Engineering Mechanics
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Module 6
Lecture No 48
Motion with drag-solved examples

In the previous lecture, we saw how to solids when they come in contact, apply frictional force on each other and how that affects their motion. In this lecture, we are going to look at another kind of frictional force when a body moves through a fluid or gas.

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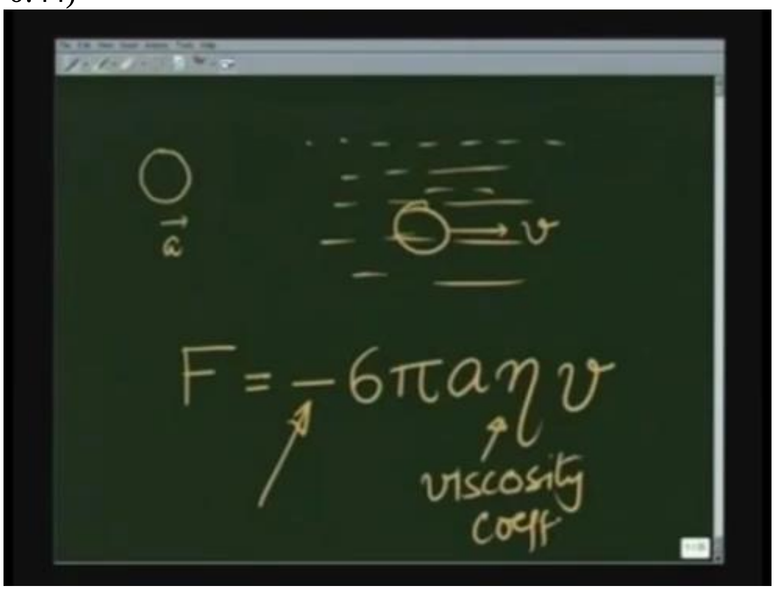

You are well similar from stoke's law that when a spherical body of radius A most through a gas or fluid at speed V, it is moving with speed V then there is a force on it due to viscosity which is F equals - 6 pi A eeta V. This - sign denotes that the force is opposite to its direction of motion. eeta is the viscosity coefficient.

This is one example of the drag force that any body moving through a fluid or gas experiences. Of course, this is the lowest order force.

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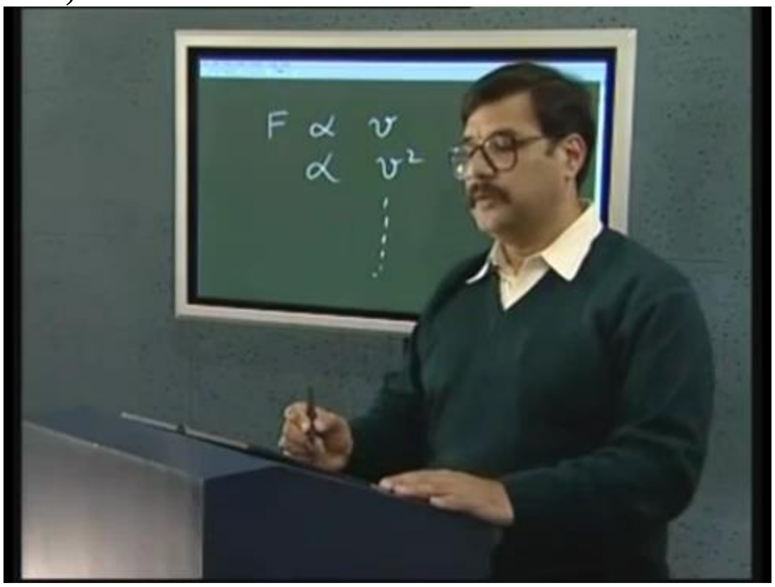

We may have forces which are proportional to V, which are proportional to V square or higher power. These forces oppose the motion and is this lecture we are going to look at how to deal with these forces when they are included in the motion of a particle.

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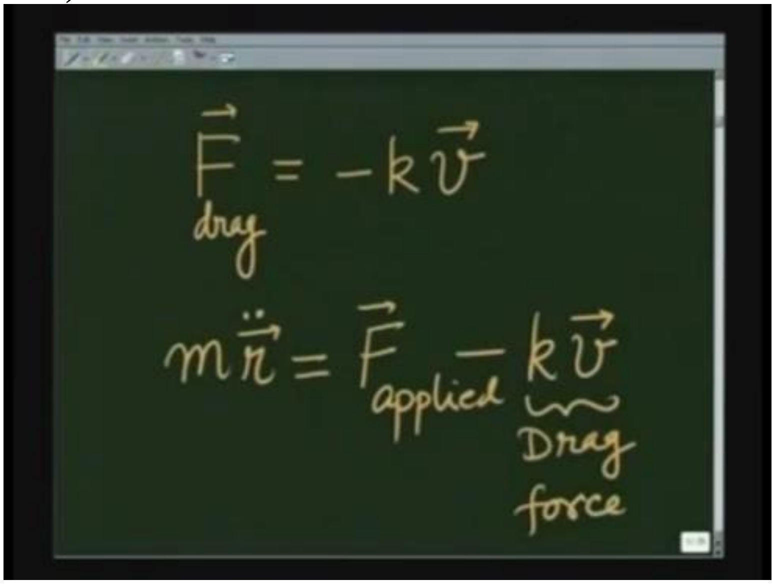

The simplest one as I said if the force which is proportional to V and I am going to write this as F equals - KV and I am going to call it drag force. K is the coefficient of drive. - sign again denotes that the force is opposite to the motion. And therefore, the equation of motion is going to be MR double dot mass times acceleration equals F applied that the force that I apply from outside - KV which is the drag force.

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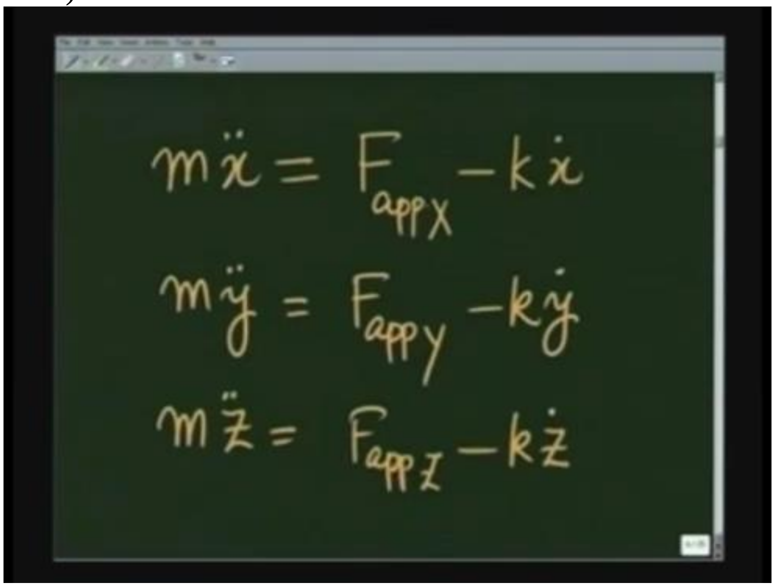

If I write these in components form, I get MX double dot equals F applied X component - KX dot M Y double dot equals F applied Y component - K Y dot. And similarly MZ double dot equals F applied Z - KZ double dot. This is the simplest of drag forces and in the next few examples, we are going to see how to incorporate this in the motion.

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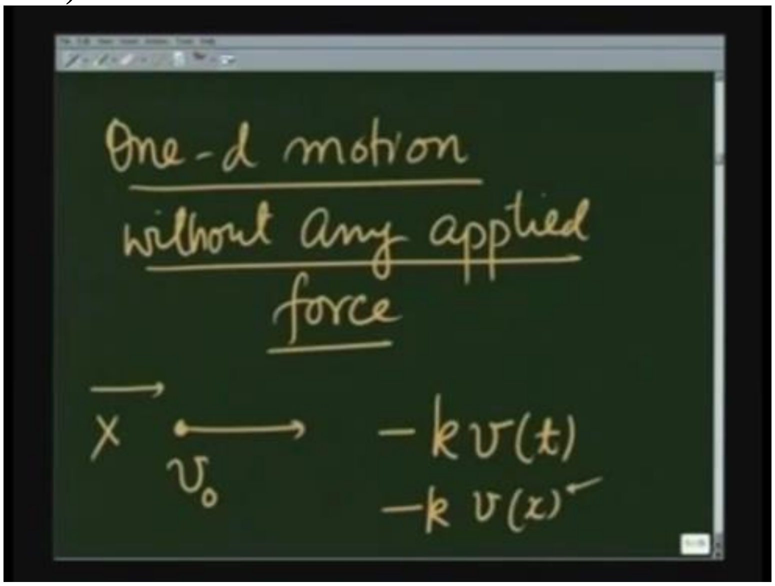

The 1st example I am going to take is going to be one-dimensional motion without any applied force. There is no applied force but drag is there. Therefore initially I have to throw the particle with some initial speed V0 and as it moves along, it experiences a force which is - KV. V is changing with time or I can also write V is changing with the distance. Let this direction be X direction. That is why I have written X here.

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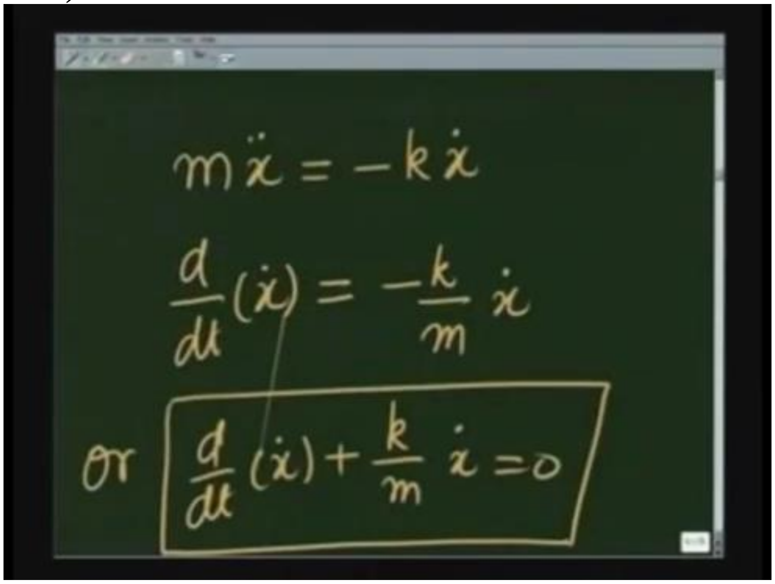

The equation of motion for this is going to be MX double dot equals - KX dot. In more familiar terms, it is going to be D over DT of X dot which is nothing but X double dot, is equal to - K over MX dot or D over DT of X dot + K over MX dot equal to 0. This is my equation of motion that I want to solve for this particle.

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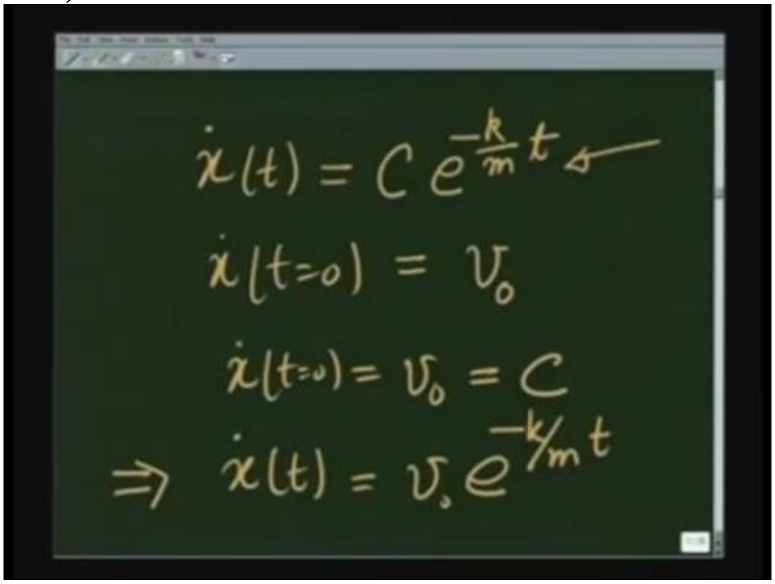

You can check by direct substitution that the solution is going to be X dot as a function of time equal to some constant C E raised to - K over MT. What is C? Let us write MX dot at T is equal to 0 that is initial time through the particle with speed V0. And therefore I must have X dot at T equal to 0 is equal to V0 and when I substitute T equal to 0 here, I get this is equal to C. And therefore my general solution is going to the X dot, that is the velocity of the particle equal to initial speed, initial velocity - K over MT.

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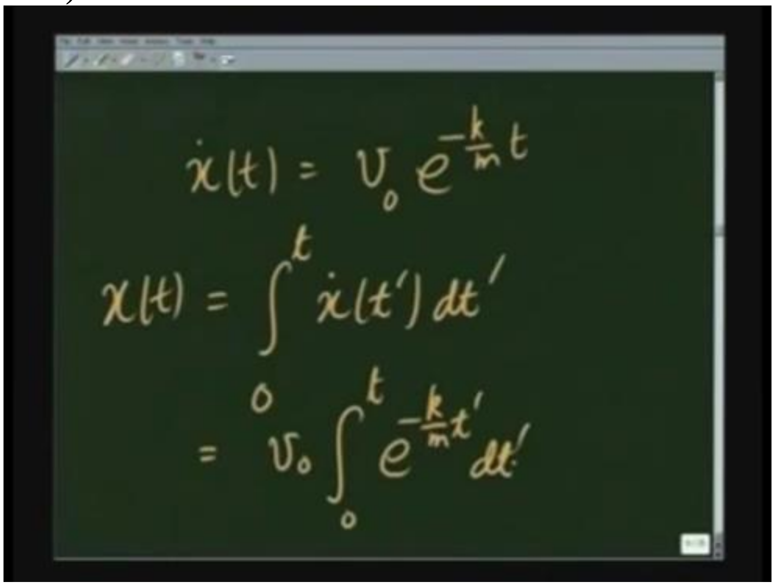

Let us plot it and see how does it look. So initially the particle started moving with some speed V0. I am plotting VT vs T and as time passed, the velocity decreases exponentially. This curve is V0 E raised to - K over MT. How about the distance travelled by the particle? That is also easy to calculate once I know the velocity. X dot T is given to be V0 E raised to - K over MT. So the distance travelled in time T is going to be integral 0 to T X dot T prime DT prime which is equal to V0 integral 0 to T E raised to - K over MT prime DT prime.

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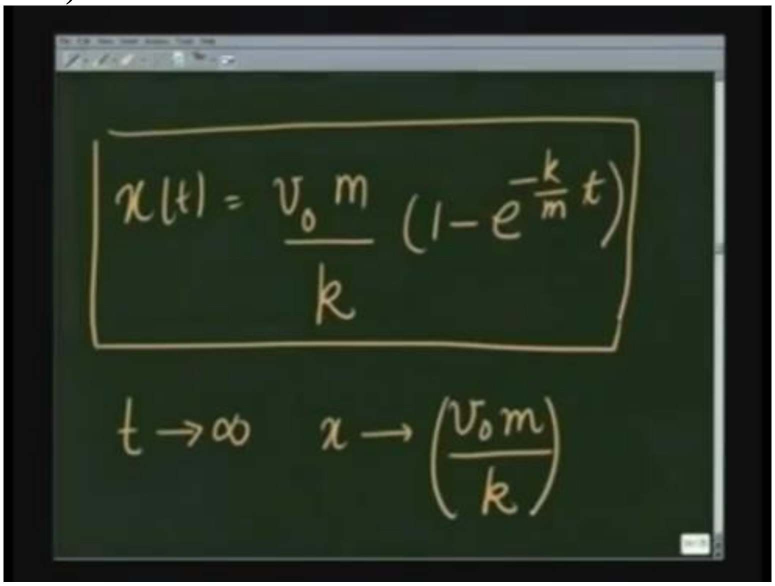

After integration, I get XT is equal to V0M over K1 - E raised to - K over MT. So I have also calculated how much is the distance in time T. As T goes to infinity, X goes to V0M over K. So that is the distance that the particle finally travels and stops after this.

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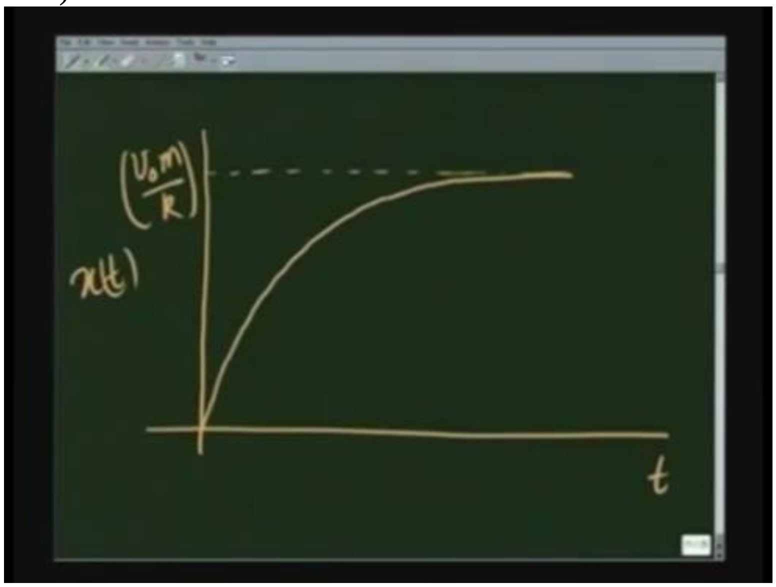
e 

Let us see how does the plot of this look. If I plot XT as a function of time, initially obviously distance travelled is 0 and slowly it moves up and goes like this. This distance being V0 and over K.

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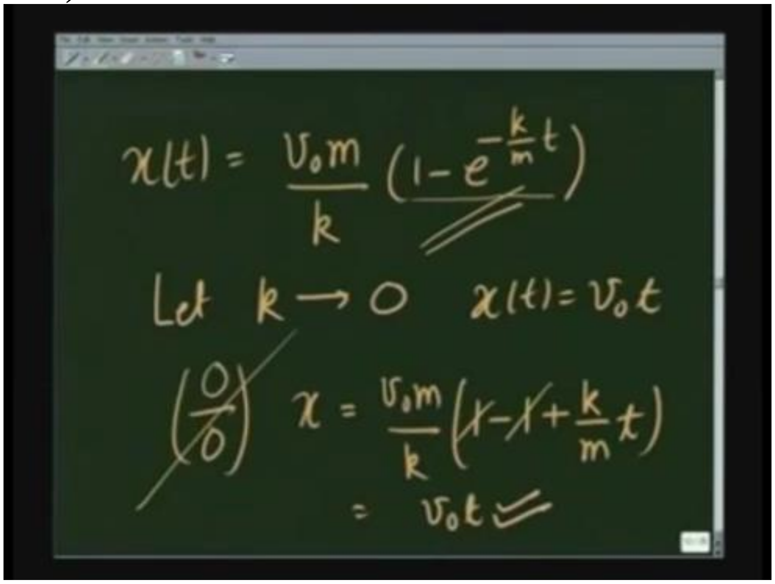

So let us look at this expression once more. It is V0M over K1 - E raised to - K over MT and ask if this is the correct answer. One way to check this is, let K go to 0 and I should get my familiar XT equals V0T. But if I substitute K directly here, I get an answer like 0 over 0 which is not correct. I have to be more careful and take the limit K going to 0 keeping terms up to order K in this expression and see if I get the correct answer.

If I do that, I find X is equal to V0M over K1 - 1 + K over MT. This cancels and I get V0T which is correct. So this is one way, when you get an expression in these, when you apply these forces you can check your answer.

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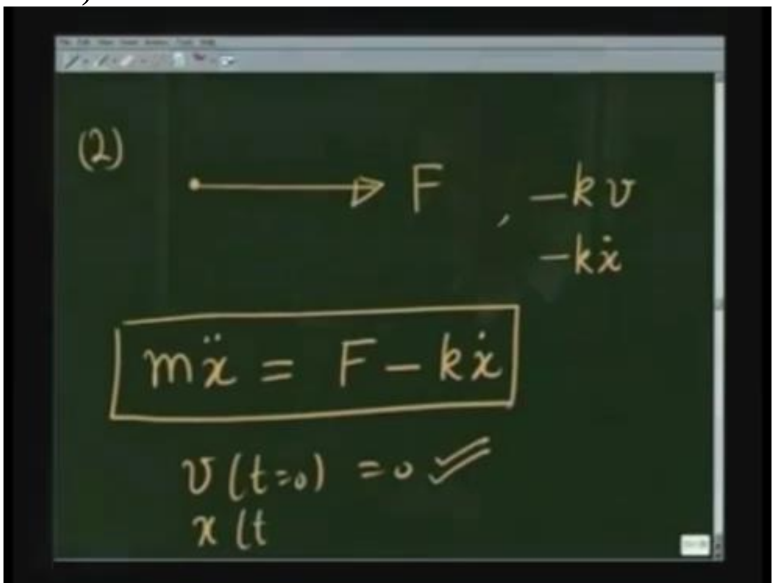

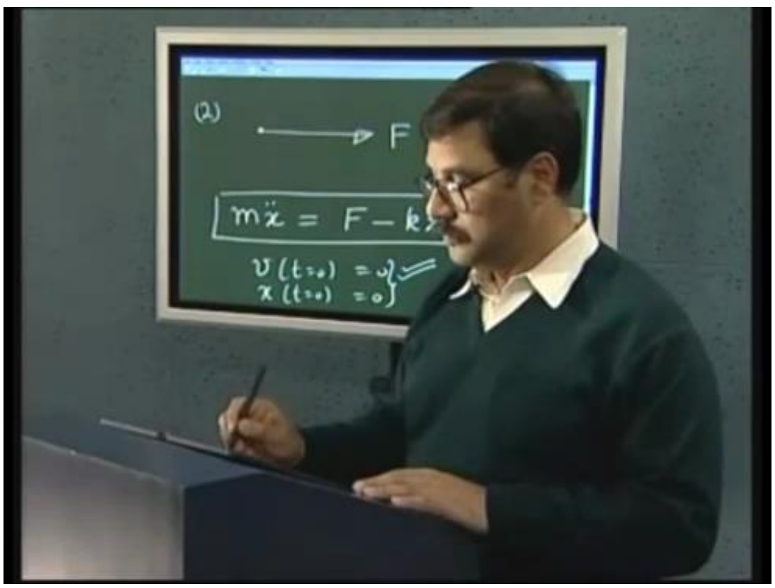

Next, let us consider a particle still moving in one dimension but applied on it is a constant force. So I have a particle. I apply a constant force F again in X direction and let there be a drag force, - KV 4 - KX dot. And I want to solve how, I want to see how this particle moves. The equation of motion is going to be MX double dot is equal to F - KX dot. In this case, notice that I do not really have to start by shooting a particle with some initial speed.

I can start with VT equal to 0 still 0. Because I am applying a force, the particle will start moving and this is what I want to see how the particle moves. So this is going to be one of my initial conditions. The other one is going to be let us also say that XT equal to 0, it starts from the origin.

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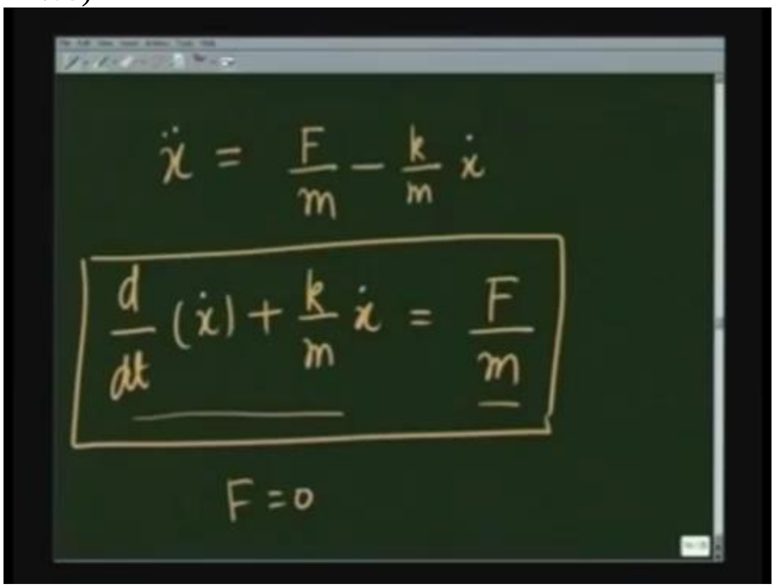

If I rewrite the equation, it will come out in the form of X double dot is equal to F over M - K over MX dot or D over DT of X dot + K over M of X dot is equal to F over M. Notice, I have already solved this problem for F equals 0 that was one-dimensional motion without any force.

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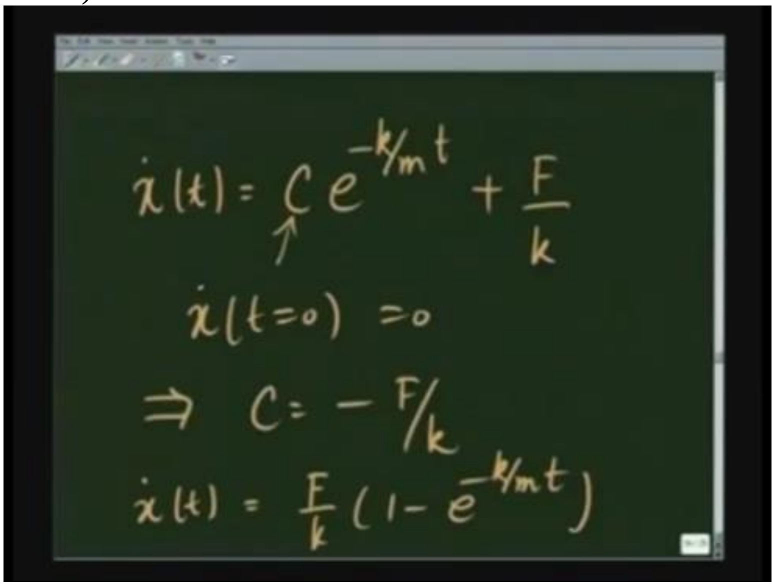

In such situations where I have one part like this and another part like this, I can write my solution as the previously obtained solution is F equals 0 which was nothing but C E raised to - K over MT. This was X dot of T + a particular solution that corresponds to the term on the right-hand side. Remember I am solving the equation, D over DT X dot + K over MX dot is equal to F over M.

This is the solution corresponding to this part equal to 0. The particular solution would refer to this term. And you can see, the particular solution is going to be X dot is equal to F over K. I substitute this here and I get on the right-hand side, F over M. And therefore my general solution for X dot is going to be X dot of T is equal to C E raised to - K over MT + F over K.

This constant C is again determined by initial condition which is nothing but initially, the particle was at rest and therefore I get C equals - F over K. And the solution with this initial condition comes out to be X dot T equals F over K 1 - E raised to - K over MT. Let us now look at the solution carefully

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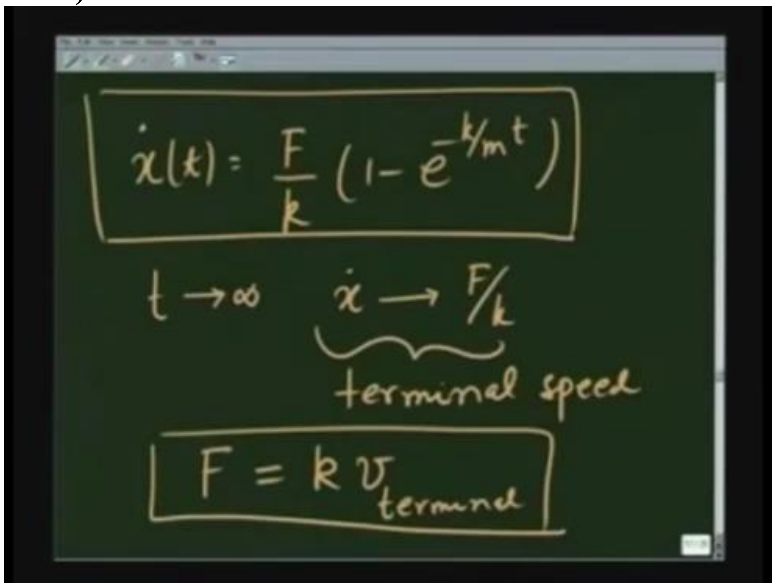
e 

So I have X dot T equals F over K1 - E raised to - K over MT. You notice, as T goes to infinity that is after a long time, I have applied the force, X dot goes to F over K. This is nothing but the terminal speed. At this speed, the applied force F becomes equal to K times V terminal. The two forces balance and therefore the particle moves with a constant speed of V terminal.

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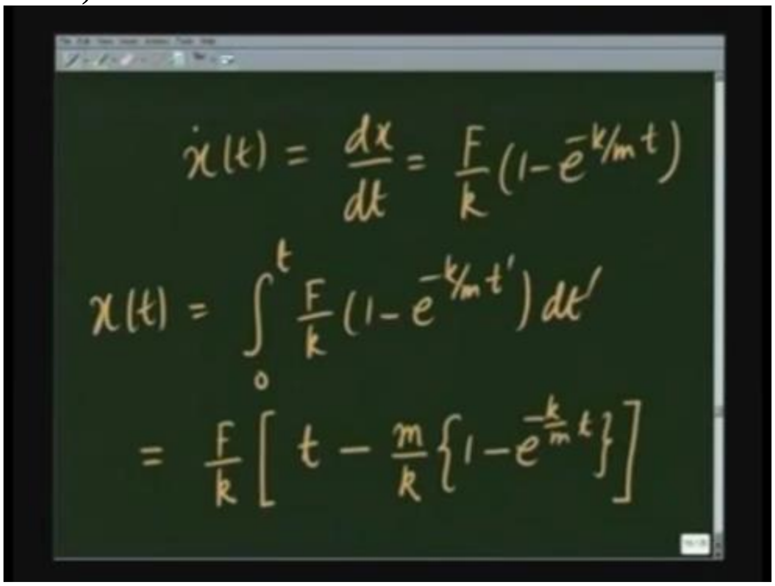

If I plot this speed, I am going to get VT or X dot T same thing vs T. The final speed is F over K and the speed slowly builds up. How about the distance travelled? Well for that I again in the great it. So I have X dot T equals DXDT is equal to F over K 1 - E raised to - K over MT. So X at T D is again going to be 0 to TF over K1 - E raised to - K over MT prime DT prime. Integrating, I get F over K T - M over K 1 - E raised to - K over MT. So this is going to be the distance travelled by the particle.

Now that we have gotten the expression for the distance travelled, let us play around with it a bit and see if it goes to correctly mix.

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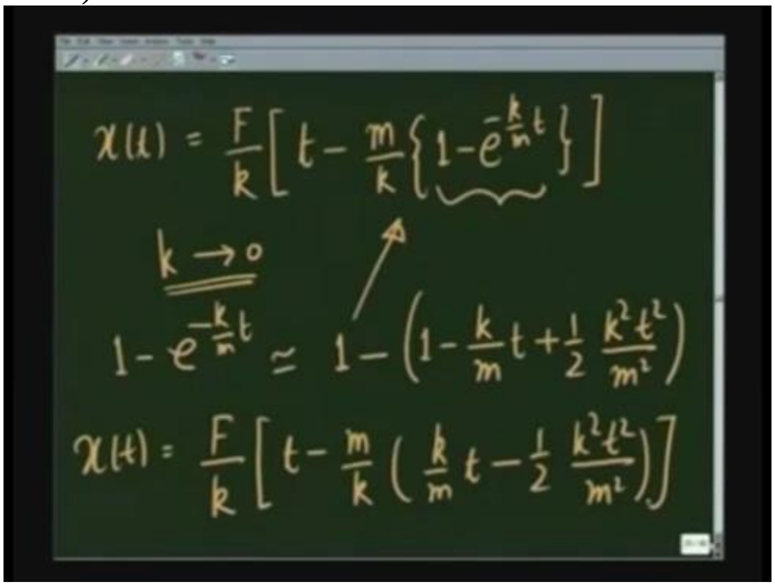

So I have XT equals F over K T - M over K1 - E raised to - K over MT. So 1st question I ask is what happens if K goes to 0? Remember, I said earlier, you have to be careful in taking limits. So I am going to take, expand this term, keep terms up to order K and see what happens. In the limit of K going to 0, I can write 1 - E raised to - K over MT if K is really small for any T, I can write this approximately as 1 - 1 - K over MT + half K square T square over M square.

Substituting this in here, I get XT is equal to F over KT - M over K K over MT - one half K square T square over M square.

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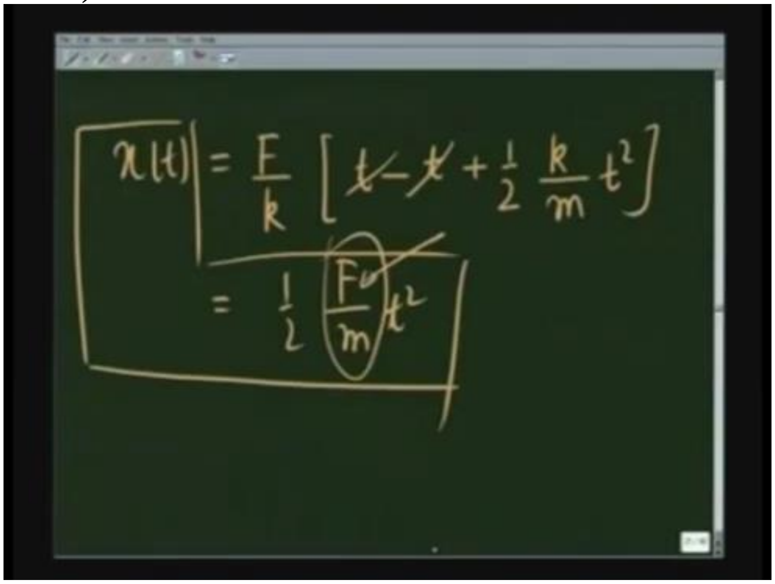

And expanding it further, I get XT equals F over K T - T + one half K over MT square which is nothing but you are familiar one half F over MT square which is the correct answer in the limit of K going to 0. That is I am taking a particle and applying a constant force F, its acceleration is F over M and then it moves distance one half F over MT square in time T. Let us try to plot this and see how it goes.

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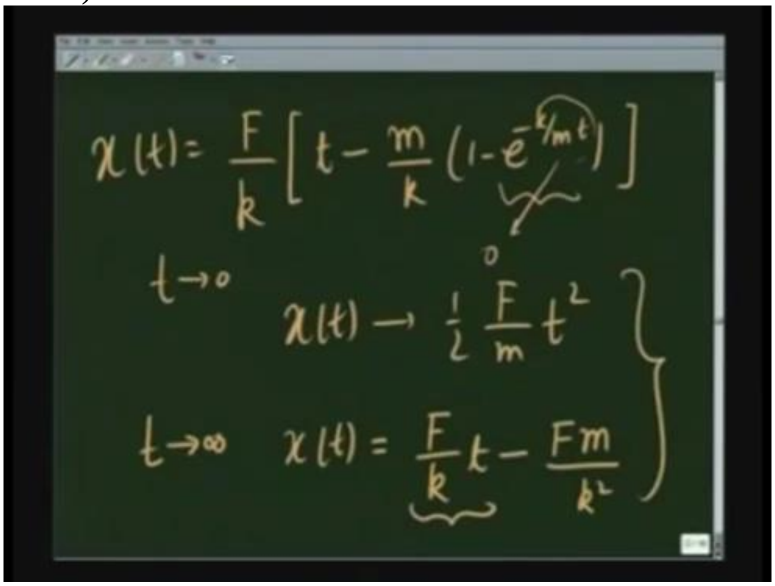

So XT was F over KT - M over K 1 - E raised to - K over MT. Again in the limit of very small T basically making this term very small, we just saw that XT goes as one half F over M T square. How about large T? In the large T limit, this term would tend to 0 and therefore I would have XT as F over KT - FM over K square.

This is nothing but movement with a constant speed. F over K, you recall from earlier is nothing but the terminal speed. So the particle has now after a long time started moving with terminal speed F over K.

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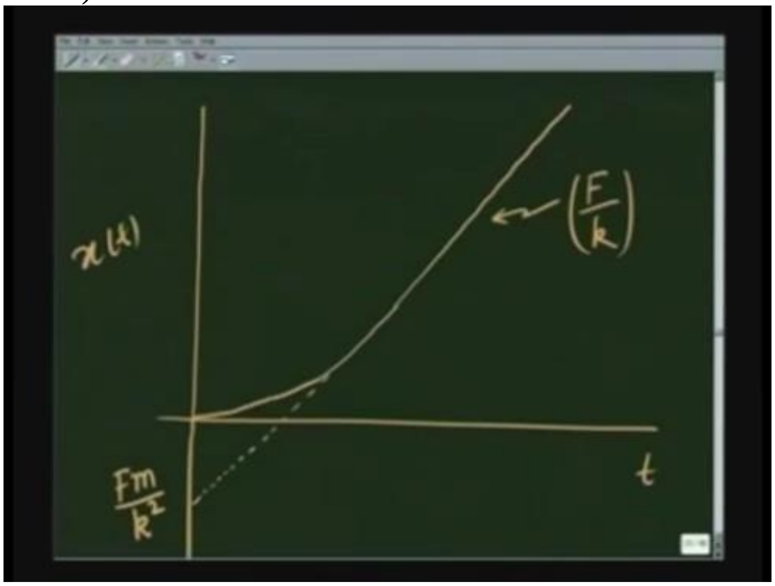

So therefore if I combine the 2 and plot XT, it is going to look like T, XT initially it is going as T square and finally it goes as a straight line. A straight line with slope F over K which is nothing but the terminal speed and which cuts the Y axis somewhere here which is nothing but FM over K square. So this is how the particle is going to cover the distance with time. This is a problem where we applied a constant force.

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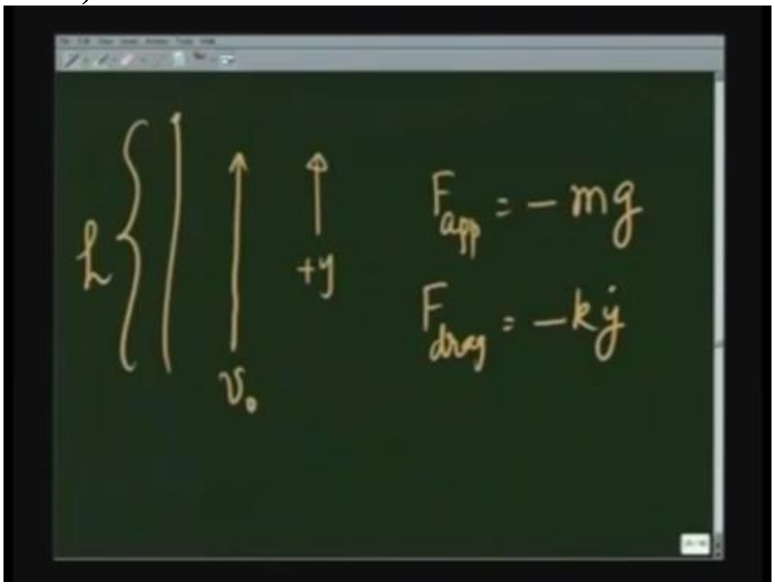

So the most familiar problem in everyday life of this is a ball being thrown up with say some initial speed V0. The problem with that I just solved. I took the initial speed to be 0. But suppose I throw a ball up then the force and to take this direction to be + Y then the force applied is going to be - MG, drag force is going to be - KY dot and the problem becomes similar to what I solved.

I leave this as an exercise to find how long does it take for the particle to reach up? How high does it go? Because I am going to solve slightly more complicated problem which is related to this and that is going to be the motion in two dimension, the projectile motion in gravitational field. So I am done with a one-dimensional motion now. Let us go to two-dimensional motion with drag force included.

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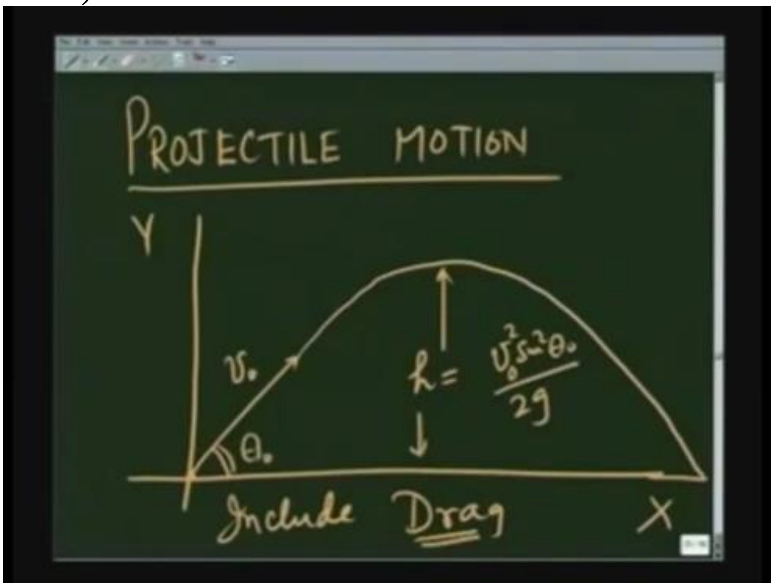

You are familiar with projectile motion which is in the vertical plane. In this motion, I take a particle and project it at some angle theta 0 from the horizontal with some initial speed V0 and without drag it follows a perfect parabola and this problem you have solved in your 12th grade. Your height comes out to be V0 square sine square theta 0 over 2G. You can calculate the time, the range and everything. What we are going to do now is make the problem slightly more complicated by including drag force in it in its simplest form.

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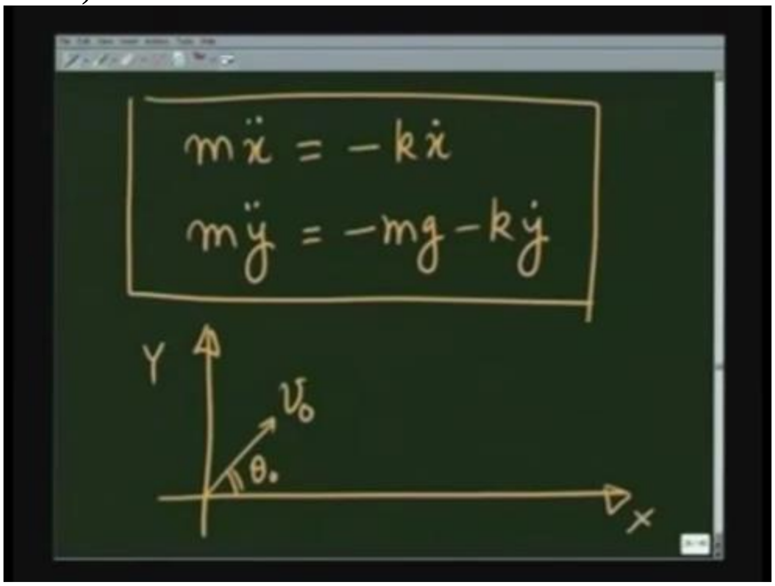

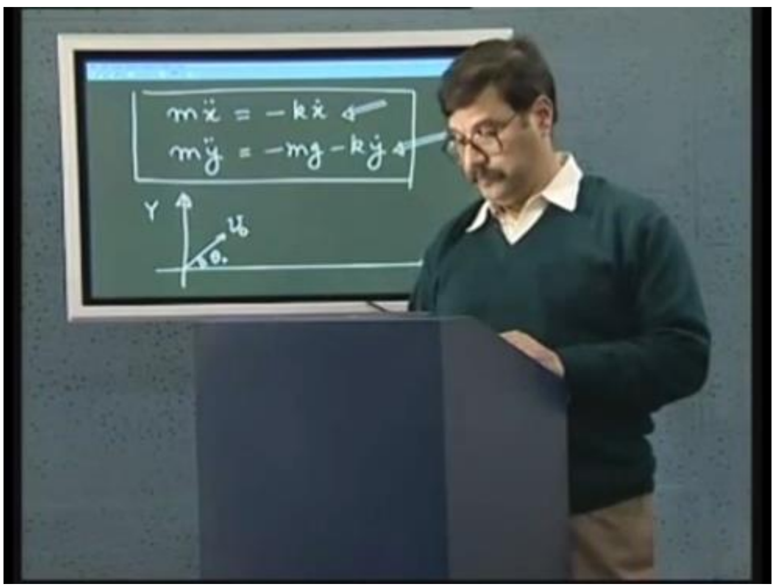

Now therefore my equation of motion for the projectile is going to be MX double dot is equal to - KX dot because there is no horizontal applied force. And in the Y direction, I am going to have MY double dot is equal to - MG - KY dot. These are my 2 equations of motion when I am solving for a particle which is being thrown at an angle theta 0 with initial speed V0.

This I am taking to be positive Y direction, this to be X direction. These 2 I have solved separately when I was solving the 1D problem so that I know the solution.

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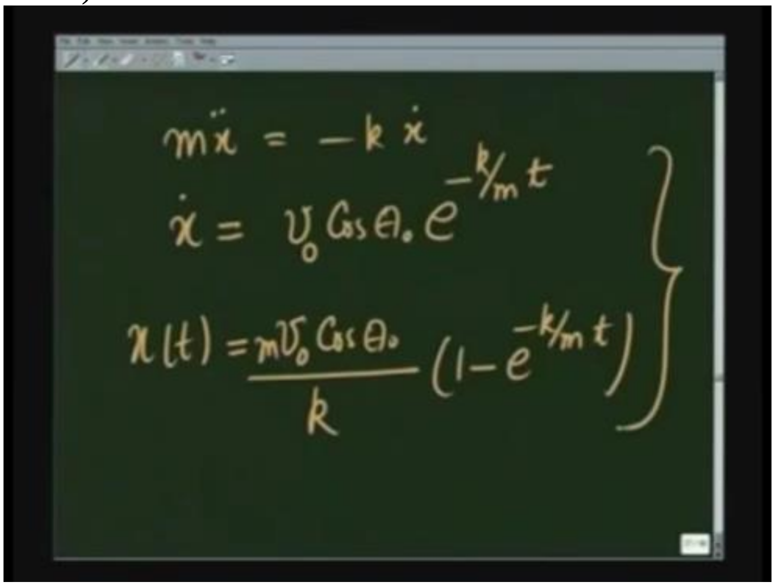

MX double dot equals - KX dot gives me X dot is equal to V initial bids in this case happens to be V0 cosine of theta 0 E raised to - K over MT and also X as a function of time comes out to be V0 cosine theta 0 M over K1 - E raised to - K over MT. This is the motion in X direction and listen going on concurrently with motion in Y direction.

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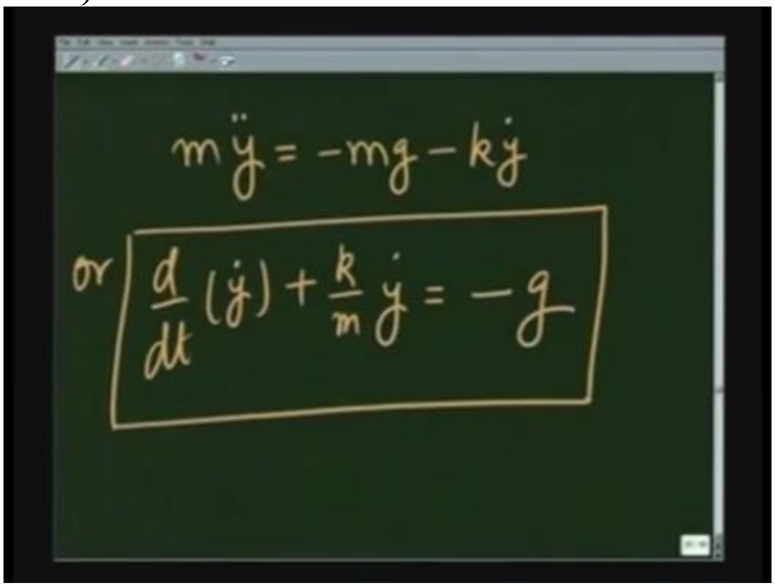

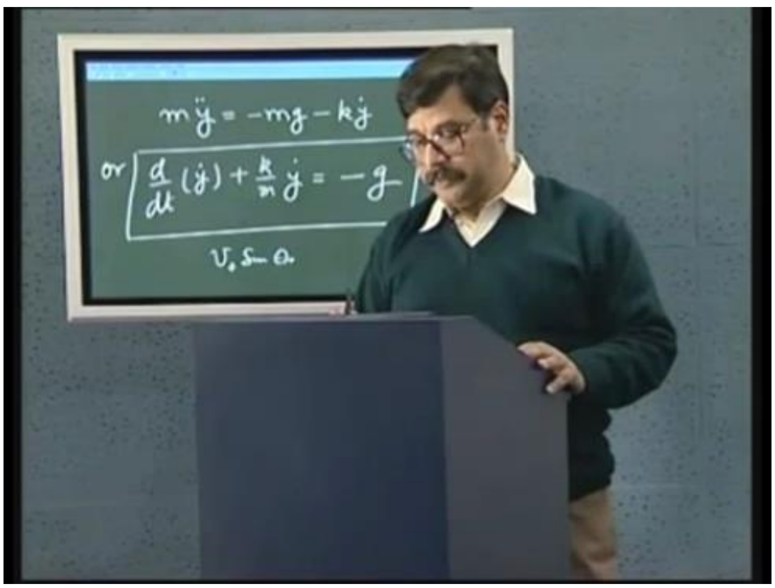

For which I have M Y double dot is equal to - MG - KY dot or D over DT of Y dot + K over MY dot is equal to - G. In our one-dimensional example, we have already solved this with slightly different parameters. Instead of this - G, I had an F there, initial velocity was 0. Here, initial velocity is V0 sine theta 0. So let us solve this for back.

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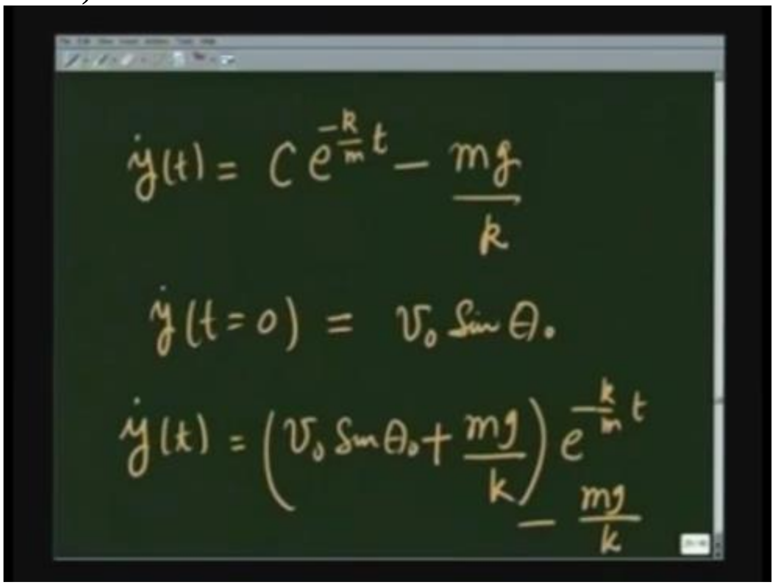

As I said earlier in the case of one dimension, the general solution is going to be summation of the particular solution and the solution of the homogeneous part and therefore Y dot as a function of T is going to look like C E raised to - K over MT - MG over K. And C is determined by initial condition which is that Y dot at T is equal to 0 is equal to V0 sine of theta 0. And therefore I have Y dot of T is equal to V0 sine of theta 0 + MG over K E raised to - K over MT - MG over K.

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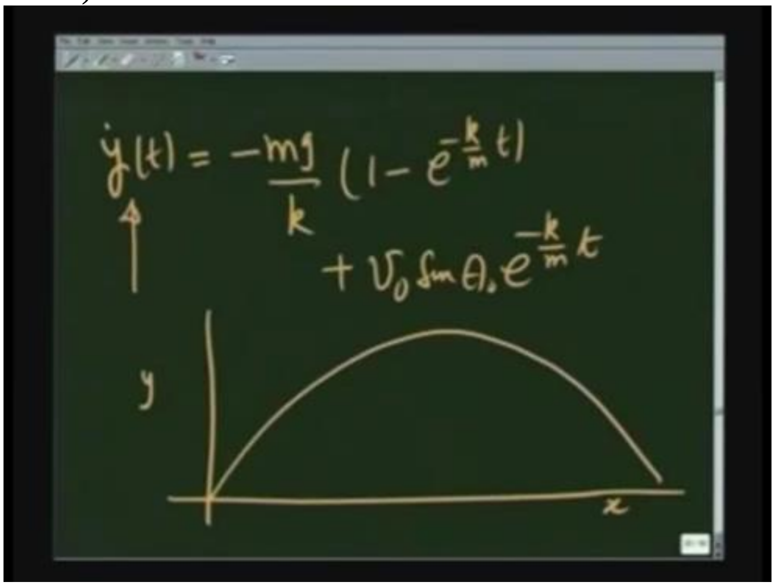

Or I can also write this as Y dot of T is equal to - MG over K1 - E raised to - K over MT + V0 sine theta 0 E raised to - K over MT. This is how the velocity of particles in Y direction varies with time. If you want to find the trajectory of the particle, that is I want to find how does it go in the space? Then I have to solve for X and Y and project Y vs X, one vs the other. However before we find the trajectory, let us play around with the expressions for the velocities that we have already obtained.

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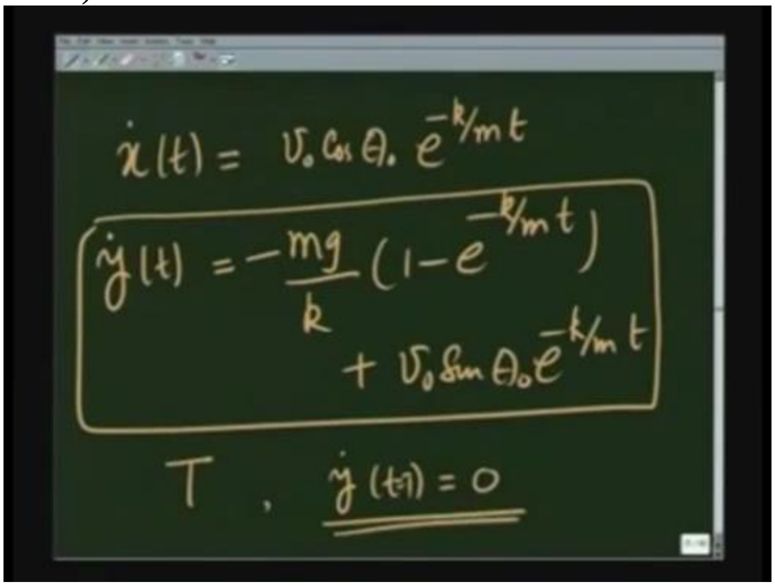

X dot T is V0 cosine of data 0 E raised to - K over MT and we have just found that Y dot T is equal to - MG over K1 - E raised to - K over MT + V0 sine theta 0 E raised to - K over MT. At this point, I may ask how long does it take, time T before Y dot becomes 0? That is, after I throw the particle, after a project the particle, when does it stop or when does it reach its maximum height?

So for that I substitute this is equal to 0 and corresponding time would get me the answer. Let us do that.

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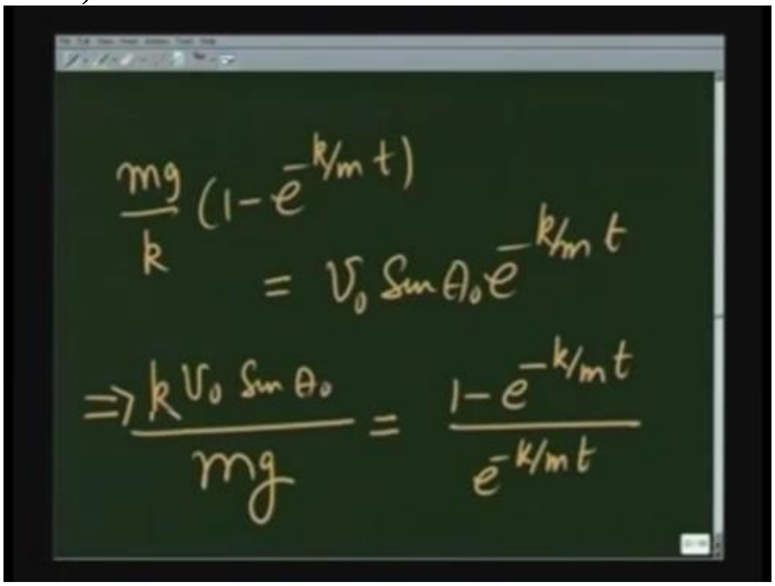

If I do that, I get MG over K1 - E raised to - K over MT is equal to V0 sine theta 0 E raised to - K over MT and this gives me V0 sine theta 0 K over MG is equal to 1 - E raised to - K over MT over E raised to - K over MT.

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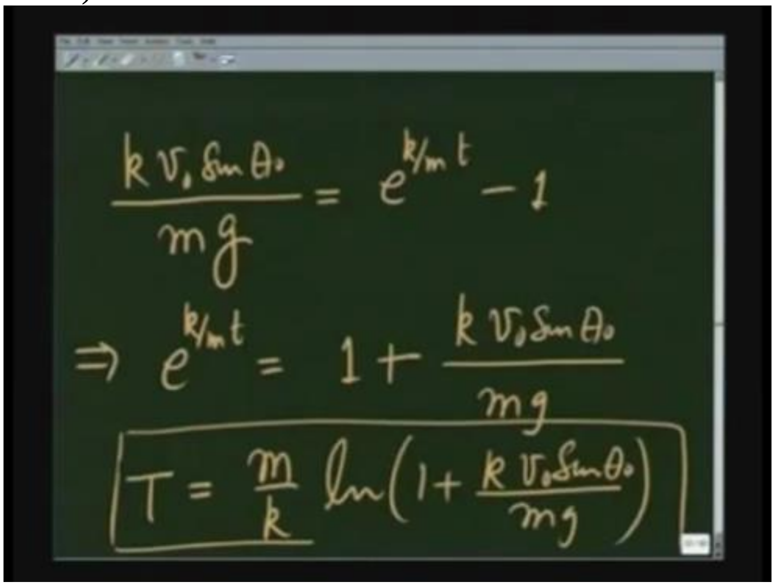

Or V0 sine theta 0 over MG is equal to K over MT - 1 which implies E raised to K over MT is equal to 1 + K V0 sine theta 0 over MG or time T when its velocity becomes 0 reaches its highest point is equal to M over K log of 1 + K V0 sine theta 0 over MG. Let me again ask and check my answer. Is this correct in the limit of K going to 0? I am sure it is.

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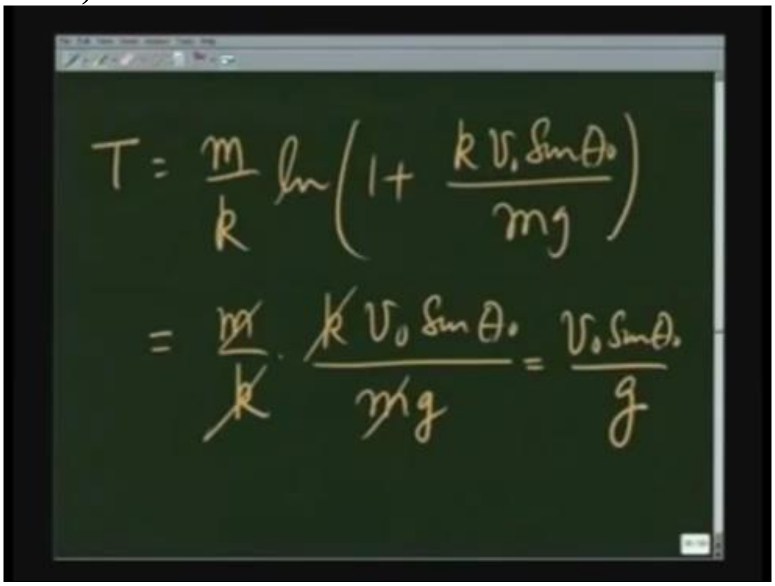

Because T is equal to over K log of 1 + K V0 sine theta 0 over MG in the limit of very small K can be written as M over K. Expanding log, I get V0 sine of theta 0 over MG. This cancels, this cancels and I get my familiar answer, V0 sine theta 0 over G. After doing this exercise, let us go on to find the trajectory.

For trajectory, what I will do is find X and Y separately and plot them. Recall that without any drag force, one could eliminate T and find X and Y, a direct relationship between X and Y. This is not possible in this case.

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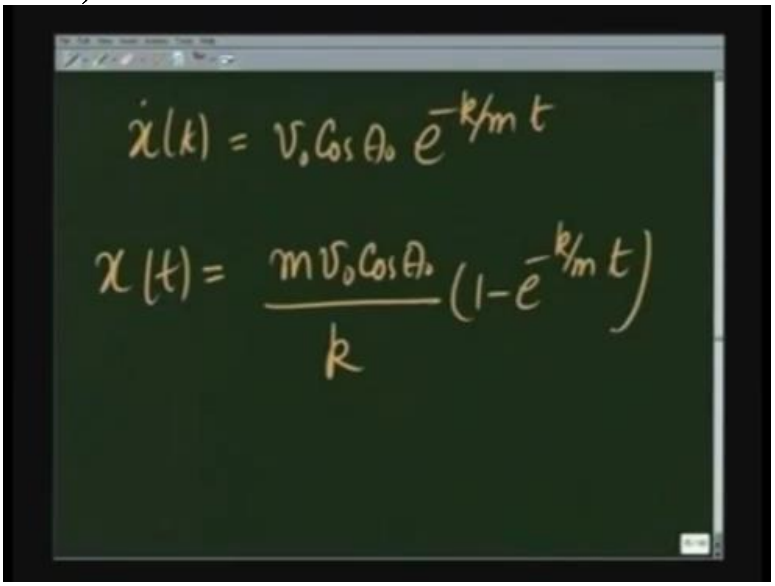

So going back to our expressions, we had X dot T is equal to V not cosine of theta E raised to - K over MT which gives me XT is equal to M V0 cosine of theta 0 over K1 - E raised to - K over MT. This is my expression for X. You are also familiar with this from the earlier part of the lecture.

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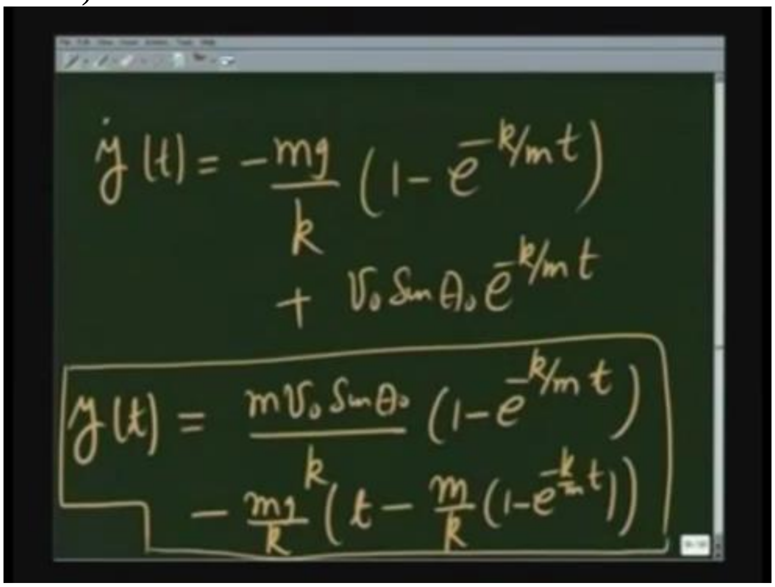

Similarly, I have Y dot T which is nothing but - MG over K1 - E raised to - K over MT + V0 sine theta 0 E raised to - K over MT. Integrating this, I get an expression for Y as a function of time and it comes out to be M V0 sine theta 0 over K1 - E raised to - K over MT - MG over KT - M over K1 - E raised to - K over MT.

You see because T appears in linear form as well as this form, that is why I cannot eliminate it to get a direct expression between X and Y. However, for a given time, I can calculate X and Y and plot the trajectory.

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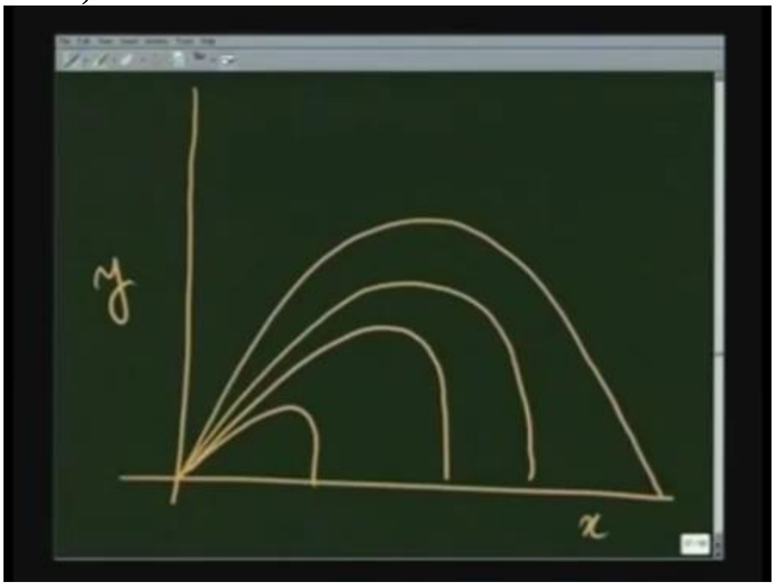

If I do that for different case and a fixed V not and sine theta not, I find that for K equal to 0, it is a perfect parabola. However, as K is introduced, that means the drag force comes into play. Its height and range, both start changing and I have got some numbers here which I will show you how for different parameters it changes.

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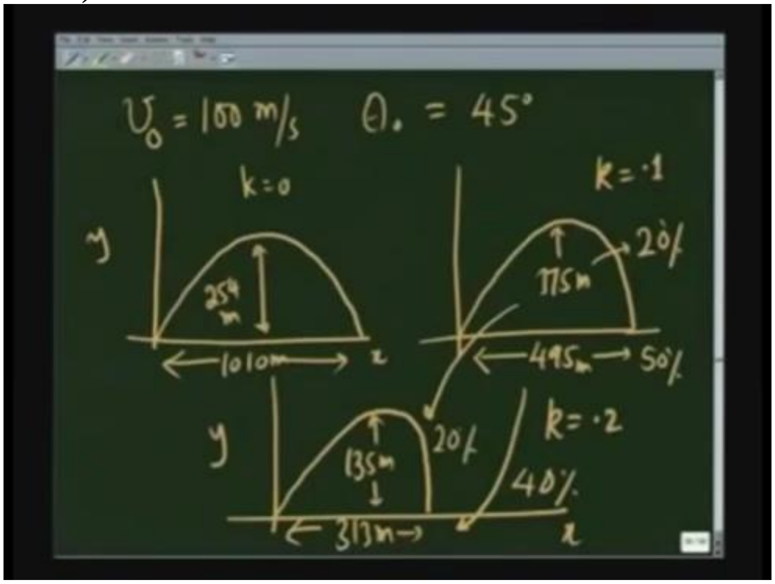

As an example of how the trajectories are affected for different values of the drag coefficient or by this drag, I have taken a sample problem of the particle being thrown with a speed of 100 metres per second at an angle theta 0 equal to 45 degrees and I will plot for the different case, the trajectories which I have calculated. I take K is equal to 0 which is the case of no drag or no viscosity.

In that case, you are well familiar that this comes out as a parabola. Its height is about 254 metres and the range comes to about 1 kilometres, 1010 metres. Now I take K to be 0.1, the trajectory changes slightly. It goes, it comes down but not as a parabola but something like this. Its height now reduces to 175 metres and range comes out to be 495 metres. Notice that the reduction in height is about 20 percent whereas the range has gone down by about 50 percent.

If I go further and make K equals 0.2, the trajectory becomes even more like this with height now becoming 135 metres and the range becoming 313 metres. I noticed that the reduction in range is much more which is about 40 percent whereas the height goes down by about 20 percent. So you see how introducing drag or this firctional force due to fluid or a gas affects the motion of a trajectory.

This is a very important problem when we consider a motion of particles or motion of say bombs or motion of artillery pieces being thrown out, being projected in a battlefield.

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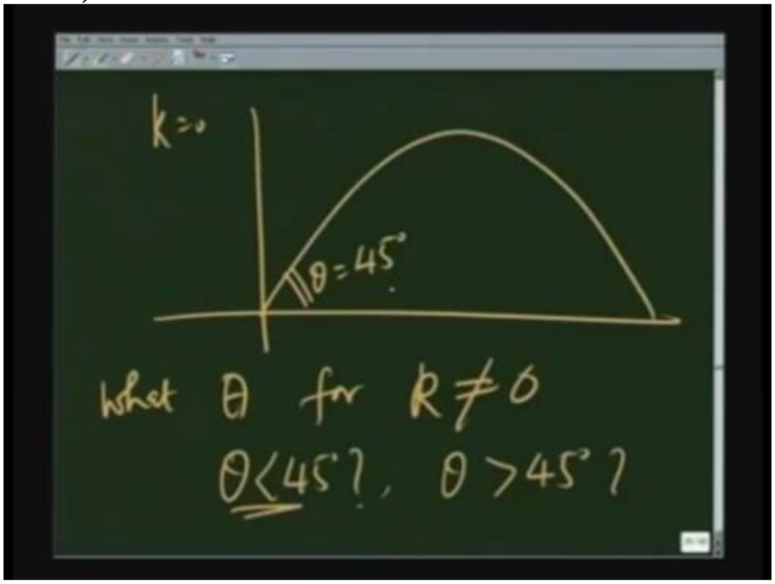

An interesting question you may ask at this point is, you all know from your 12th grade that if K is equal to 0 the maximum range which is attained is when theta is equal to 45 degrees. What theta for K not equal to 0? Is theta less than 45 degrees? Is theta greater than 45 degrees?

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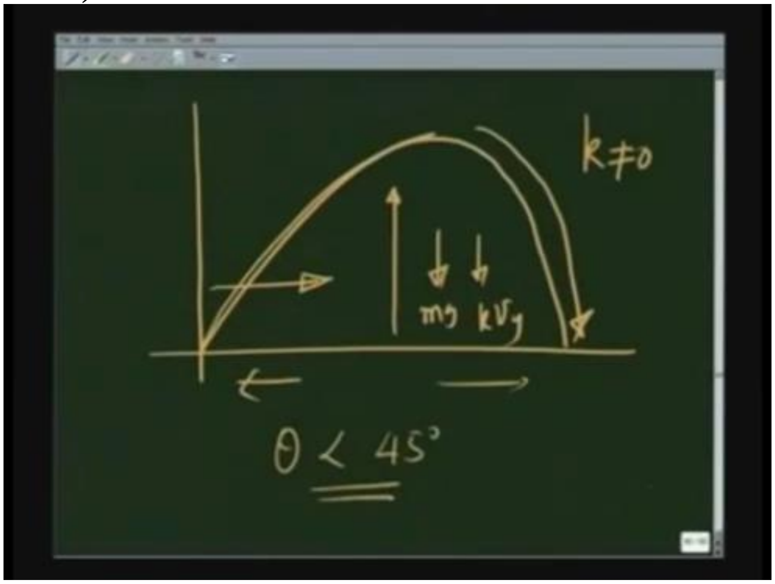

The answer to this question is that theta should be slightly less than 45 degrees because you saw that the height is not affected so much by introducing of K as the range. So what I want to do is, give slightly more of X component to the velocity if I want to fire the projectile farther and farther. The other way to look at the same problem is that when the particle is moving up, it spends more time because it is being pulled down by MG, it is also being pulled down by KVY.

Spends more time in going to the same height than when it is coming down. So at that time you want to cover as much distance as possible because X is X component of the velocity is also going down. So to attain maximum range, I should fire at theta less than 45 degrees. So far, I have considered the drag force which is proportional to the velocity and in the opposite direction and shown you through some examples, how it affects the motion. This is the simplest possible drag force.

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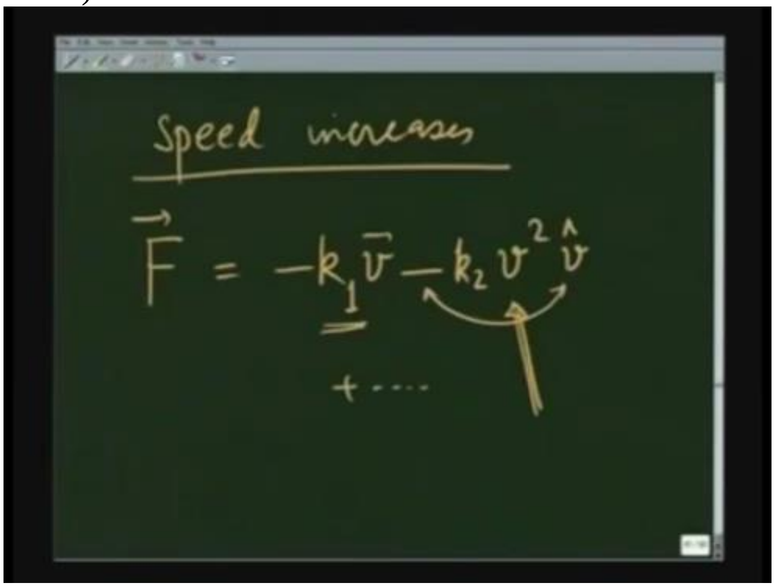

As the speed increases, more dependence on velocity may come in the form of F may be equal to - KV - let me write this K1, K2 V square let us write a unit vector V so that it tells you this is opposite to the velocity vector and higher-order terms. This being the simplest. When these terms come into play, problem becomes non-linear.

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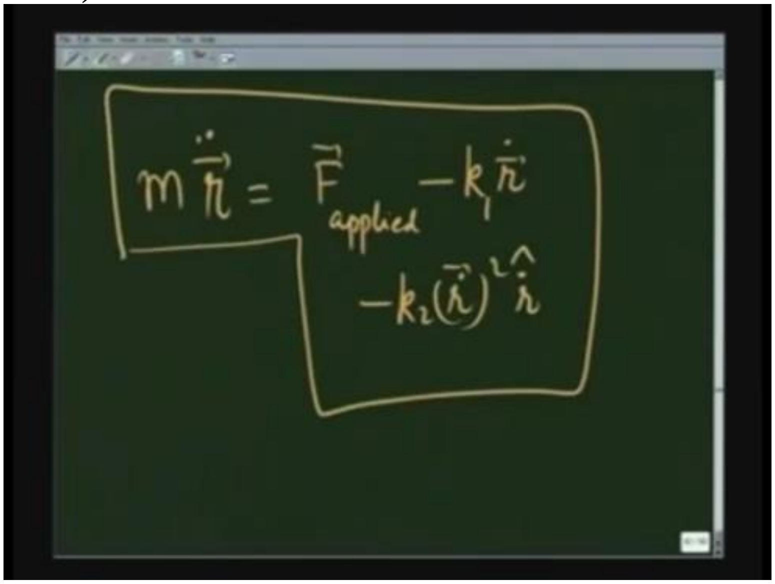

Because now you have MR double dot is equal to F applied - KR dot K1 R dot - K2 R dot square R dot vector. And these are not easy to solve. In most of the cases, you have to apply numerical methods to get the final solution. However, there are some cases where you can apply some techniques and get solutions. I will solve one example to illustrate this and that would complete our lecture on drag forces.

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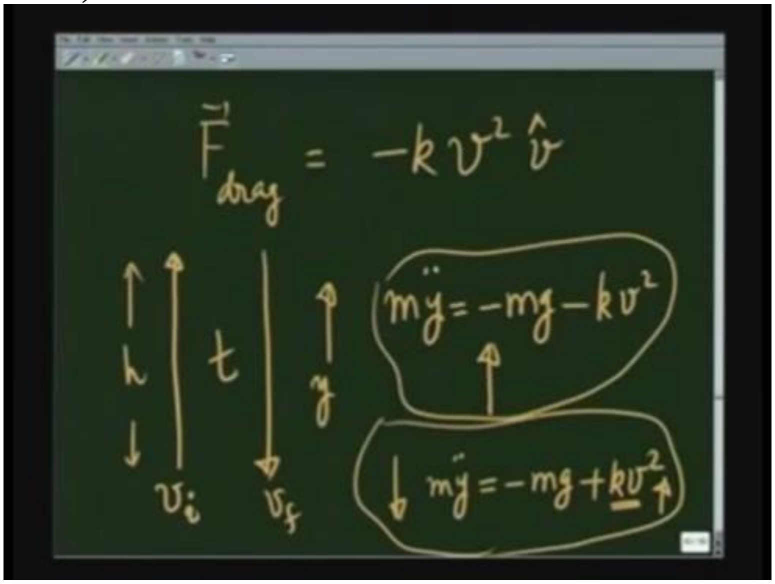

The example I take is that of F drag being proportional to V square and obviously it is in direction opposite to its velocity. And I want to find, if I throw a particle up with some initial speed, V initial, how long does it take to reach up? What is the height it can go up to and what is its speed when it comes down, V final? Let us see what all answers can we get.

If I write the equation of motion, since this is a one-dimensional case, let me take Y to be going up. I can write MY double dot is equal to - MG - KV square. This is when the particle is going up. Notice, in this case I have to be careful when considering the motion of particle coming down.

When the particle is coming down, I would have MY double dot is equal to - MG + KV square because in that case the force, drag force would be acting in upward direction. That is why I took this particular example. So let us see how do we go about solving this.

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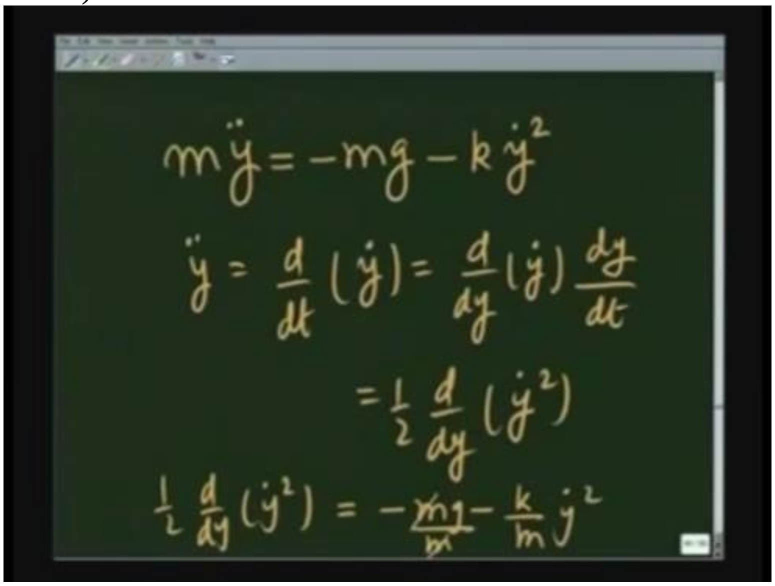

While the particle is moving up, I write MY double dot equals - MG - KY dot square and I go back to my technique of writing Y double dot as D over DT Y dot which is same as D over DY Y dot DYDT which is same as D over DY of Y dot square one half. And therefore, I can write my equation as one half D over DYY dot square is equal to - MG over M - K over MY dot square.

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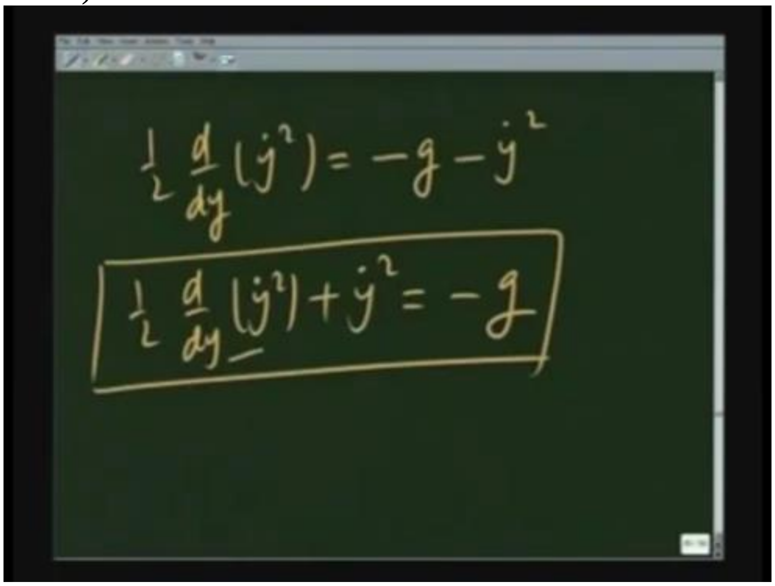

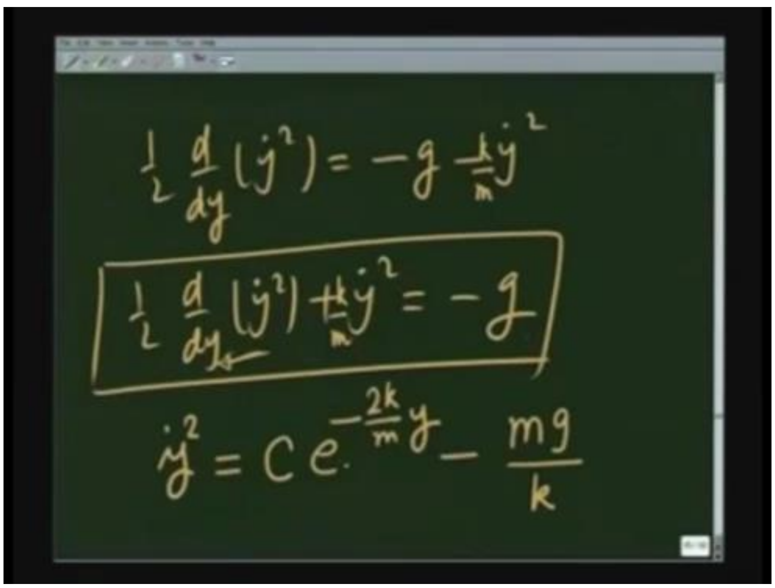

And I have one half D over DY Y dot square is equal to - G - Y dot square. One half D over DY Y dot square + Y dot square equals - G. I can solve for Y dot square directly in terms of Y. You see, I have eliminated time. So in this problem it would not be possible for me to calculate how long does it take before the particle comes to stop but I certainly calculate how high the particle goes before it comes to stop.

You can directly see that the solution oh I forgot K over M here. So there is going to be a K over M here. Y dot square is going to be of the form C E raised to - 2K over MY - MG over K. You can directly substitute this and see. If I substitute Y dot square with MG over K, it gives me - G and this satisfies the homogeneous part.

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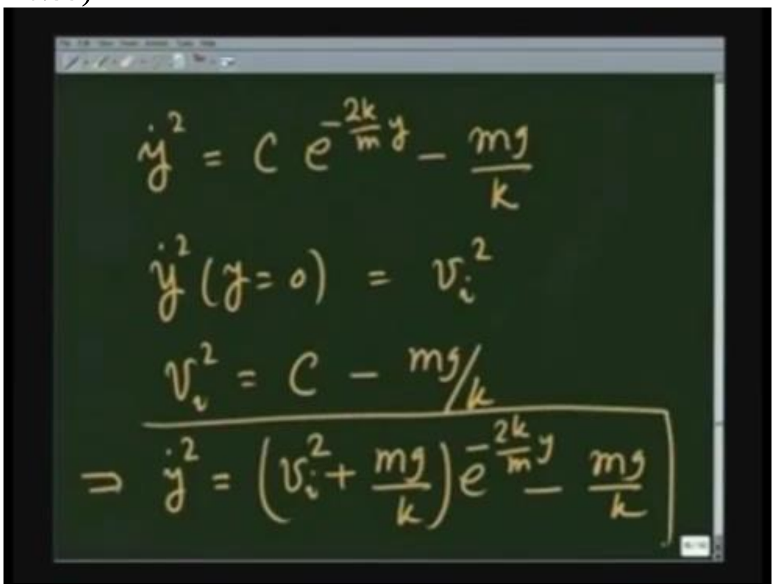

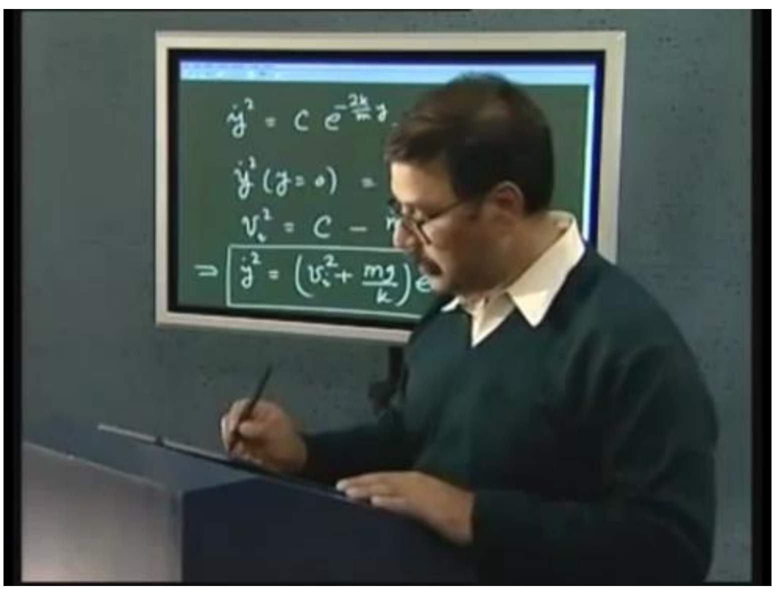

Now to initial conditions. So I have Y dot square equals some constant E raised to - 2K over MY - MG over K. And I know the initial velocity Y dot square when Y is 0 is equal to V initial square. And therefore V initial square is going to be equal to C - MG over K and that gives me Y dot square is equal to V I square + MG over K E raised to - 2K upon MY - MG over K.

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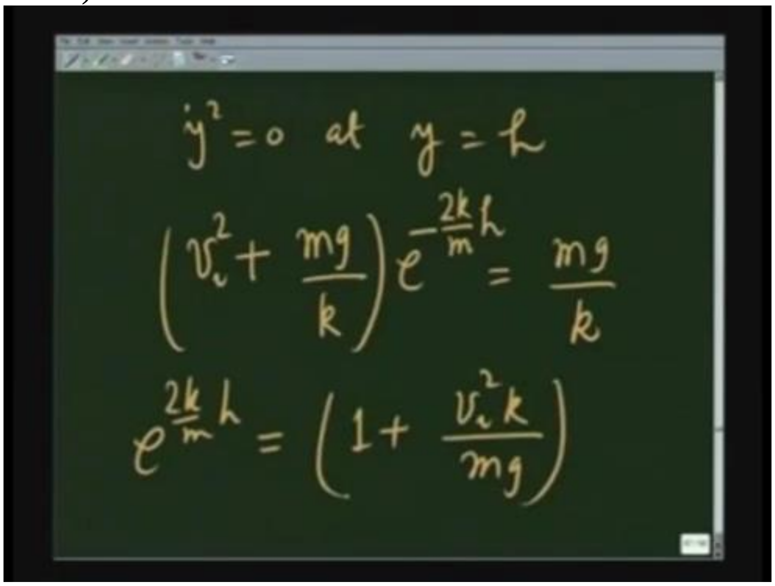

To find how high the particle went, I take Y dot square to be 0 at Y equal to H and therefore I get VI square + MG over K E raised to - 2K over MY over MH equals MG over K. And that tells me that E raised to 2K over MH is equal to 1 + VI square K upon MG.

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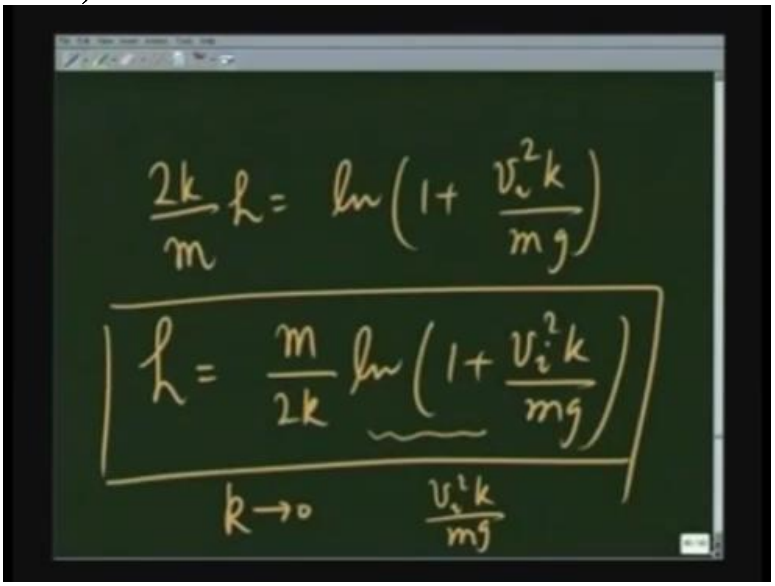

Or 2K over MH equals log of 1 + VI square K upon MG. Or the height it went up to is equal to M over 2K log of 1 + VI square K upon MG. This is the height at which it stops. Again I will go back to limit of K going to 0 and see if my answer is correct. When the limit K equal to 0 is taken, I can expand the long-term and write this as VI square K over MG.

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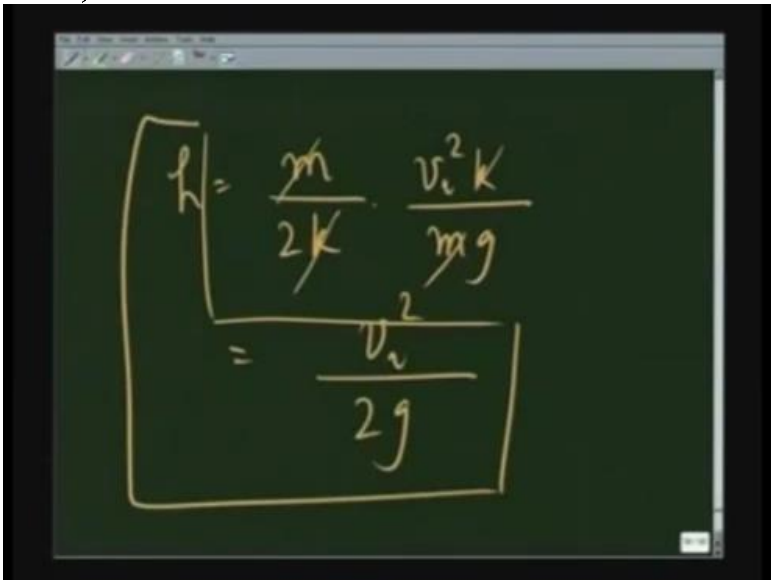

And therefore the height that I get is H equals M over 2K VI square K over MG which is our familiar result from the previous classes. So I have calculated the height up to which the particle will go, will go with the drag included.

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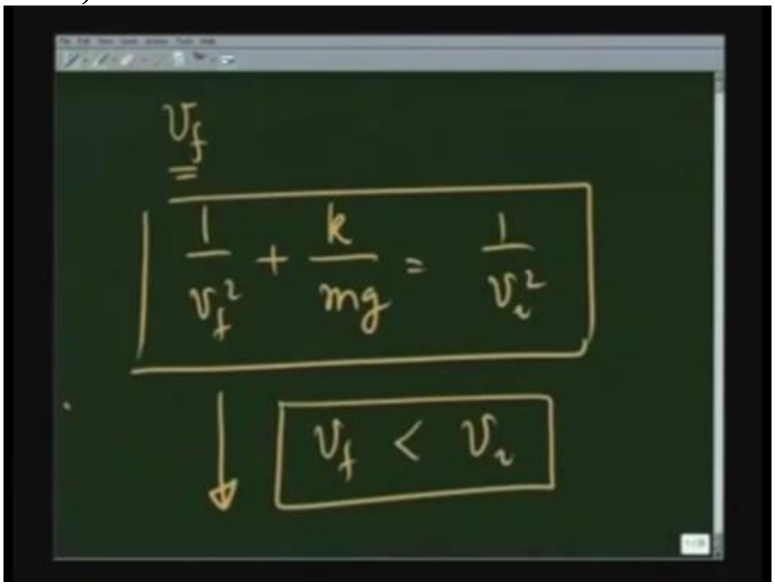

I leave it for you as an exercise that when it comes back with velocity V final then the relationship between V final and V initial square is going to be 1 over V final square + K over MG is equal to 1 over V initial square. A word of caution though, when considering the motion of particle coming down, you should take proper account of the force of friction.

The force of friction changes direction and as I showed in one of the previous slice that you have to change the sign of the force. V final of course is going to be smaller than V initial because in this friction, the particle has lost energy. So what we have done in this lecture is, shown through some examples how the drag force affects the motion of a particle.

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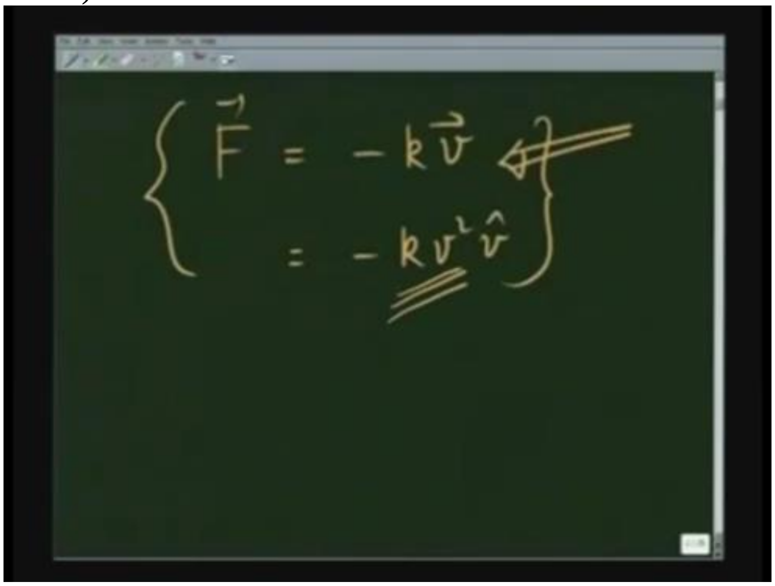

I have considered the simplest possible form F is proportional to the velocity itself and another form where F is proportional to the velocity square. This is encountered when velocities are small and suffices in most of the cases. Another mathematical advantage or analytical advantage with this is that in most of the cases, I can solve it analytically.

If the speeds become higher, then higher order of V's power come into play and then one has to solve the problem numerically. With these, we finish the 1st part of these lectures on engineering mechanics.

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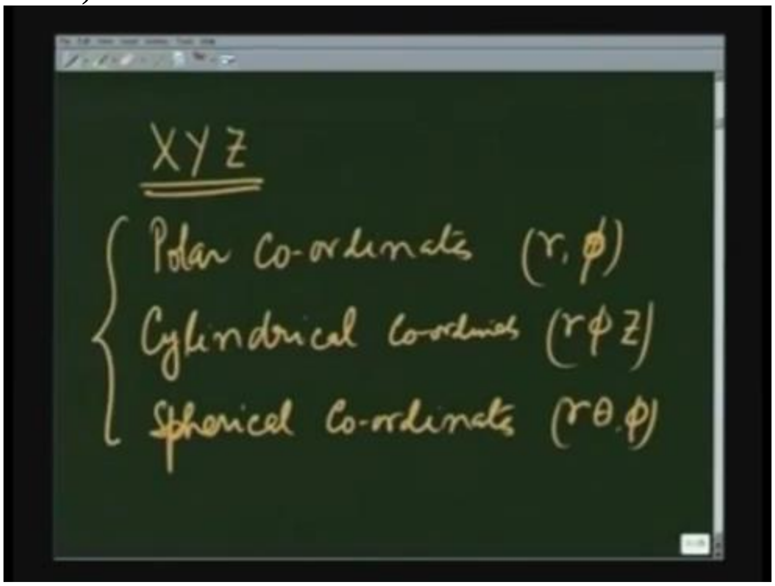

In this part I made you familiar with different coordinate systems that we use to stop besides the XYZ coordinates systems, Cartesian coordinate systes, we considered the planar polar coordinates, R and phi. We considered cylindrical coordinates R, phi and Z. We considered spherical coordinates R, theta and phi. And we used some of them while solving problems.

Not only these coordinate systems are useful in solving mechanics problems, you will find them very useful in solving problems in electrodynamics and wherever there are such symmetries.

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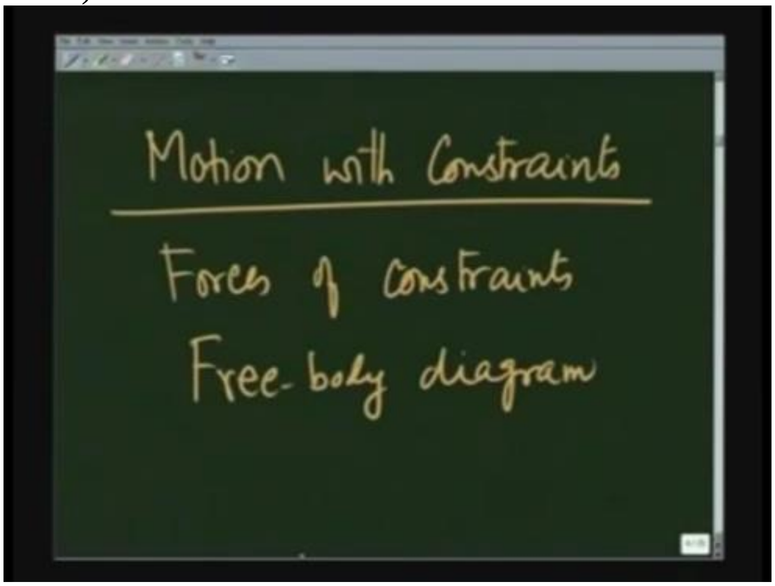

After this introduction to coordinate systems, we considered motion of particles, single particles with constraints and learnt what are forces of constraints? What is free body diagram? How to take constraints into account and how to solve problems when constraints are there.

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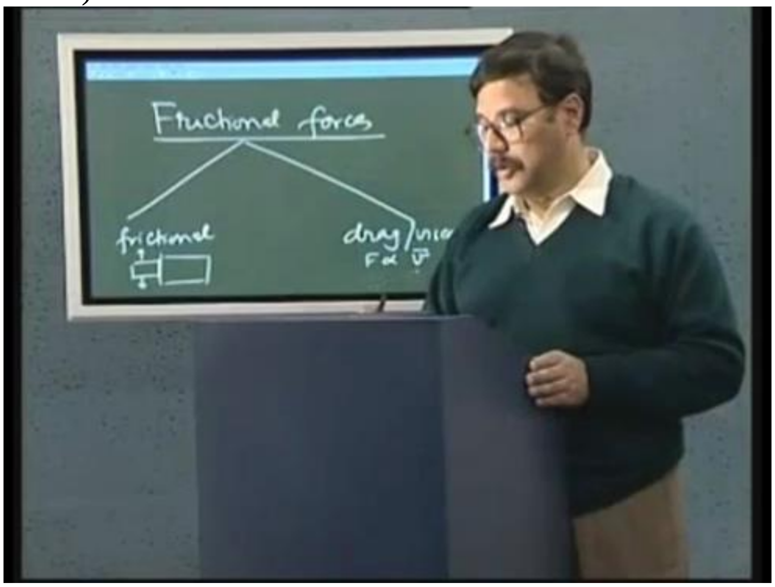

Following that we considered a very special class of forces called frictional forces and took them into account when solving dynamical problems. And these we considered regular frictional force that you are familiar with when one body slides over the other body, solid body and we also considered drag or viscous force in their simplest possible forms, one where F is proportional to the velocity and also proportional to the V square.

In the coming few lectures we are now going to get more sophisticated. Consider the work done by forces, do work energy theorem and then go to many particle systems where we will consider rigid body motion and things like those.