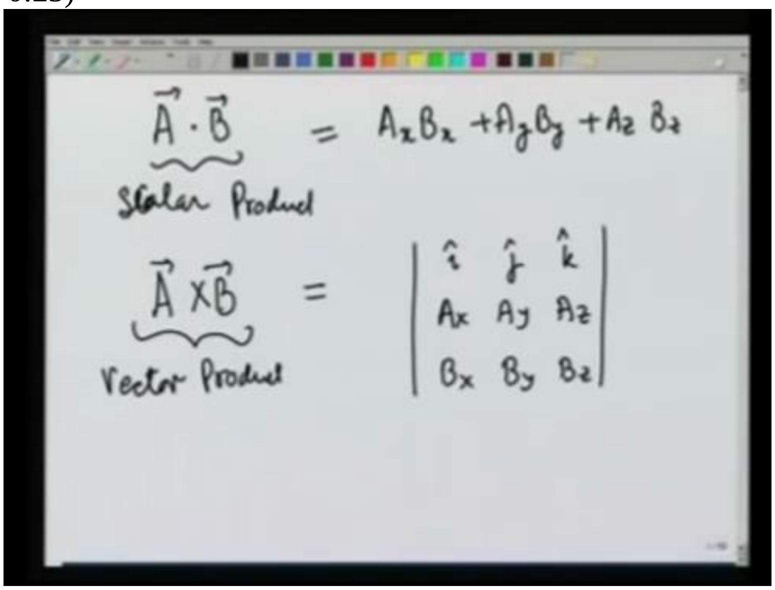
Engineering Mechanics
Professor Manoj K Harbola
Department of Physics
Indian Institute of Technology Kanpur
Module 1
Lecture No 06
Vector products and their geometric interpretation

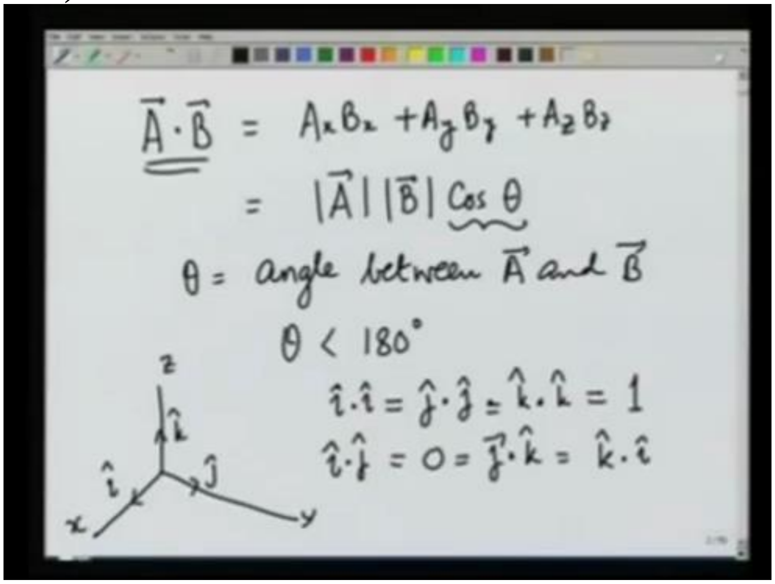
In the last lecture we had been looking at vectors. In particular how to represent them algebraically.

(Refer Slide Time: 0:23)



And then we looked at the product of vectors and defined the scalar product which we wrote as AXBX + AYBY + AZBZ. And a vector product which gives me a vector A cross B. And in a short way I wrote this as determinants of AX, AY, AZ, BX, BY and BZ. I had had asked you to check that this gives the correct answer. We define these 2 products in this manner because this satisfies certain transformation properties of vector components under rotation. Let us look at these products are slightly more carefully.

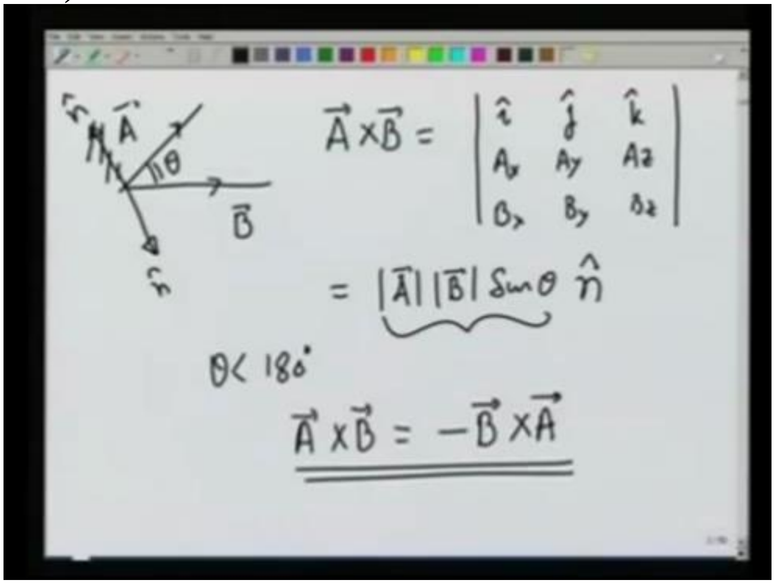
(Refer Slide Time: 1:40)



So let us look at the dot product A dot B which is AXBX + AYBY + AZBZ. This can also be written as magnitude of a times magnitude of B times cosine of theta where theta is the angle between A and B. And theta is the smaller of the angle. So theta is less than 180 degrees. In particular, if I look at the unit vectors along X direction, Y direction and Z direction, we find that I dot I is same as J dot J.

They are called equal to 1 because their magnitude is 1 and the angle between J and J or K and K is 0. Similarly I dot J is 0. So is J dot K. And so is K dot I. Notice that the dot product or the scalar product can be both negative and positive because cosine theta can be negative or positive.

(Refer Slide Time: 3:37)

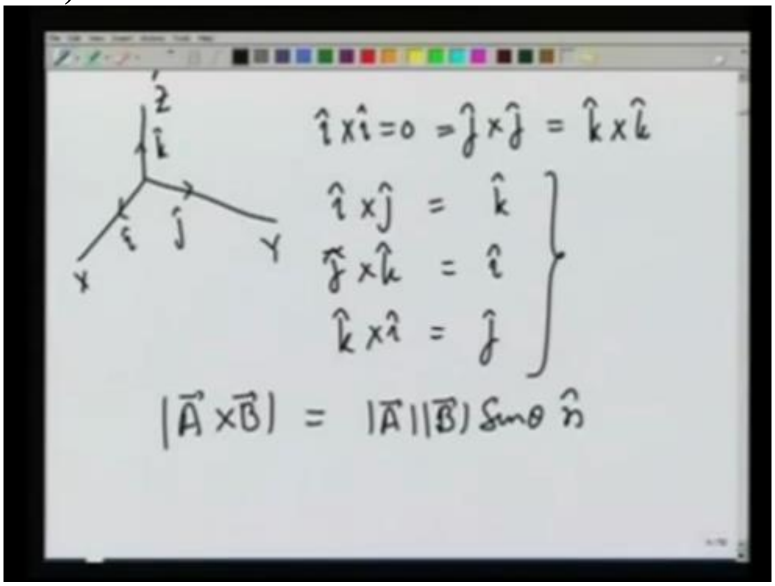


Similarly, if I look at 2 vectors A and B, then A cross B which I can write as the determinant I, J, K, AX, AY, AZ, BX, BY, BZ is also equal to magnitude of A times magnitude of B times sine of theta where theta is the smaller of the angle between A and B. So theta is less than 180 degrees, times a unit vector in the direction N where N is perpendicular to both A and B. So it is perpendicular to the plane formed by A and B.

So it will be somewhere like this. Maybe coming out of this plane of the board. How about whether N is this way or this way? This is answered by the right-hand rule. You turn A towards B through the smaller of the angle and the thumb gives you the direction of the vector product. So in this case, N would be going down.

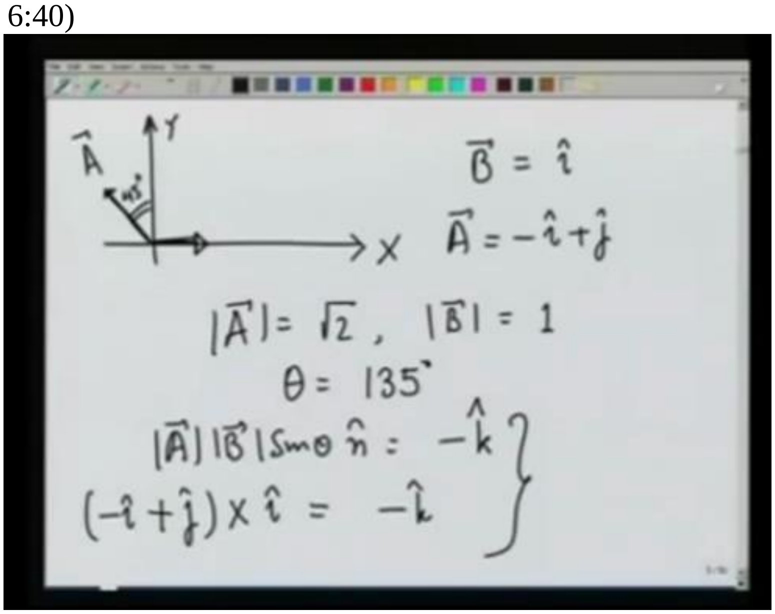
You can see that A cross B is going to be equal to - B Cross M because the magnitude will remain the same but the direction of N would change because now you will be turning B towards A. This is also clear from the components if you look at them carefully. So this is how the cross product works.

(Refer Slide Time: 5:29)



In particular if I look at the unit vectors, I, J, and K in X, Y and Z direction. Then I cross I is 0. And so is J cross J because the angle between them is 0. And so is K cross K. On the other hand, I cross J is equal to K. J cross K is equal to I and K cross I is equal to J. These are the relationships between the unit vectors. Let us now look at 1 or 2 examples where you see these relationships carefully. 1st, I want to show you that A cross B through an example is really equal to A B sine theta times N where N as I told you is in the sense of rotating A towards B through the smaller of the angles.

(Refer Slide Time: 6:40)

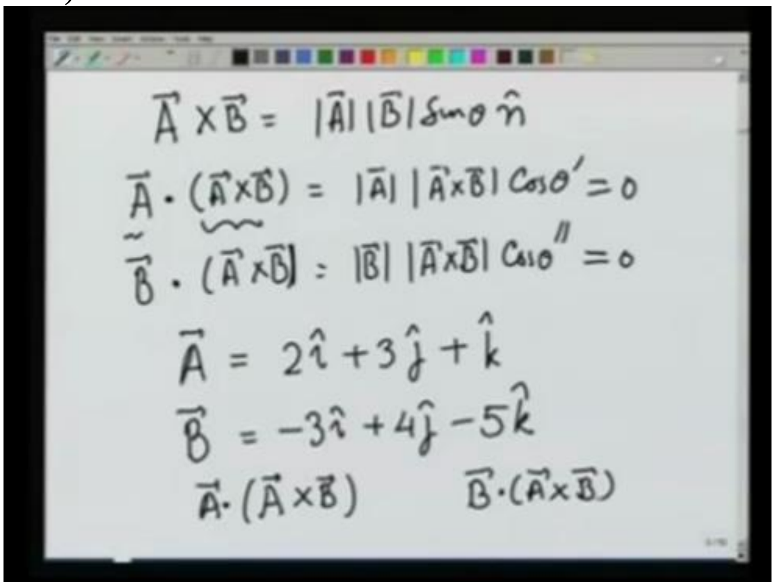


So let us for simplicity take this as X axis, this as Y axis. Consider B to be a vector, unit vector in X direction. So B is equal to I. And consider A vector to be in this direction which is 45 degrees from the Y axis or the X axis. So A is equal to - I + J. You can see that the magnitude of A base square root of 2, magnitude of B is one.

And the angle theta between them is 135 degrees. Therefore AB sine theta N is going to be equal to sine of 135 over root 2 is going to be equal to 1 times N. Turn A towards B through the smaller angle and that gives you a direction which is opposite to z-axis. So this is going to be - K. Let us see if you get the same answer if I explicitly do component wise vector product.

In that case, A cross B would be - I + J cross I. I cross I is 0 and J cross I is - K. So you see both give you the same answer. Now I want to demonstrate to you that A cross B gives you a vector that is perpendicular to both A and B.

(Refer Slide Time: 8:34)

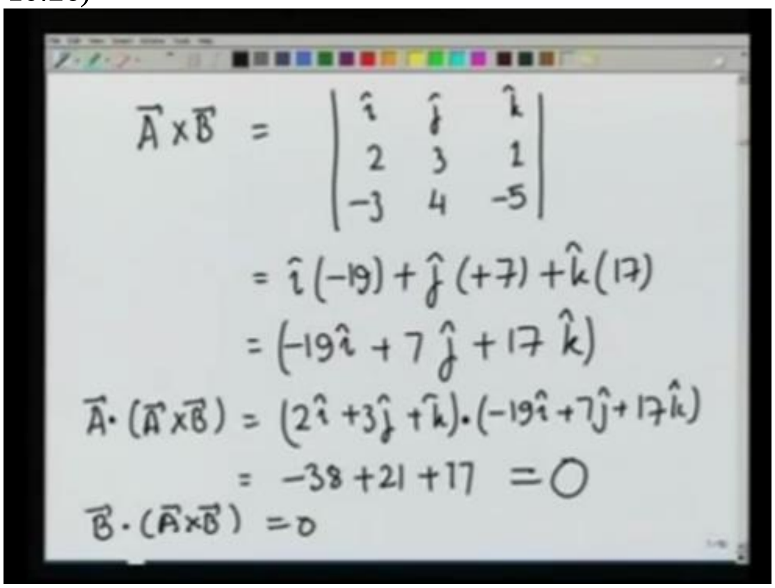


Since A cross B is equal to modulus A modulus B sine theta N where N is perpendicular to both A and B. That means that A dot A cross B which is equal to modulus of A cross B cosine of theta, let us call it theta prime because now this is the angle between A and A cross B should be 0. Because A cross B is in a direction perpendicular to A. And so, it should be B dot A cross B which is equal to modulus or agnitude of B magnitude of A cross B cosine of theta double prime.

Again, since A cross B is supposed to be perpendicular to A and B both, they should also be 0. Let us take an example. Let us take A to be a vector. Let us say 2I + 3J + K and B be a vector let us say - 3I + 4J - 5K. We want to take their cross product or vector product and show that the resulting vector is really perpendicular to both A and B. So what we will do?

We will calculate A cross B and then take dot product with A and take dot product with B. So let us do that.

(Refer Slide Time: 10:26)

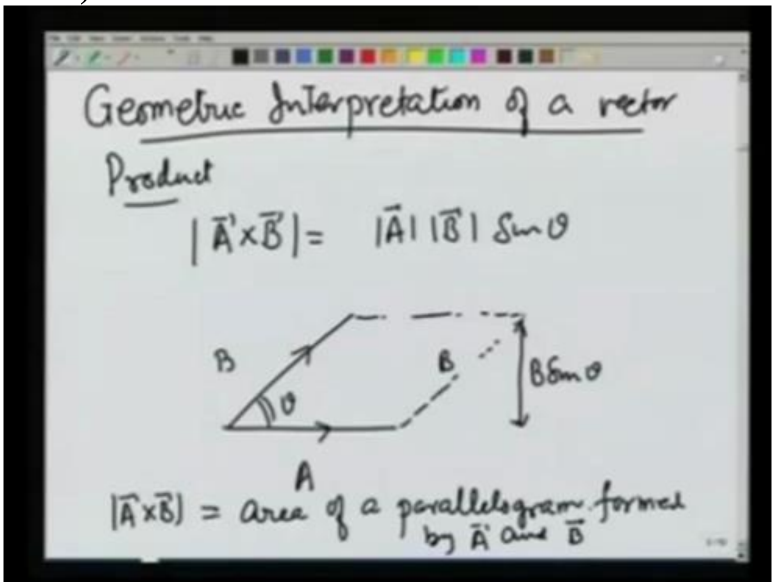


So A cross B is going to be equal to I, J, K. Let us see what the vectors were? 2, 3, 1. So 2, 3, 1. And - 3, 4 and - 5. - 3, 4 and - 5. So the cross product is I - 15 - 4 so - 19 + J - 3 + 10 so + 7 + K 8 + 9, 17. So this gives me a vector - 19I + 7J + 17K. This is A cross B. Let us take A dot A cross B which is equal to 2I + 3J + K dotted with - 19I + 7J + 17K.

Now I can use the distributive property and shown that I dot I is 1. So this is going to be - 38, I dot J is 0, I dot K is 0. So no contribution from here. + 3 would multiply only with the J component + 21. And K would be + 17 which is equal to 0. Since a and A cross B both magnitudes are nonzero, this can only happen if the angle between A and A cross B is 90 degrees.

I leave for you to show that B dot A cross B is also 0. Thus we see that A cross B is perpendicular to vectors A and B both.

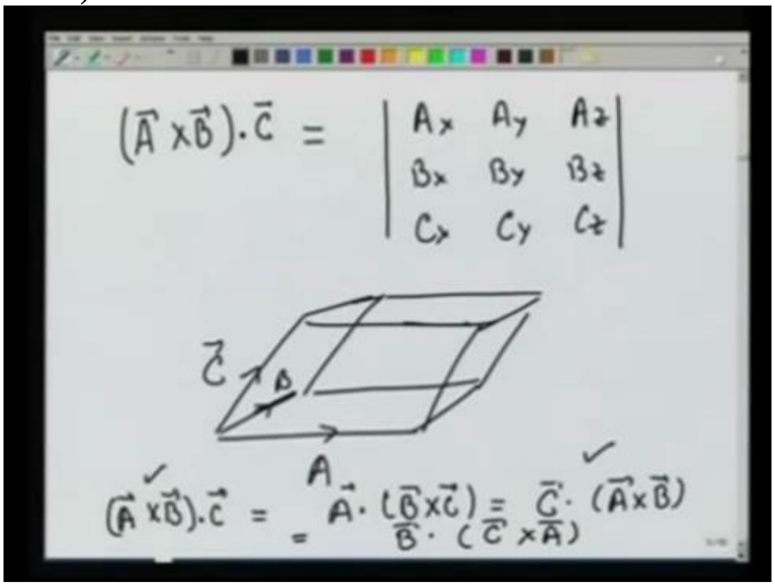
(Refer Slide Time: 12:44)



Let us now look at geometric interpretation of a cross product of a vector product. A cross B magnitude is nothing but magnitude of A magnitude of B times sine of theta where theta is the angle between them. So if there is a vector K and there is a vector B, angle between them is theta, you can see that A cross B is nothing but A times B sine theta. This is B magnitude.

So this is B sine theta. So this is the base times the height of this parallelogram. And therefore magnitude of A cross B is the area of parallelogram formed by A and B.

(Refer Slide Time: 14:22)



Next, we look at A cross B dotted with C. This I leave for you to show that is AX, AY, AZ, BX, BY, BZ and CX, CY and CZ determinant, its value is equal to this. And this has a geometric interpretation of if I form a parallelopiped by vectors A, B and C. It is the volume of this parallelogram. So that is the geometric interpretation.

Since I can write A cross B dot C as this determinant form, this also shows since determinant does not change if I change its rows twice, this is also equal to A dot B cross C. This is also equal to C dot A cross B which is same as this. And this one more term, this is also equal to B dot C cross A. All these are equal. This is cyclically, the dot product and the cross product go like this. So this is a general review of vectors we obtain.