Engineering Mechanics
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Module 1
Lecture No 07
Kronecker Delta and Levi-Civita symbols-I

So far we have looked at vector operations like vector sum, subtracting a vector from another vector, and product of lectures. In this lecture, we are going to look at how symbolically right in a compact way and I am going to introduce something called the delta function.

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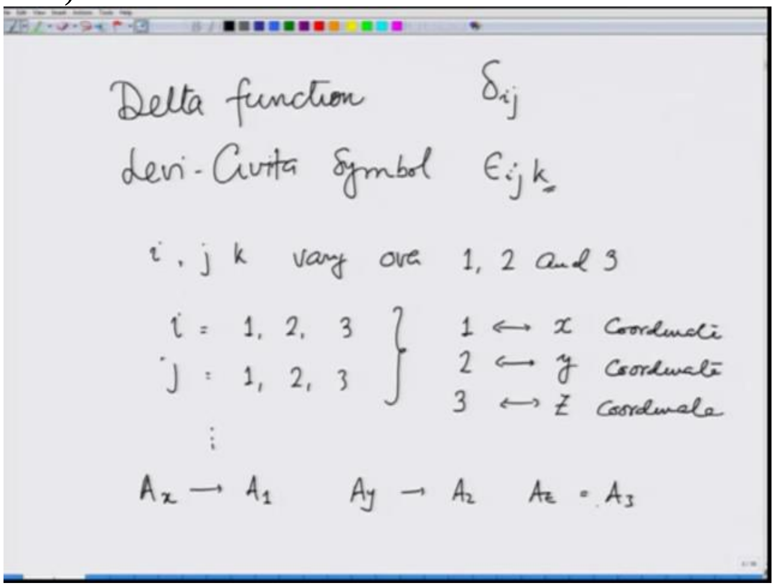

And Levi-Civita symbol. This I am going to denote as Delta IJ and this I am going to denote as IJK where I, J or K vary over 1, 2 and 3. That means I could take values 1, 2 or 3, J could take values 1, 2 and 3 and so on. So any any subscript that I use here can take values 1, 2, and 3 and we are going to denote 1 we are going to use 1 for X coordinate, 2 for Y coordinate and 3 for Z coordinate.

So just to make you familiar, for example I am going to write AX for a vector as A1, AY as A2 and AZ as A3. Writing in terms of 1, 2 and 3 makes life easy when we want to express quantities in a compact form.

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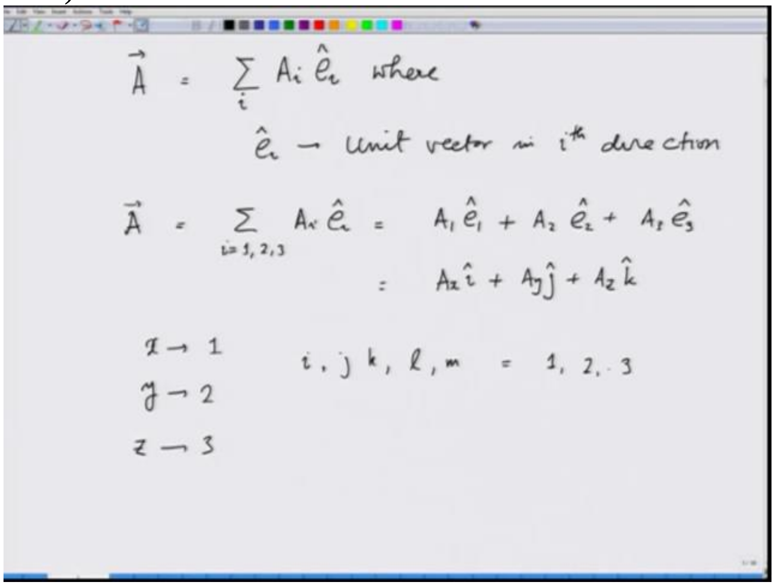

So for example vector A then could be written as summation I AIEI where EI's unit vector in ith direction. So you can see that if I expand it, it will be summation I varies from 1, 2, 3 AIEI and that becomes A1E1 + A2E2 + A3E3 which when translated into our regular language of X, Y and Z it will become AXI + AYJ + AZK. So this is just getting used to it that I am going to denote X as 1 I am repeating it, Y as 2 and Z as 3.

And any symbol subscript I, J, K, L, M, all of these very over 1, 2 and 3. So let me now introduce the delta symbol and Levi-Civita symbol.

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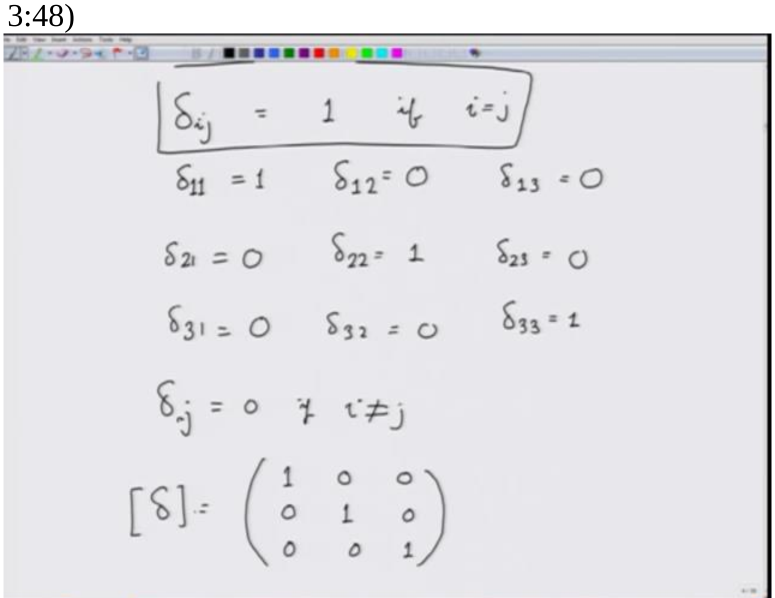

1st the delta symbol. Delta IJ is going to be defined as it is going to be 1 if I equals J. That means Delta 11 is 1, delta 12 we will see, delta 13 we will see what it is, delta 22 we will see what it is, delta 33 is 1, delta 23, delta 21 and delta 31 and delta 32. So this part we have taken care of by writing delta 11 equals 1, delta 22 equals 1, delta 33 equals 1. And if delta delta IJ is equal to 0, if I is not equal to J.

And therefore, all these terms are going to be 0. If I were to the present this delta symbol is an matrix, does going to be 1, 0, 0, 0, 1, 0, 0, 0, 1. This is delta matrix if you like. So this is an identity matrix. How does it help?

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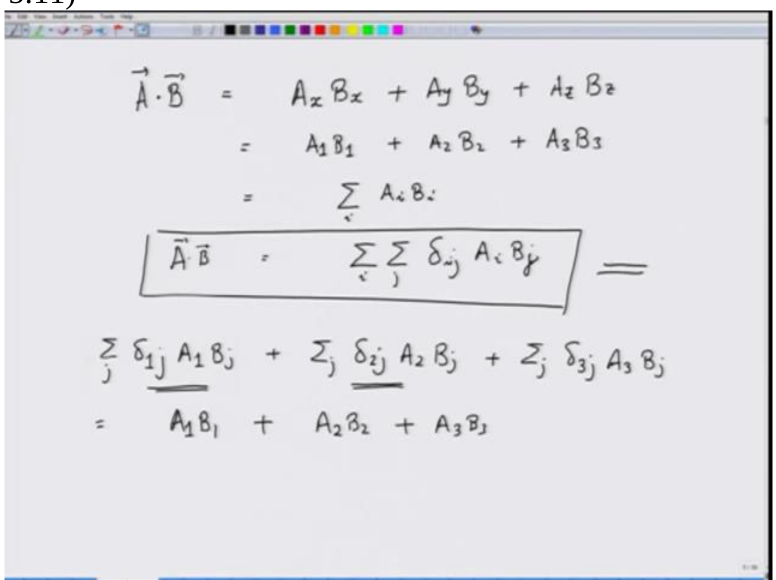

Let us look at that now. If I take A dot B, recall from earlier exercises or earlier lectures that this is nothing but AXBX + AYBY + AZBZ which I can write as A1B1 in our new notation + A2B2 + A3B3 which I can also write as summation IAIBI. And in terms of Delta symbol I can write this as summation over I summation over J delta IJAIBJ.

So this is A dot B. You can very easily see that it actually is the same thing as AXBX + AYBY + AZBZ. Let us do that just to get more familiar with it. So I have summation over J delta 1 J A1BJ + I is 2. So that is going to be summation over J delta 2JK2BJ + summation over J I equals 3 delta 3JA3BJ. When J is one, I get delta11 in the 1st term. Let us look at this term, J is 1, I get 1.

So I get A1B1. When J is 2, delta 12 is 0, delta 13 is 0. So the only contribution for the 1st term that comes is A1B1. Similarly in the 2nd term when J is 1, delta 21 is 0. When J is 2, delta 22 to 0. So I get A2B2. + in the last term similarly I get A3B3. So you can say this is a very compact way of writing a dot product.

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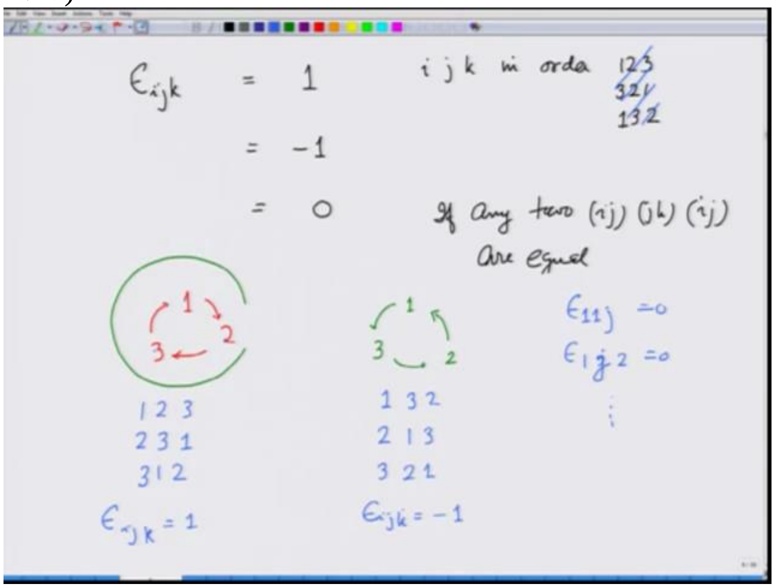

A more useful symbol is what I am going to define now, I, J and K. Again, I, J and K vary over 1, 2 and 3. This is equal to 1 or - 1 or equal to 0 depending on how I, J and K combine. So if I have I, J, K in order 1, 2, 3 or its cyclic order, that means 3, 2, 1 or 1, 3, 2, then is going to be 1. This in the other order is going to be - 1 and 0 if any 2 IJ or JK or IJ are equal.

If you find it a little difficult to remember, now let me give you a trick. Let us write, we will take a different colour 1, 2 and 3 in clockwise fashion. And let us also write this in counterclockwise fashion 1, 3, 2, traverse this in counterclockwise way. If 1, 2, 3 appear in any order of this circle, that means if I have 1, 2, 3 or 2, 3, 1, 2, 3, 1, or 3, 1, 2. So this was wrong.

Then I am going to have epsilon I, J, K equal to 1. If I am on the other hand 1, 3, 2 or 2, 1, 3 or 3, 2, 1, then I am going to have Epsilon I, J, K equals - 1. And Epsilon 11 J is 0 no matter what J is. Epsilon 1J2 30 and so on. So you can see very easily that this is a very easy way of remembering it. You put 1, 2, 3 if you are going clockwise. Put it like a clock 1, 2 and 3.

If you are going clockwise, any combination, 1, 2, 3, 32 3, 1, 2 or 2, 3, 1 gives you Epsilon I, J, K equal to 1. And any counterclockwise combination, 1, 3, 2, 2, 1, 3 or 3, 2, 1 gives you - 1 and rest 0.

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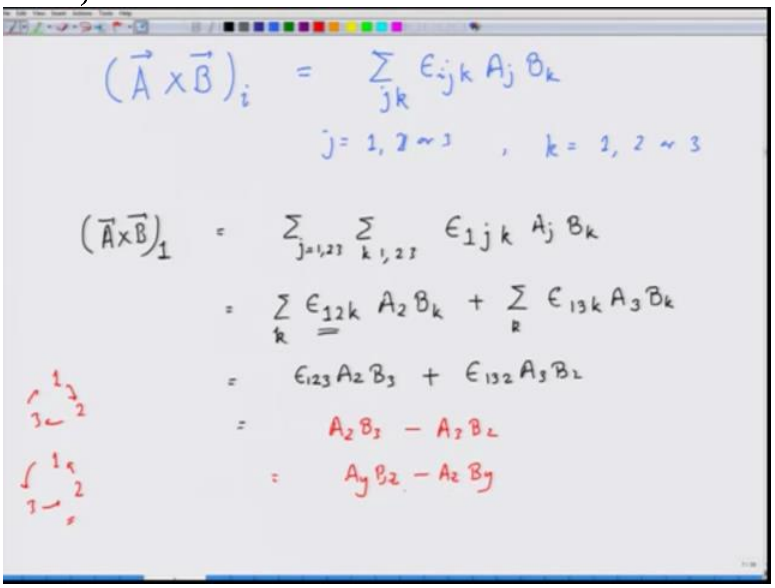

So now I am going to express the cross product. If I take A cross B, it is Ith components, I could be 1, that is X component, I could be 2, that is Y component, I could be 3, that is Z component is going to be expressed as summation over J and K Epsilon IJKAJBK where J equals 1, 2 or 3. So I am going to sum over 1, 2 and 3 and K also equals 1, 2 or 3. Let us see that. Let us calculate A cross B component 1.

That means I am going to calculate component number X component of this which is going to be summation J equals 1, 2 and 3 summation K1, 2 and 3 Epsilon 1JK AJBK. Let us expand this. If J equals 1, J is 1. If J is 1, for J equals 1, Epsilon 11K is 0. So J could be either 2 or 3 only. So I am going to have Epsilon 12K summed over K I have taken J equal to 2 A2BK for J could be 3. So summation J is 3 and K would be 1, 2 and 3 Epsilon 13KA3BK.

Now, so we have done the sum over J using the fact that Epsilon 11K is 0. Right? So now if I look at Epsilon 1, 2 and K, that is this term. If K is 1, this is 0 because Epsilon 121 is going to be 0. So the only term that I am left with is 1 2 3A2B3 + Epsilon again the 2^{nd} term 13K. If K is one, it is 0. If K is 3, it is 0. So only choice I have is 1, 3, 2A3B2. Now 1, 2, 3, let us make again our 1, this is a clock 1, 2, 3. If I go this way it is 1, 1, 2, 3.

If I go this way, it is - 1. So Epsilon 1, 2, 3, I am going clockwise. So it is going to be 1 so it becomes K2B3. 1, 3, 2 is the other way. So that is going to be - A3B2 which is nothing but AYBZ - AZBY which the correct answer. So you can see by using the symbol I can very easily express all components in one simple formula like this. And then we are going to use the properties of these 2 proved various relationships.

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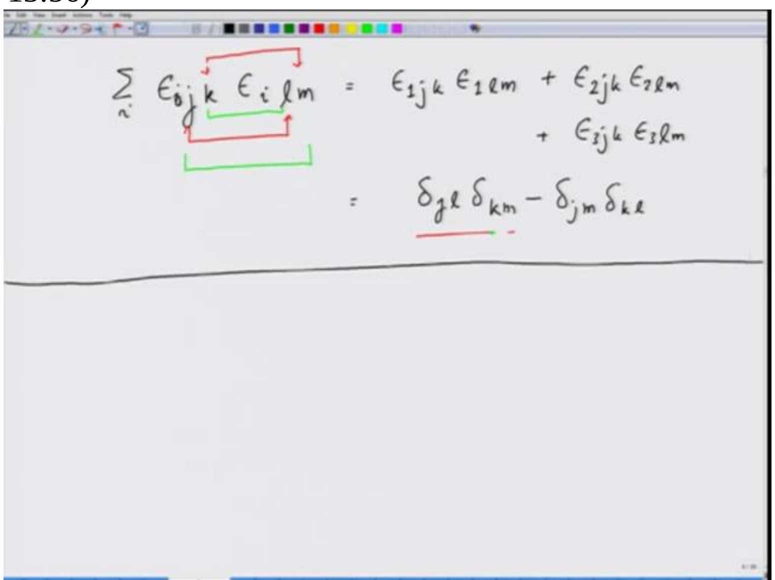

One property that we are going to use is going to be Epsilon IJK Epsilon ILM sum over I. That means if I take Epsilon 1 JK Epsilon 1 LM + Epsilon 2 JK Epsilon 2 LN + Epsilon 3 JK Epsilon 3 LN, it is going to be equal to and I am summing over I, so I have summed over I, delta JL delta KM - delta JM delta KL. What I have done is, I have taken J and L in the 1st term and K and M, made their delta symbols, put them here.

Then I took the other combination, that is JM and KL, put a - sign can put the other delta functions. This is a very useful identity. So let us now see how these definitions or expressing vectors like this.