline over Lagrange polynomial is that, for Lagrange polynomial if I want to fit with n point I will get n th order polynomial know or n minus 1 th order polynomial, and there will be lot of oscillations higher order problem you say oscillations and that is not so, just not nice.

If you are in fact, it is impractical if you have n equal to 50. So, 50th order polynomial is not a good thing. So, we want smooth function and it is cubic in each piece. So, it works quite higher order you can do higher order spline, but this is this is very popular. So, is that clear and also for estimate the zeta for Lagrange polynomial lies between first point and the last point. So, zeta is between first point and the last point. So, then you have to estimate max value of f prime f n th derivative of zeta with. So, you have to find the minima and maxima between the 2.