Engineering Mechanics

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Module 1
Lecture No 05
Transformation of vectors under rotation

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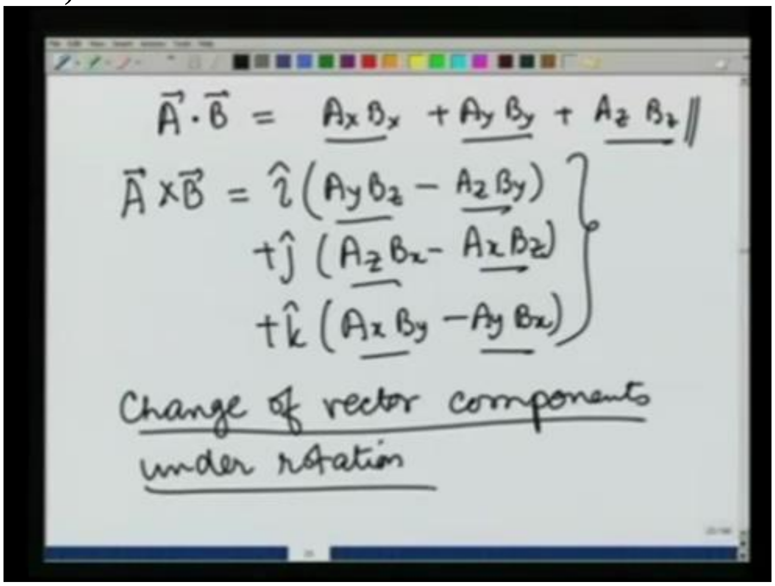

So so far we have just looked at the vector quantity as something that has a direction and that has a magnitude but now we are going to look at some more properties.

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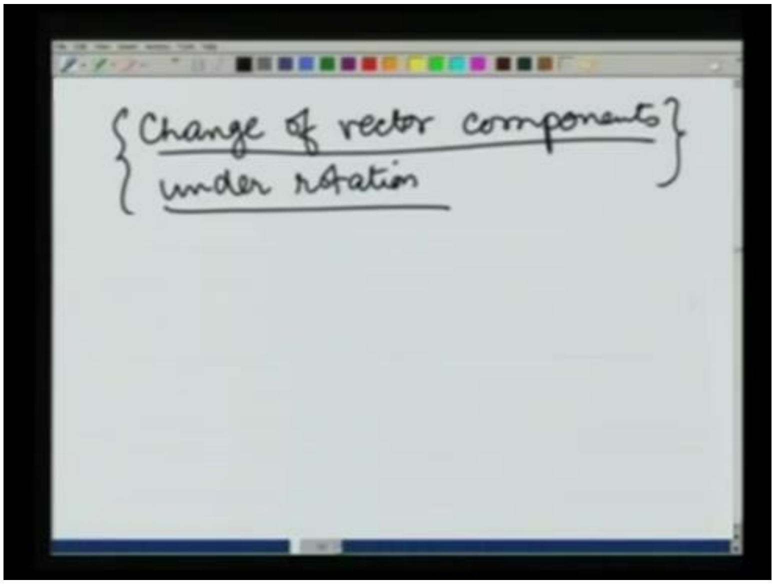

And one property that we specifically look at is the change of vector components under rotation. Let us see that.

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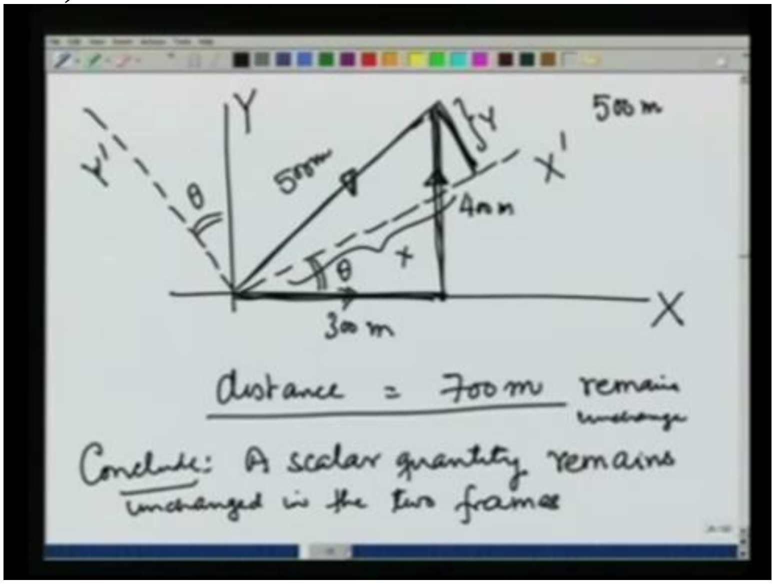

Suppose you have a friend who asks you, how far is the house of another friend? And you say, it is 500 m or you say, you walk 300 m to the east I am just taking the previous example, and 400 m to the north. So that the direct distance of that house is 500 m. But you have to walk in a zigzag manner. In that you have to go east 1st and then north. Total distance you cover is 700 m.

Now suppose you have another friend who is looking at it from a different frame. So for the previous friend, you had the X axis along the east and Y axis along the north. But another friend has his axis X prime and this at some angle Theta from the east and Y prime at some angle Theta from the north.

Although the distance travelled by this person in going to the other house is going to be 700 m in both the frames, however this vector you can see is going to have components which are not going to be 300 and 400 m. But this is going to be its X component and this is going to be its Y component. So although the vector remains the same, it is 500 m in this north-east direction but its components along different frames are different.

On the other hand, for the scalar quantity, distance, that remains the same, unchanged. So what we conclude? A scalar quantity which is nothing but a number remains unchanged in the two frames.

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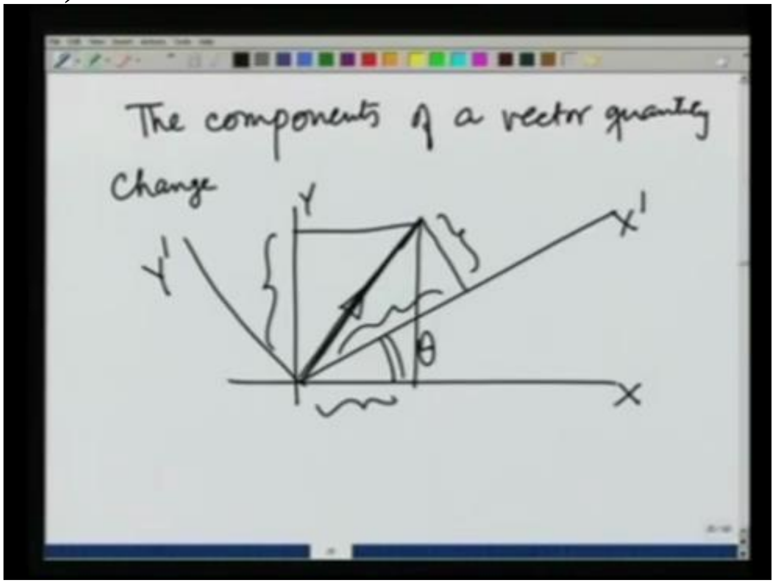

On the other hand we also see that the components of a vector quantity change when we go from one frame, say XY to another frame which is rotated with respect to the 1st frame. X prime, Y prime which is rotated by an angle Theta with respect to the 1st frame. For the same vector quantity which in space is still pointing in the same direction, has the same magnitude.

This is going to be X component, this is going to be Y component. On the other hand, the X prime component is going to be this and Y prime component is going to be this. And the relationship between X prime and Y prime component and X and Y component is well-known and that is what we are going to derive now.

So a vector quantity must follow that relationship. Its components should change according to that relationship when we go from one frame to another frame which is rotated with respect to the 1st frame. So let us derive the relationship.

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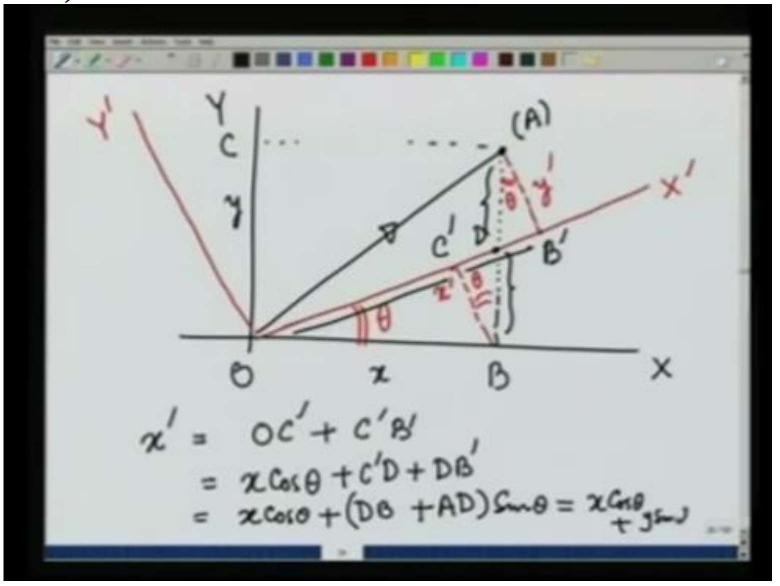

So let us say there is a vector pointing in a certain direction in the original frame X and Y. So that this is its X component and this is its Y component. I am looking at the same vector from another frame. Let me now use a different colour. X prime and Y prime which is rotated with respect to the 1st frame by an angle Theta. The component X prime is going to be given here.

This is going to be X prime and this is going to be Y prime. You can also see that this angle is also Theta. Let me also draw a perpendicular from here to here. This angle is also Theta. Let this point be A, B, C, this is the origin. Let this point be B prime, let this point be C prime. So we see that X prime component is going to be equal to OC prime + C prime B prime, that is this + this.

And let me also call this point where it intersects as D. OC prime OC prime is nothing but X cosine theta. So this is going to be X cosine of theta. + C prime B prime is nothing but C prime D + DB prime. So this is nothing but C prime D + D B prime which is equal to X cosine of theta. C prime D is nothing but DB sine of theta. DB sine of theta. I am deliberately writing it like this.

+ DB prime AD sine of theta. Sine of theta I have taken out. Now you see, DB + AD is nothing but Y. So I can write this as X cosine of theta + Y sine of theta.

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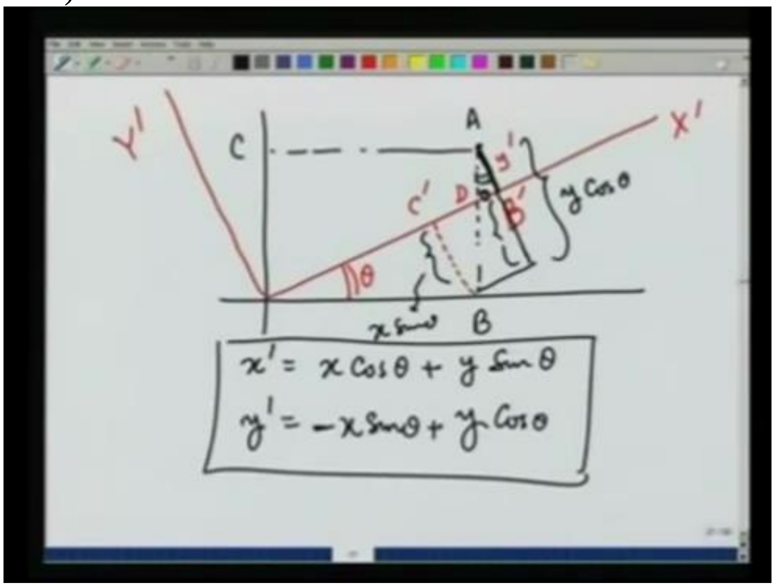
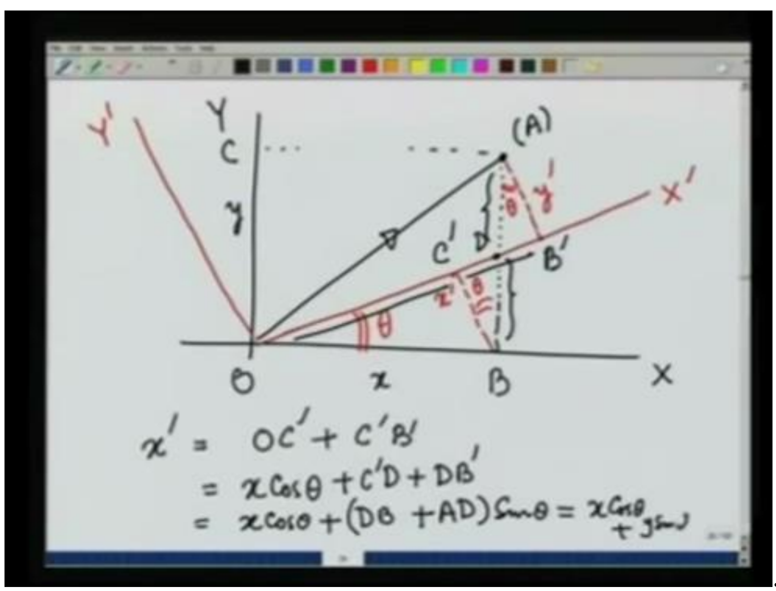


And what I have shown you in the previous page is that X prime is X cosine of theta + Y sine of theta. Similarly, if I were to calculate Y prime this length, this, let me extend this here, this distance is nothing but, this angle is theta. Y cosine of theta - this distance which is the same as this distance which is nothing but X sine theta.

So Y prime therefore is going to be Y cosine of theta - X sine of theta. And this is how the X and Y components of a vector are going to change. I wrote this for a displacement vector X and Y.



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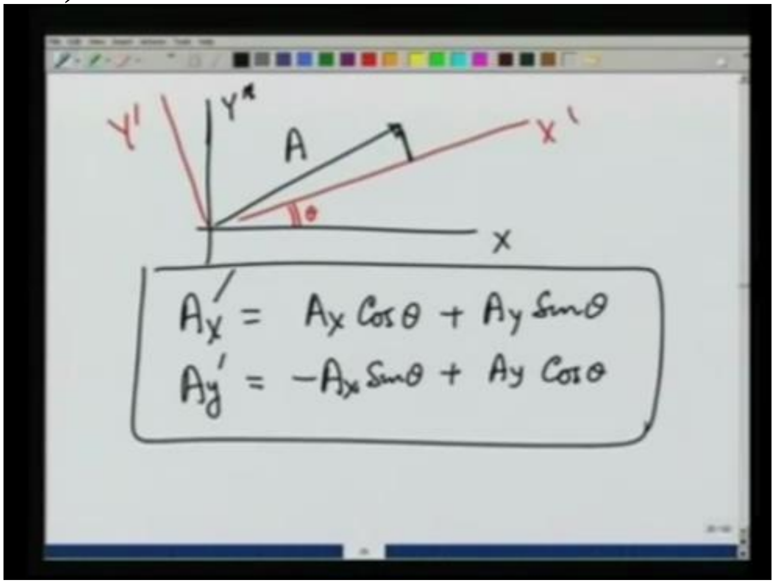

But in general you can do the same exercise and you would find that if I look at a vector A in the two frames which are rotated with respect to each other, then AX prime that is the component of A in the rotated frame is going to be equal to AX cosine of theta + AY sine of theta and AY prime is going to be equal to - AX sine of theta + AY cosine of theta. This is how the components of a vector transform.

On the other hand a scalar number remains the same number in both the frames. We have looked at, in this case, when we rotated the frame about the Z axis because only X prime and Y prime axis changed. We can do the same thing for the other axis also. So for example, suppose we rooted about the X axis.

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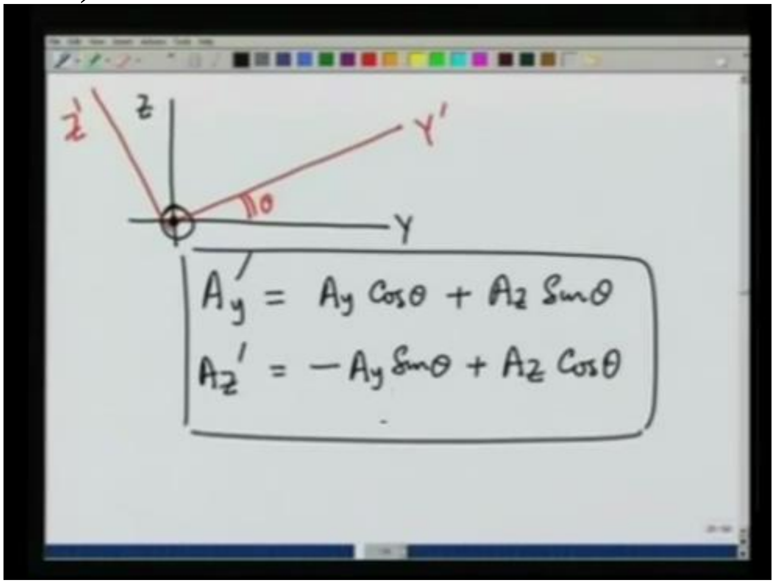

So let us say we have X which is coming out, Y and Z. And suppose we take the new frame by rotating about the X axis. XY and ZX is the same. So this is going to be Y prime and this is going to be Z prime by an angle theta. Then you can see that what we did earlier, by that AY prime in the rotated frame when I rotate about the X axis is going to be AY cosine of theta + AZ sine of theta.

And AZ prime is going to be equal to - AY sine of theta + AZ cosine of theta. Similarly, if I rotate about the Y axis, let us do that.

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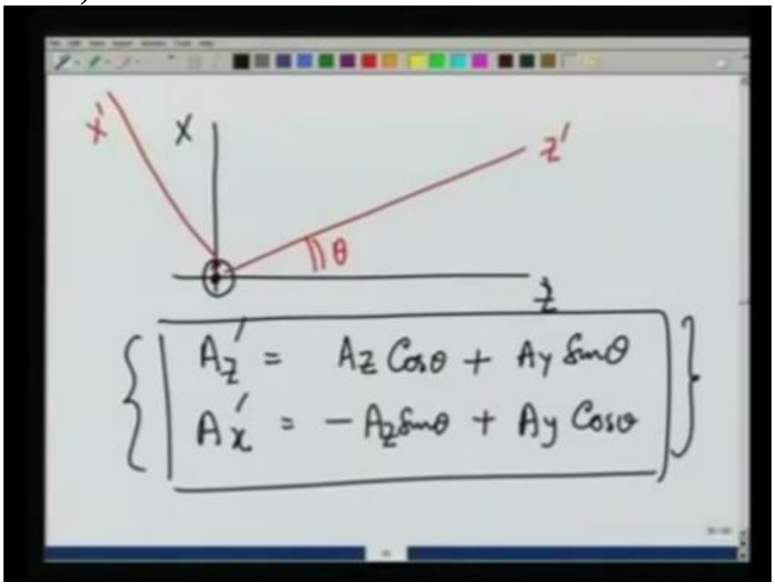

If I rotate about the Y axis, this is going to be X axis, Z axis, Y axis coming out and make my new Z prime and X prime axis like this by rotating about the Y axis, you will see that AZ prime is going to be AZ cosine of theta + AY sine of theta and AX prime is going to be - AZ sine of theta + AY cosine of theta. So this is how the vector components of a vector transform under rotation with respect to X axis, Y axis or Z axis, that is if I rotate about the Z axis, I change X prime and Y prime components.

If I rotate about the X axis, I change Y and Z components and if I rotate about the Y axis, I change X and Z components and they change in a very very specific manner. On the other hand, the scalar quantity remains unchanged even if it is looked at from a rotated frame and that we are going to use now to define our products.

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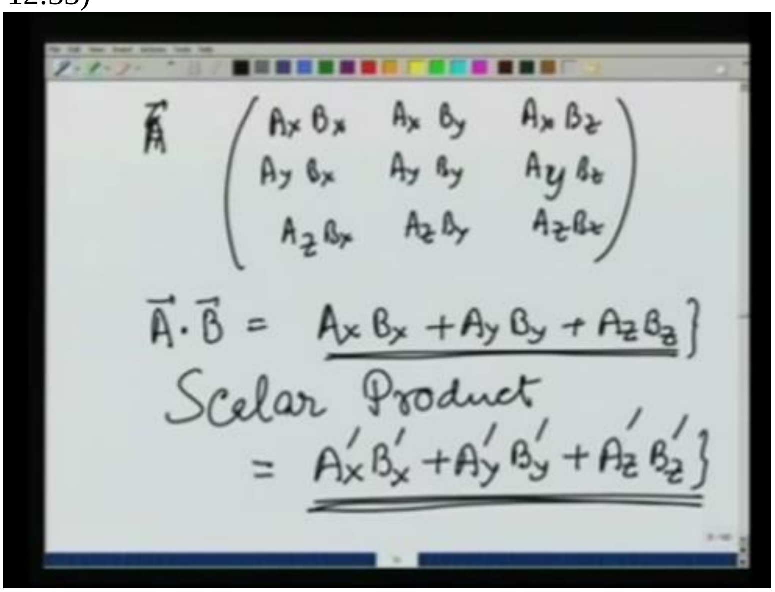

So 1st quantity, I told you about is of those, let me write all the components. AXBX, AXBY, AXBZ, AYBX, AYBY,AYBZ, AZBX, AZBY, AZBZ. Of all these 9 components, we took 3 of them and wrote a scalar product as the sum of AXBX + AYBY + AZBZ and we call this as a scalar product. That is, if I take these components and add them up, this gives me a number which is scalar. How do I prove it?

I prove it by going to a new frame in which the scalar product is going to be AX prime BX prime + AY prime BY prime + AZ prime BZ prime. And this AXBX + AYBY + AZBZ must be the same as AX prime BX prime + AY prime BY prime + AZ prime BZ prime if this is a scalar. Let us show that for one specific case and rest I will leave for you as an exercise.

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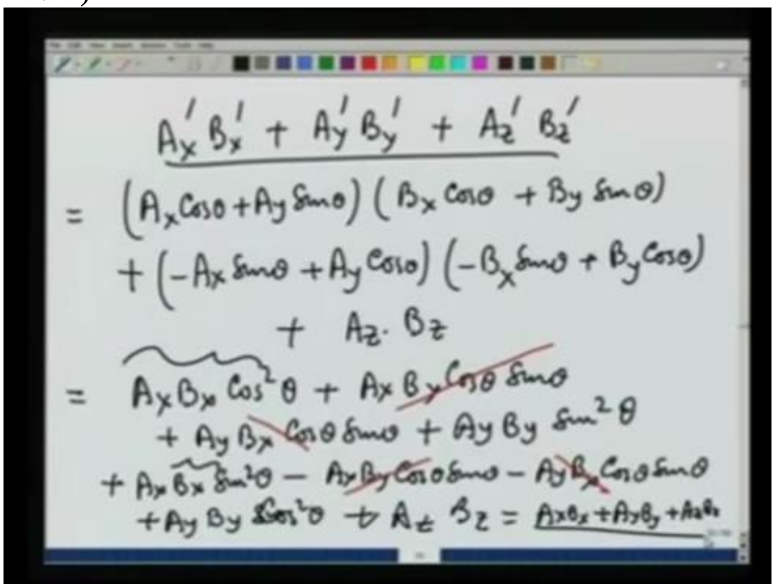

So let us look at rotation about Z axis so that in that case, AX prime would be equal to AX cosine theta + AY sine theta, BX prime would be similarly BX cosine theta + BY sine theta. AY prime would be - AX sine theta + AY cosine theta. BY prime would be equal to - BX sine theta + BY cosine of theta. AZ prime would be same as AZ because Z axis is not changing.

I am rotating about the Z axis. BZ prime would be same as BZ. Now let us calculate AX prime BX prime + AY prime BY prime + AZ prime BZ prime and this would come out to be equal to AX cosine theta + AY sine theta times BX cosine theta + BY sine theta + - AX sine theta + AY cosine theta times - BX sine theta was BY cosine theta + capital AZ times BZ.

This gives you, AX times BX cosine square theta + AY times BY sine square theta. The 1st gives you AXBX cosine square theta + AXBY cosine theta sine theta + AYBX cosine theta sine theta + AYBY sine square theta + AXBX cosine square theta - AXBY cosine theta sine theta - AYBX cosine theta sine theta + AYBY cosine square theta + AZBZ. Now you see, this term cancels with this, this term cancels with this and AXBX cosine square theta + AXBX sine square theta gives me AXBX.

Similarly AYBY sine square theta + AYBY cosine square theta gives me AYBY + last term is AZBZ. So what I showed you just now by transforming the different components that AX prime

BX prime + AY prime BY prime + AZ prime BZ prime even under transformation remains the same as AXBX + AYBY + AZBZ and therefore this is a scalar quantity.

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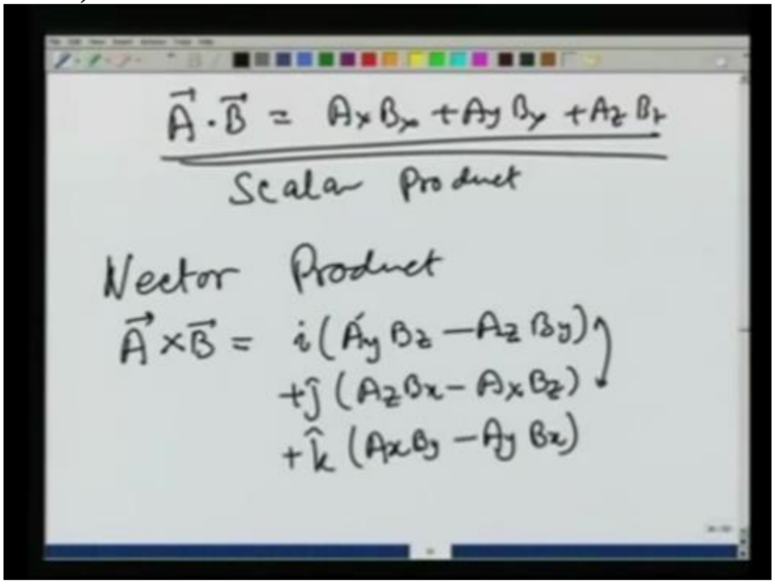

And that is precisely why we called take this particular, AXBX + AYBY + AZBZ and call this a scalar product. Similarly, the other quantity that we defined is the vector product in which case A cross B is given as I BX component is given as AY BZ - AZ BY + the Y components is given as AZ BX - AX BZ + the Z component is given as AXBY - AYBX.

What happens under transformation? Under transformation, if I transform each component, AX, AY, BY, the way we did earlier, these components should also mix according to the rule of vector transformation.

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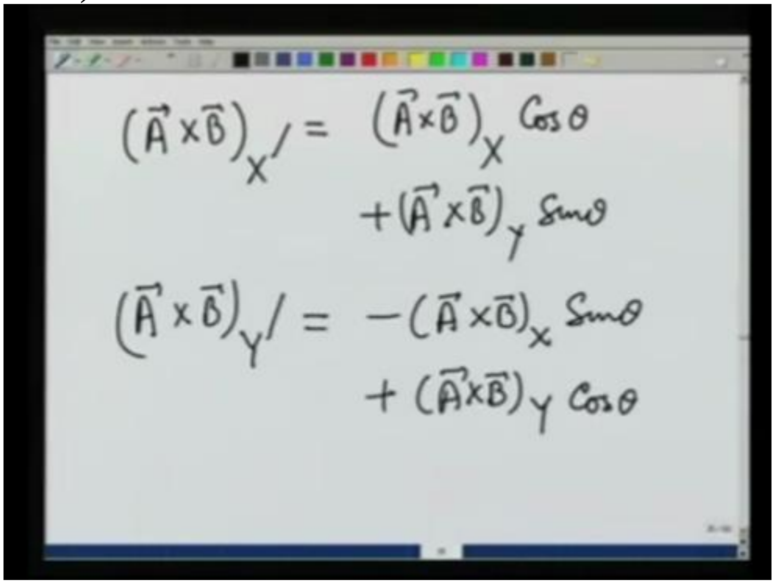

And I leave it as an exercise to show that A cross B, if I take its components in X prime direction, it is equal to A cross B. X component cosine of theta + A cross B the Y component sine of theta. Similarly A cross BY prime component if I am rotating, if I am only changing X prime and y prime axis by rotation about the Z axis should be good to - A cross B the X component of sine theta + A cross B Y component cosine of theta I leave this as an exercise war you.

So it is this combination, AXBY - BYAX and things like those that defines the vector product for us.

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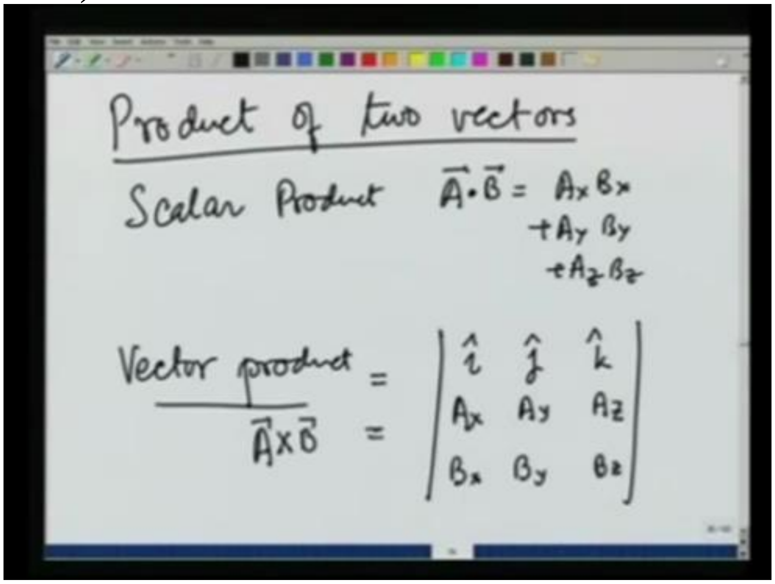

So product of 2 vectors. One is a scalar product which is A written as A dot B which is nothing but AXBX + AYBY + AZBZ and the other is a vector product which gives me a vector quantity. In a short-term, I can write as a determinant I, J, K, AX, AY, AZ, BX, BY, BZ, a determinant of this. This is A cross B. I will leave it for you to check that this gives the same expression as we used earlier.