Computational Science and Engineering using Python

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Lecture – 09
Lagrange Interpolation

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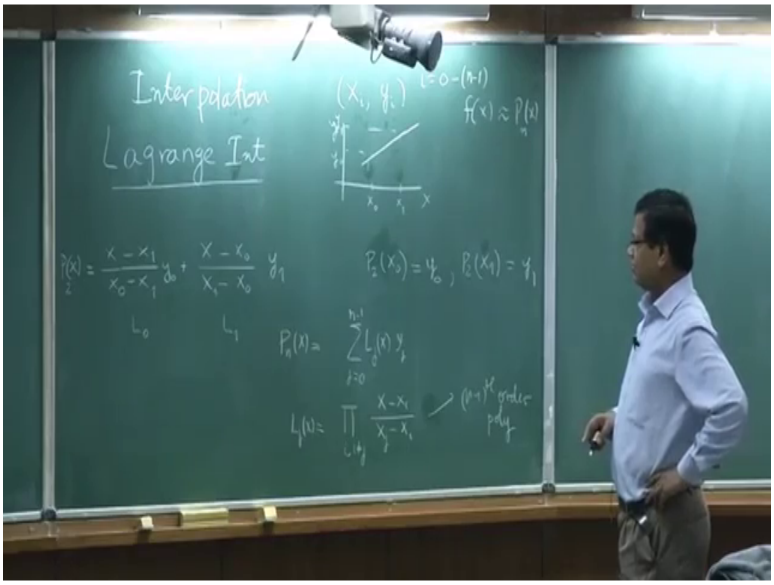

So, what is interpolation? It is when I do experiment or simulation, I get results for some finite number of points. For example, temperature I may not record every second. In fact, nobody records every second unless you are doing the experiment or related to a which requires one second. Now only required for 15 minutes, but sometimes I may require temperature in between or I want to forecast for temperature tomorrow. So, that is called interpolation.

Interpolation is general word, it is also use for extrapolation tomorrows temperature. So, we have given I am going to denote my variables as x i this is my independent variable like time. For example, and this is my function value at x i. So, these are dependent variable and we get some n points and using these n points and using these n points I should be able to interpolate any value in between it was very useful. So, for your mathematical calculations in the past. So, the person we will discuss Lagrange in this class, but the other people also involved in this game hermite interpolation also very important interpolation, but I will not do in the class. But all these guys were really wanting very efficient interpolation scheme.

The reason being any function for example, a sin x i need sin x for my trajectory of planets know. So, we need for particular phi which could be in seconds or minutes. So, they will interpolate accurately. So, for that reason it was very important problem, which was of interest to lot of very famous mathematicians faces in past. Now with computer we are lazy and we will make computers do always task, but they are doing by hand 500 years back 400, 400 years back.

So, I am going to cover Lagrange interpolation and the same Lagrange of mechanics. So, the idea is quite simple. So, suppose I have 2. So, let us first understand for 2 points. The formula how it how we can write this interpolation. So, I am going to draw this x axis y axis. So, I am going to call them x 0 and x 1 write by following this notation. And value of the function is y 0 and y 1 y 0 y 1. Now I may want value of the function anywhere in between for this x or this x or outside. So, I need a formula. So, for 2 points it is easy liner interpolation. So, I just make the function which is linear well. I do not have a choice I do not have any other point. So, liner interpolation is a valid interpolation or in fact, or only available interpolation with 2 points.

So, now the way to write this in a in a very systematic or very transparent manner is. So, so you pay attention. So, how I am going to write? So, x, I am going to write like x minus, so x minus x 0. So, now, if I put x 1 I should get y 1. So, this is. So, well of course, you can make a liner function know passing through these 2 point, but I want to can automate this for more points. So, this is linear in x it satisfies the value of the function at the condition, I need to demand is the function I am going to write is well let me just make another remark, the function which is really the function like temperature. For example, passing through these 2 points is f of x, but I do not know that function I really do not have access to that function, but I am approximating this by a polynomial and is a polynomial. So, what I am writing is p 1 f x. So, first order actually p 2 f x first order polynomial involving 2 points, now this is one of them. So, what is should be other point.

So, should be x 1 minus y 0. So, this is the function and this is not the actual function because actual function I do not know and we call it p 2 f x. So, this is the first order first order polynomial going through the 2 points. Now you can easily see what are the properties of this function that p 2 at x 0 should be if we put x equal to x 0, this becomes one y 0 and this is 0 right, p 2 of x 1 y 1, so that is satisfy. Now this part this function we call it l 0 and l 1, l 1. Now let us generalize to end points now this motivation is clear how you write the function. Now I want to write for n points. So, my points are xi to xi comma y i, i goes from 0 to n minus 1, a following python notation. So, it is index starts from 0.

So, the idea to write this this function, we write this p n of x as it involve lots of point. So, I am going to write this as y of j sum l j of x. So, j is a one of the indices. And j goes from 0 n minus 1, my l j is written quite nicely as a l j of x is a product function, so product. So, in very similar fashion, this is x is here x i, j. So, these are product. So, is this product our all i's except i equal to j, when i equal to j this will blow up i, but i not equal to j. So, how many of these, I will get for every point i will get one a polynomials and there are it involves n minus 1 product. So, this well this is our n minus 1 th order polynomial.

Now, does it satisfies relation like this.

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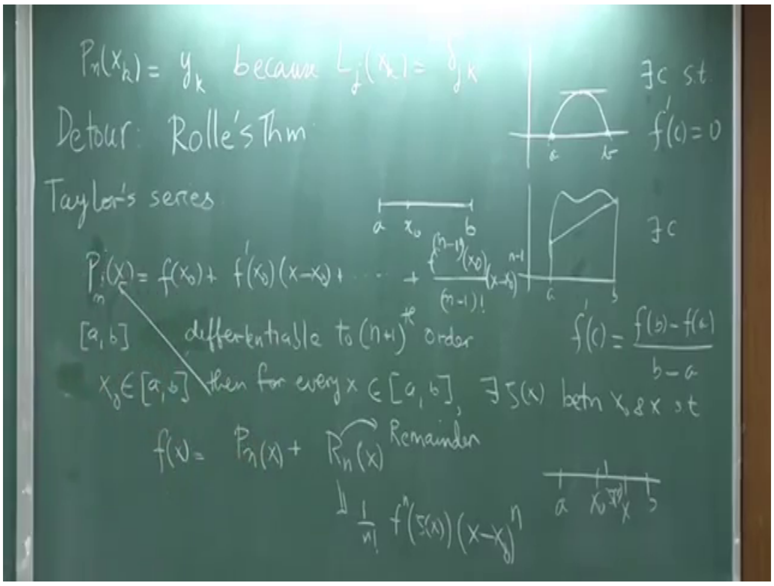

So, let us look at p n of x k, p in f x k. So, for which one it will be non 0. So, x k is one of this points know. So, in the top it will become 0 if so, the one which will not be 0 is when involving j equal to k, because that will not have x k here in the product. So, this is all this points except one point. So, x minus x 0 x minus x 1 (Refer Time: 09:42) all the way up to there except one point. So, if x k is involved in any of this products. So, there are n there are. In fact, n of them there will be 0 the one which is non 0 is where x k is not there and then. In fact, that becomes one.

So, this function will just give you y of k because this l j of x k satisfy the property delta function delta j k. So, this is a very useful function. And your polynomial required polynomial which Lagrange (Refer Time: 10:27) it says, is this now using this we can interpolate to whatever x you like of course, the some x the error is more some x error is less. Now how do I estimate error for this function.

Now, somebody give you the answer, but now we should be able to get the error. So, estimating error is not a I mean, I will do full mathematical derivation for this what is the error estimate for this which I will not do for lot of other algorithms for this I will show you how to estimate the error and is done by using r rolles theorem. So, this part of the analysis know first year or. So, we will use rolles theorem to get an estimate on the error. So, I will do little bit slowly I will also prove Taylors theorem for the which is not exactly related to this, but I will prove Taylors theorem and that will be spring board for analysing error for Lagrange polynomial. So, is the Lagrange polynomial clear. So, this is my interpolated function.

So, we are going to take detour. So, detour is to first get error for Taylor's theorem. Now Taylor theorem is just Taylor series and Taylor series written is slightly mathematical manner which involves error as well.

So, first we need to rolles theorem. What is rolles theorem? So, suppose I have a function which is 0 at 2 points is 0 at point a and point b then I can claim is a continues function and differentiable function they can claim that there exists a point c, so the lot of symbols in mathematics. So, please remember this a product like series this like everybody knows is this symbol product. So, this involves. So, let me just write this. So, this symbol is very critical. So, if I have 2 3 points then I will have term like x minus x 0 x minus x 1. So, this product now I have the symbol call there exists. So, this is shortened. So, there exist c such that f prime the derivative at c is 0. So, there is a point. In fact, the point is here and where it will be 0. Now this can be generalized when. So, I assume that f a is 0 and f b is0, but it may not be 0 for some other function.

So, it has value f a and f b, then whatever function it may be there I can get there exist at least one point there exist a point c such that f prime c. This call mean value theorem and what is that. So, this is a and this is b. So, f b minus f a (Refer Time: 14:16), there exist a point c, I do not know where that point is, but there is a point c. And this we will lead to prove our error estimate or well get a formula for error estimate, I wants a prove get a formula for error estimate.

The Taylor series is also useful for our course as well, but this is part of your analysis, but I am going to re do this thing. So, again in mathematics you will you say lot of mathematical symbols. So, these are interval a b, interval a b. Now I want to get value at any point x. So, Taylor series you know that I do near a point x 0, there is some point x 0 and about which we expand right. So, that is Taylor expansion. So, you write f of x as f of x 0 plus f prime like this and this goes up to higher order terms. Now the Taylors real Taylors statement is. So, the function is continuous between a and b.

So, let me just write this statement. So, now, this is not the (Refer Time: 15:53) Taylor it is only approximate statement of Taylor. What Taylor says is that function is continuous between a and b and it is differentiable to n plus 1th order. So, you can differentiate that many times this function. Then we say that there exists and of course, a interval. So, I will I will choose the point x 0, which is my which is the point about which I am going to expand. Then it says which x not belongs to a b and let me just get this. So, I am going to write this here then for every x belonging to a b. There exist a zeta x between x naught n x such that f x is so, these were real function, but I am approximating this by a polynomial p n of x nth order polynomial plus a reminder these are reminder and so this formula for both of them. So, p of x is. So, f x is not written like this. In fact, this is p n of x. So, this p n of x is here and the last term of this is a n th order polynomial sorry n ones minus n ones n minus 1th order polynomial.

So, this is n minus 1 x 0 divided by n minus 1 factorial x minus x 0 n minus 1. So, this is a polynomial of the order highest power is n minus 1. And what is a reminder this one this guy is 1 by n factorial f n zeta of x. So, this zeta of x is coming here that point there exist a point. So, you should take the derivative nth order derivative at zeta x not at x.

So, this is exact this is exact relation. Now problem is of course, you do not know what is zeta x there exist does not mean that I know where it is, but we can pick mega bound on that. So, I can we can say that well this number lies between a window and that will give a bound on the error. So, that is what you can say you can not quite figure out most of the time what is zeta x. So, shall we do the proof you done in your math course and one math proof I would like to do my function f x which I do not know. I am approximating it by a polynomial and a reminder and polynomial is that Taylor series that we use, but these are something which is an error and the error will be of this form as zeta x lies between x between zeta x lies between x naught and x. So, if I draw this. So, a is here b is here x naught is here x is here and my zeta will be somewhere here, zeta and that will is a function of x that is a problem. So, at every x zeta x will different.

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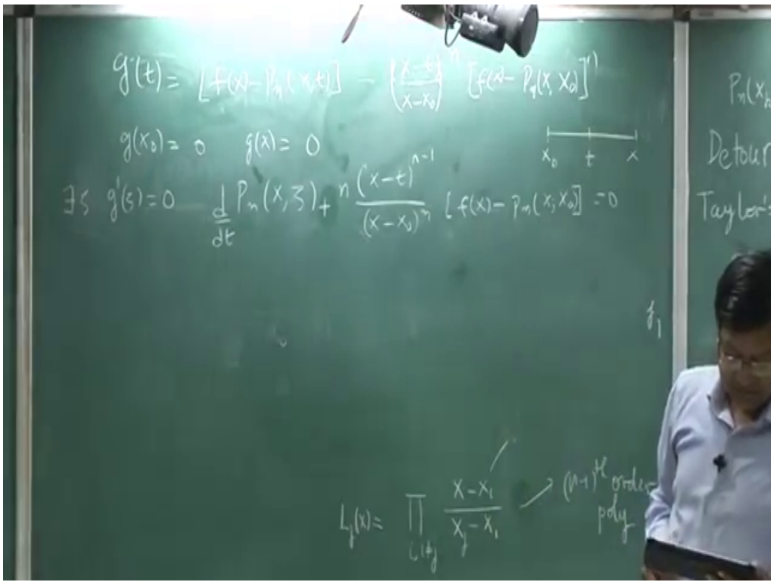

So the proof, so I have to look at this notes. So, construct a new function g of t, which is f x minus x of t. So, instead of x 0, I put t it will become clear, why I am using t minus x minus t by x minus x 0 to the power n f x minus p n x (Refer Time: 20:55) So, this is my function. So, x naught xi am focusing on this region and my point t is somewhere here. In fact, I want t to be a zeta. Now what is what is the value of g of x 0. So, at x equal to x 0 this becomes one and this becomes f x 0. And so it is 0. What about g of x. So, I make I make. So, this is 0. So, t of x 0 and here g of, so p of x is same as, so x equal to x0.

Student: (Refer Time: 22:05)

So, that is f of x right. So, that is f of x by definition. So, that is 0. So, I can use rolles theorem now. So, at this 2 points my value of g of t is 0. So, use rolles theorem and seeing that there existed point they exist a point c, where the derivative of g prime is 0, I will call it zeta instead of c. So, there exist say a point zeta where g prime zeta is 0. So, now, instead of t I substitute zeta and I take the derivative. So, I will get f prime of x. So, I have to well. So, I have to take derivative with related to t. So, this is not there, so p n prime x. So, these related to t is zeta equal to so this should be 0. So, I am gonna take it right hand side. So, take the derivative of this with related to t. So, that will be n x minus t n minus 1 divided by x minus x 0 to the power n.

Actually let us put this plus sign here. So, the minus will be plus. So, f x, the nothing happens to this know p m x comma x this equal to 0. Now what is this? So, let me write this is d by d t and I substitute t equal to zeta now what is this object. So, my p n is here either take with related to so the p has 2 arguments. So, right now I dint write it. It was x 0 along which point are you doing it now. Instead of x 0 I do t and take the derivative. So, if I take the derivative what will I get. So, let us just look at few terms.

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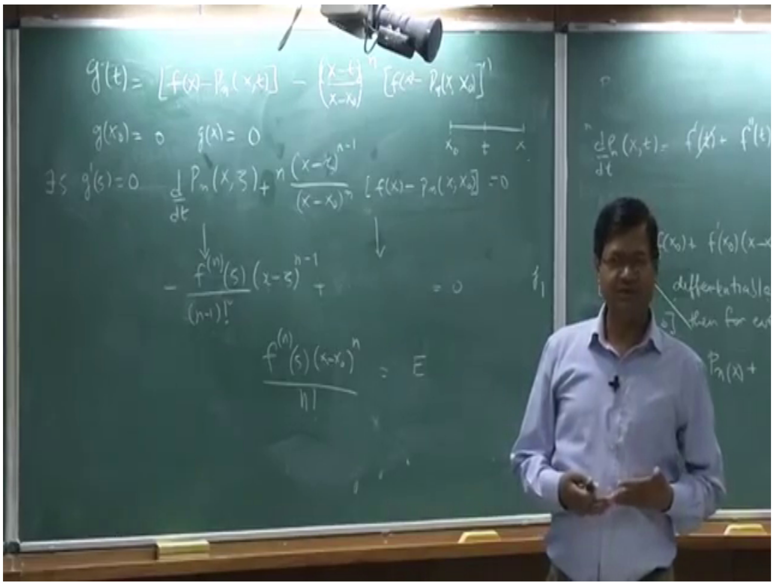

So, d by d t of x comma t, this is a prime. So, instead x 0 you should put t. So, f prime of t plus, now this is going to be f double prime f double prime of t, x minus x 0 plus product rule. So, this is a prime at p sorry and derivative of this that gives me minus and this will cancel. So, lot of this cancelation will take place. So, f double prime will cancel with the next term. And all the way and finally, we are left with derivative of this. So, this part you do it as homework you just do it at home.

So, this term when I do all that analysis comes out to be f n zeta by n minus 1 factorial x minus zeta to the power n minus 1. So, this is a derivative at zeta. Now did I miss the minus sign I thing it is right there is a minus, there is a minus sign here. This is a minus sign here oh this one has a minus sign this is a minus sign this is a minus sign. So, that now is fine. So, to the do the homework and you will get a minus sign here plus this is equal to 0. This also now this not a simple proof of course, this that is why Taylor series is famous, but now you can find what is the error know. So, where is the error? This is a error right. So, what is the formula for error?

So, error here this e, now this I have to take to the left side error equal to so there will be cancelation. So, this t equals. So, this will cancel with this right this is exactly same term. Now this will go to the left. So, this error is f n zeta x minus x naught 2 power n. Now this is already n minus 1 factorial. And this n will come n factorial. So, this is error and I know the zeta lies between x 0 and x. So, this is a proof this how you Taylor proved it that there is a Taylor series, but there is an error and error is like this understood last, and I have send this notes just do it once more and it is just using this Rolle's theorem you can prove it.

Now let us apply this to Lagrange interpolation. So, I have to construct a new g of t this one work this for Taylor series a same idea what used in new g of t the new g of. So, I am going to erase this part. So, it is quite similar. I have to copy this I do not remember this formula. So, the polynomial we already have written. So, what does the polynomial let us write the polynomial and it (Refer Time: 28:27) p n of x is the product x minus xi (Refer Time: 28:33) x.

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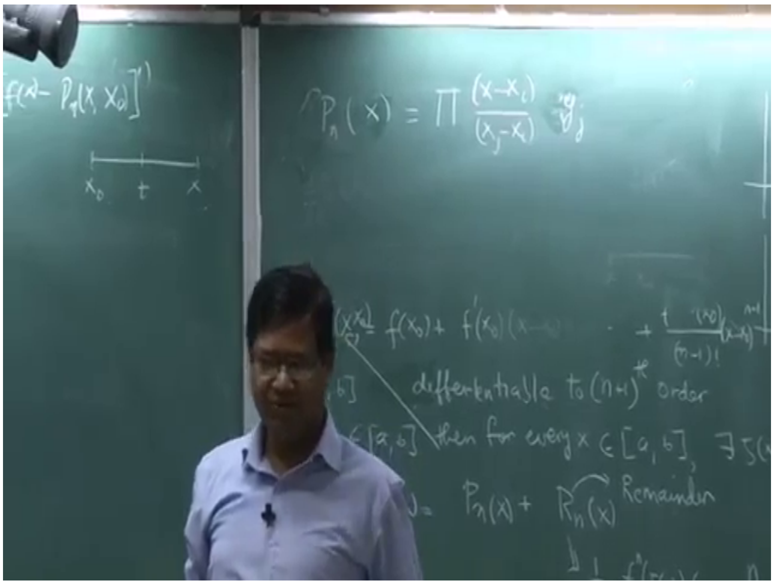

So, this is my polynomial which I am approximating which using which I am approximating function.

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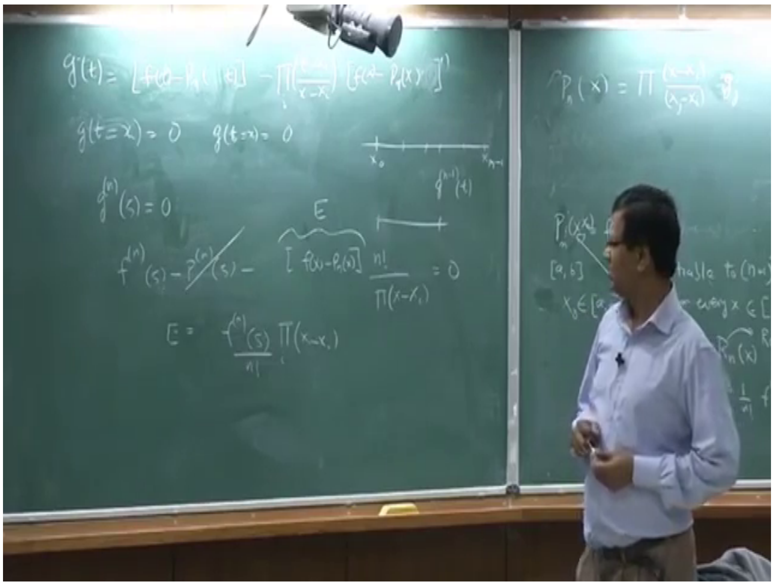

So, this is my polynomial. So, I am just going to write this as a p of t, now f x minus, so this p of x. Now it is 0. I am not expanding around and (Refer Time: 29:08) 0 and this function is replaced by product t minus xi x minus x naught. And this product for all I s there is no division by 0 problem. Now all end points, this is my g f t now this g f t is 0 at many places. So, it is 0 at what about g t equal to xi s see if I put 2 equal to xi this is one.

So, this becomes equal that f x minus p n of t. So, is 0 at all xis and how many are xis there n of them, 0 at n points. So, if I make this line at all this points x 0, this function is 0 what about g of x equals. So, if put this equal to x. So, this again is one did I make a no I did it, I did. So, this is one and this is 0 this equal to that 0. So, I think I am did make a mistake.

So, at t equal to xi this is 0. And f of x equal to p n of x i. So, that is. So, it is 0 all this points actually I made a mistake this another mistake f of t sorry that was a mistake is it clear. So, at xis this is 0 and this equal to that this n is 0. This equal to this at xis at those points a function value of the function equal to polynomial value of the polynomial and at t equal to x this matches with that. So, there are n plus 1 points where my g of t is 0. So, now, apply rolles theorem how many times, I keep applying n times. So, first time I get polynomial n minus 1 and a third polynomial that n zeros not n plus 1 n zeros. So, so keep going down and you have you take derivative n times.

And then I can claim, that after nth derivative there is a point zeta such that g n zeta is 0 is that clear. So, apply this rolles theorem again and again and again till you see that 2 points. Finally, I will see that g of n minus 1 of x of t has is 0 at 2 points. We do not know which we which where are the points I say 0 2 points. And then I say that my derivative will be 0 the next derivative at some point in between that 2 point. So, now, I have to set this 2 0. And now take the n th derivative n th order derivative. Now this is bit of task, but it is it will come out quite simple after this.

So, if I take the derivatives let see. So, I have to take with related to t. So, I will get f n of zeta minus p n of zeta. I can see that lot of you are not enjoying it. Now take the derivative of n nth derivative of this. So, this order t to power n right is n th order polynomial. So, highest power is t to the power n and any other power lower power will be 0. So, this is going to be n factorial divide by the product x minus xi is 0.

Now, this p n p as of what order? This p of t n minus 1 th order highest power is x n minus 1. So, this is 0. So, now, I have error know. So, what is the error? So, this is my error, error equal to f n of zeta product x minus xi by n factorial. So, it is not very difficult, but you just had to go through this several steps. So, let us do some examples. So, I am going to illustrate one simple example first for Taylor series how to estimate error. So, Taylor series I will just do it first, then I am going to run this python code and I will just show it here.

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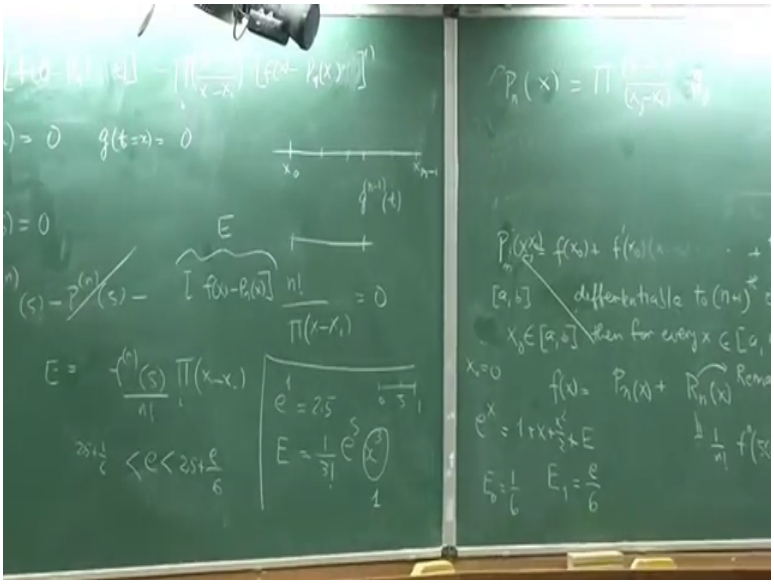

So, let us do it for x now I am going to expand near 0. So, my x equal to x 0 equal to so 0. So, Taylor series is for a. So, let us keep only 3 terms. So, first term will be, this 1 plus x. So, everybody know then the x divided by 2 plus error r.

Now, let us try to estimate value of the function e to power one. So, what is according to this what is the value without if I do not worry about the error one plus 1 plus half 2 point 5. Now how do I estimate the error? So, I have to see these errors. So, can I get an estimate of this? So, my error is one by n factorial. So, I have done up to 2. So, n should be this 2 factorial know n is 2. So, so I have to do 2 factorial is it, 2 or 3 3 3 it should be 3 factorial know. So, this is I am go on up to so yeah.

So, I am gone up to square. So, that this is 3 third factorial x minus x 0 square. So, next x minus x 0 cube. So, 3 factorial a derivative n th order derivative. So, it is going to be just e to the power x, but at point zeta which lies between 0 and 1. So, zeta is somewhere here x to the power n will be x cube, but x is 1. So, this becomes one now I gets then e to was zeta by factorial 3. So, what should do with it? So, I make a bound on this. So, what is the maximum value of e?

Student: e.

So, I have to say well this function zeta well zeta could be close to 0. Or close to one anywhere. So, I have to just estimate from here this is close to 0 then value of the e is one by c factorial, 1 6 e near 0 is zeta is near 0 what if e is around 1, e by factorial 3. So, that will give you e by 6. So, my function is bound am I getting it right. So, the function is bound between. So, well this is bigger value.

So, yeah I have put a bound my error is lying between one by 6 to e by 6, am I making any mistake that seems correct. So, point one. So, this is correct. So, this is point yes you put it right it is correct. So, e is 2.7 1 yeah. So, these are lower bound. So, es well I mean just to get this thing right e lies between 2.5 plus 1 6 and 2.5 plus e by 6. Now after this there is a trivial problem, where I can get this bound quite easily, but you can also get this I mean if you do it carefully you can get the bound, I mean this not big problem. Now let us do if I have only 7 minutes. So, let us quickly do one problem with Lagrange polynomial.

Let me just first without going through the code I can I can run the code very easily.

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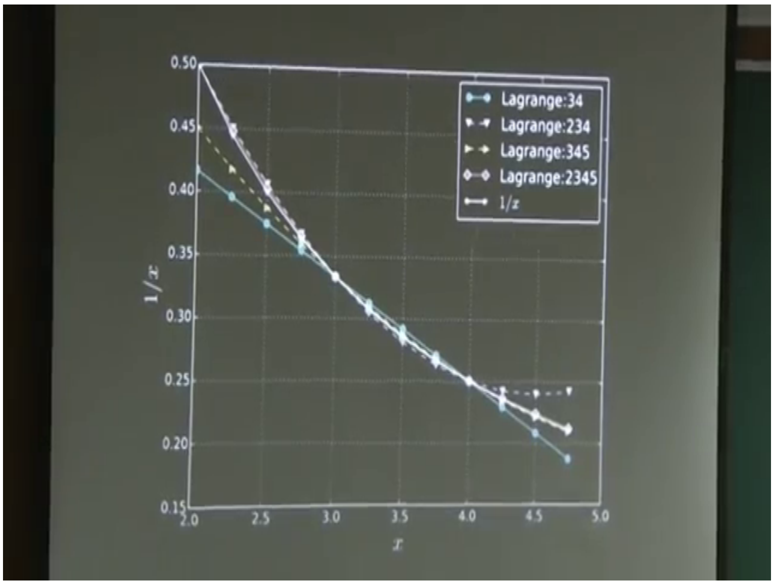

So, I am taking 1 by x an 1 by x is a function. So, this is for your illustration that I will suppose I know the function one is 1 by x. Now I take several options. So, I take 2 points, please remember.

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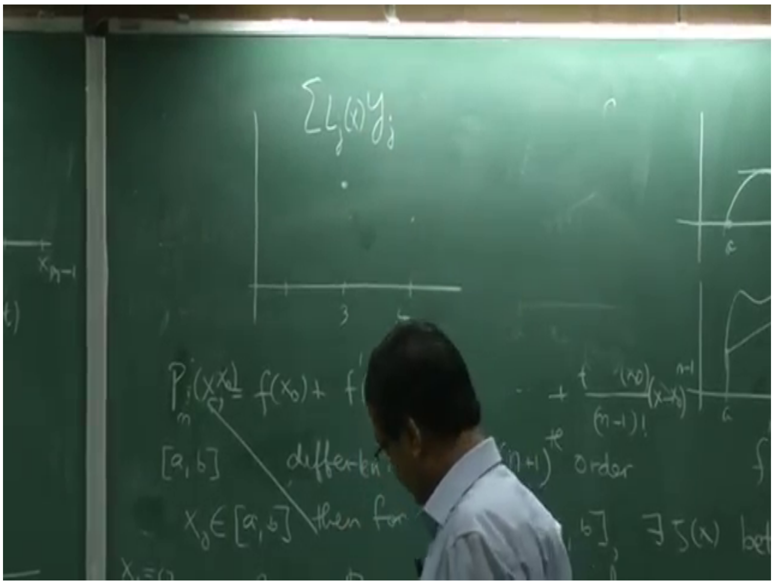

So, 3 4, I am computing. So, I am going to construct a function using 2 points 3 points 4 points that is what I will do. So, here the blue curve is using point 3 and point 3 and 4. So, at point x equal to 3, I get value one third which is here one third is here. One-third here and at 0.4. I get one quarter which is here is that clear to all of you. So, I take this 2 points and I have 2 values one third and one quarter.

Now, well I means I have given only these 2 values. And I have make an interpolation well suppose you do not know what this function f of x is, but I given these 2 points, I can make interpolation and that will be a straight line the blue line. So, there is a straight line which is the first order polynomial p 2. Now somebody says well I will give you more points. So, I will give you point say values at 2 3 4. So, 2 3 4 is I give the value at 2 which is 1 by 2 which is 0.53 and 4. Now there is a line going through that a line going through is pink the pink line which is so 1 by x is white line the real is one, white line. And my interpolation with 3 points is doing quite well right. Do you agree with that the pink line and white line are quite close I can also do interpolation with 3 4 5?

So, 3 4 5 instead of using 2, I can use 3 4 5. So, that is a yellow line yellow line is also doing this simply well pink line is off here and yellow line is off here. So, interpolation is good for the points in between those points given to you extrapolation is always bad and the reason is because of this product x minus xi, if here x is outside this region of xi s; that means, lower than lower minimum of xi s or higher than maximum of xi s you will get more error. So, you have to be careful interpolation is good extrapolation is not. So, good now, if I use 4 points which is red or light red brick colour which is fitting well all the way in the in this band so 4th order is because I am using 4 points. I can get cubic order accuracy is a polynomial. So, I am able to approximate my function 1 by x by a cubic order function. So, this is how I have show you the code in 1 minute.

Now, error, these are the error for these 4 polynomial the error you can see that for 2 points is bad lot of error, I am putting with sign I really should put absolute of you like.

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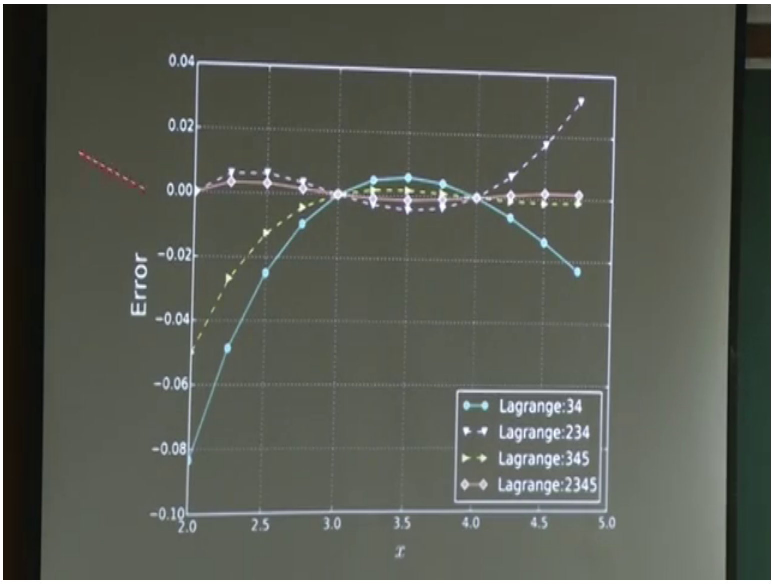

But, I am putting a sign it is not good yellow is not good in this regime and the pink is not good in that regime, but red is good all the way. So, this is still in the error these error was computed by I know the real answer minus, what Lagrange problem will gives me, I didn't use this this formula well I do not know what zeta is. So, I cheated I mean I am not getting error from the exact value.

And these are the value of the function now. Here min max I have applied this this ideas I now putting a bound.

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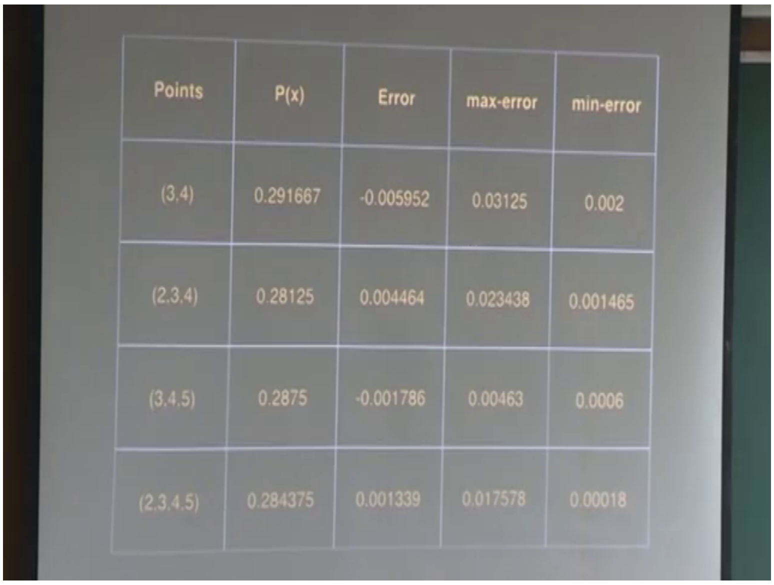

Points	P(x)	Error	max-error	min-error
(3,4)	0.291667	-0.005952	0.03125	0.002
(2,3,4)	0.28125	0.004464	0.023438	0.001465
(3,4,5)	0.2875	-0.001786	0.00463	0.0006
(2,3,4,5)	0.284375	0.001339	0.017578	0.00018

So, now I know the derivative since I know the function f of x. In fact, if you do not know the function then derivative calculation itself is a tricky part know how you compute the estimated derivative, but I know the function. So, this is the table which tells you that these are the value of the function error is decreasing. So, my error is always between these 2 bounds at the midpoint is not or none of them are doing. So, badly though this is the value of the function at 3.5 and 3.5 all of them are doing reasonably well I mean the difference is not, much because it is just lying in between of 3 and 4. So, let us look at code quickly.

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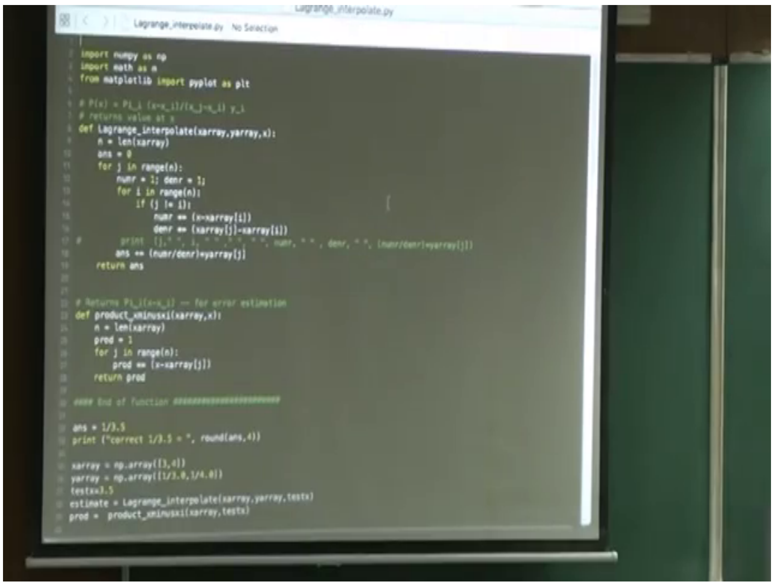

Now it is probably too tiny for you to see. So, this is the loop now Lagrange interpolate given the x array and y array. So, I give the array x. So, this is set of xis and that is set of y I s and I have to give the value of x where I want to interpolate. So, this will give you a number it returns answer now. This is denominator xi minus x j minus xi remember x j minus xi and this numerator j not equal to y 1 is product other one is the sum. So, here this is a product this one. Now the outside is the sum I am summing our all js. So, all y j remembers. So, this is the formula I am using.

So, quickly y j sum l j of x. So, this part is l j, this will give me l j. And then I have to sum over I sum over j. So, sum over j. So, please use index i and j are result for loop indices and this will do the work to estimate error. I have put this product function which is x minus x i. So, I put some comment here. So, this this one does the job and this how I generated those plots.

So, for 2 points it is easy. So, my x array is 3 4 y array is 1 by 3 1 by 4 my answer is one by 3.5 I am estimated it 3.5. So, I can estimate a Lagrange interpolate the function. So, I should use function for this. Because I can re-use it all the time and my answer is estimate and test ex test ex is 3.5. So, this gives me that stuff. So, it is easy I can do it with any point. So, I will leave this code, but you should be able to write this code we can also do it for 2 dimensions. So, this is interpolation 1d which sees easy, but I mean you are all again. So, all these schemes know it may look too simple or it may look too complicated. You have just stare at it once more and code it yourself, write your own program and then you will understand it. So, I will stop.