Engineering Mechanics
Professor Manoj K Harbola
Department of Physics
Indian Institute of Technology Kanpur
Module 5
Lecture No 46
Motion with constraints-solved examples

I am going to demonstrate this now with a slightly more complicated problem. It will be like Atwood's machine but slightly more complicated.

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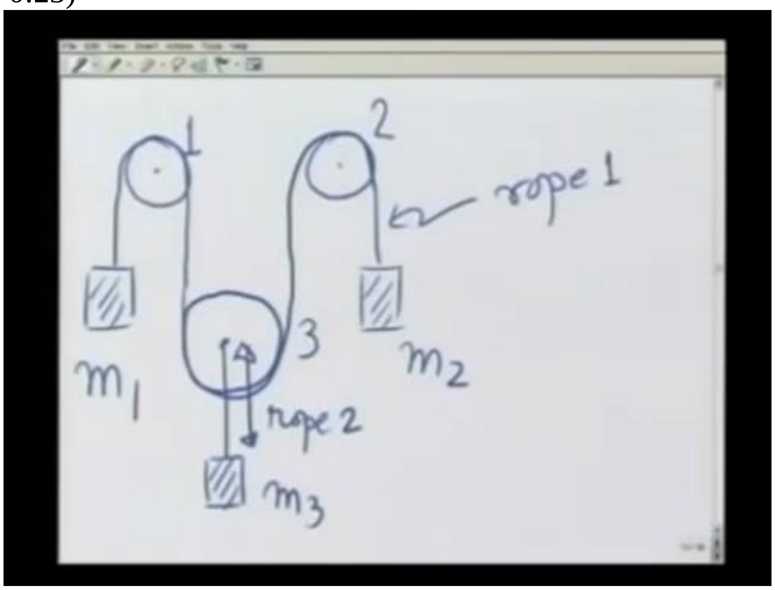

So what we are going to do is take 2 fixed pulleys. They are fixed at some height, pass a string of fixed length over them and the string turns around with the help of this 3rd pulley. Let me number them. pulley number 1, pulley number 2 which are fixed. Pulley number 3 which can move up and down. And let me also attach a mass here, attach a mass here.

This is mass M1, this is mass M2, this is mass M3 and this is attached again by a rope. Call it rope 2. Call this rope 2 and both are of fixed length. And what I want to do now is when I release this system, how does the motion take place?

For example, where does M1 move? What is its acceleration? Where does M2 move? What is its acceleration? How does M3 move? What is its acceleration? So we will go back to our steps and the 1st thing we want to do is identify the constraints and forces of constraints.

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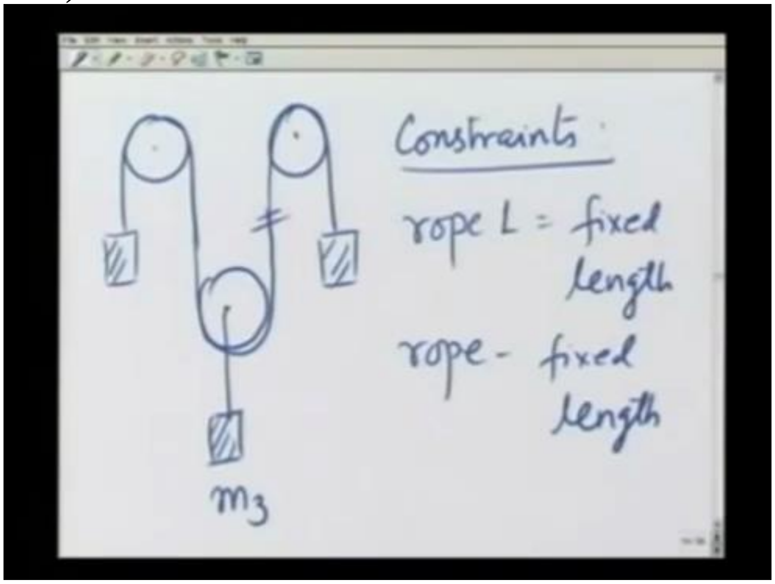

Let me make the figure again for you. The constraints are that the length of rope 1 is fixed. So constraints are rope 1 which is this one, has fixed length. Similarly, the other constraint is that rope 2 from which mass M3 is hanging, has fixed length. And I got to use them later on writing the constraint equation.

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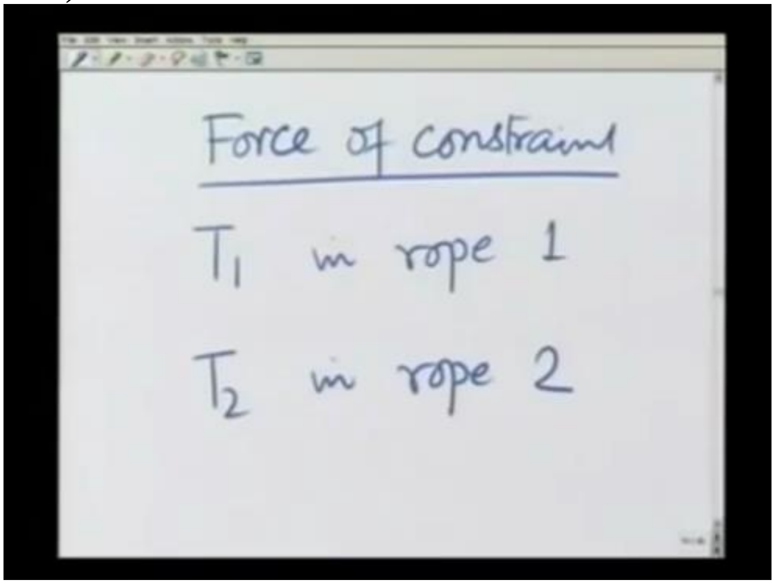

Forces of constraint are tension T1 in rope 1 and tension T2 in rope 2. These are my forces of constraints. Let me show them again in the picture.

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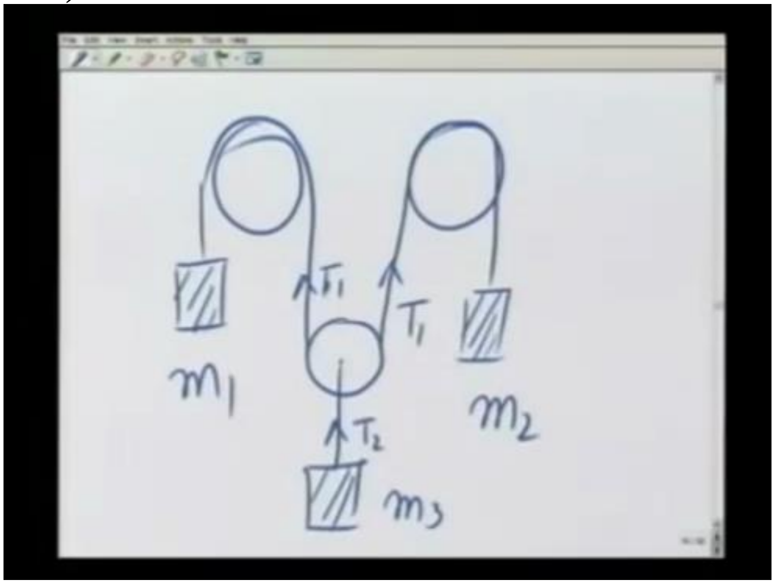

This is my pulley, mass M1, here is the pulley, here is the other pulley, mass M2, M1, M3 here. The tension works this way, T1, T2, these are the force of constraints. Again I am assuming T1 and T2 to be the same because I will be considering for simplicity massless and frictionless pulleys and assume that all the ropes are massless.

The next step in the game is make free body diagram were all moving parts. You can see that these 2 pulleys are fixed and therefore they are not going to move. This mass is going to move, this pulley is going to move, this mass is going to move, this mass is going to move. And therefore I will have 4 equations of motion. For that again, I take reference point from which I am going to measure my distances.

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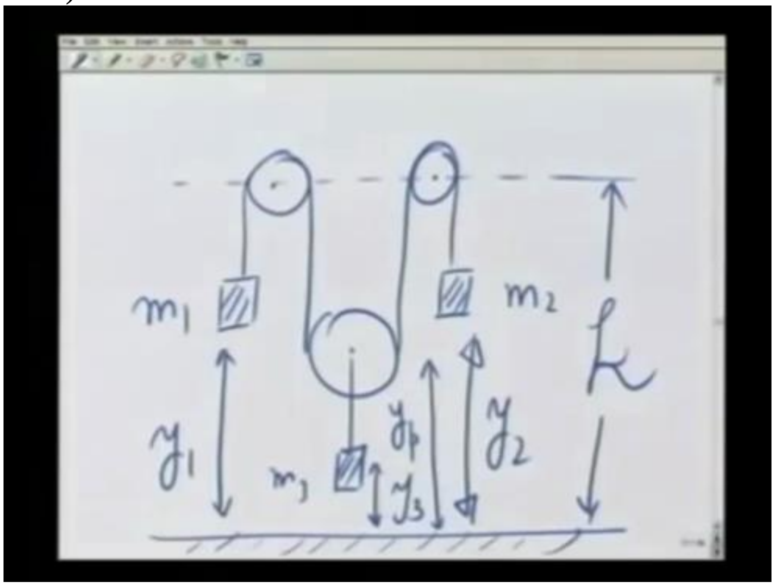

To make that picture again, this is mass M1, pulley number 3, mass M2, mass M3. I will measure the heights from the ground. Since these 2 pulleys are fixed, they are at height H which is a constant from the ground. This distance, let it be Y1 for mass M1, let it be Y2 4 mass M2, let it be Y3 for mass M3 and let it be YP for the pulley because I am considering all the subsystems of this whole thing which are moving. To write the equations of motion, let me again go back to the free body diagrams and I will take them up one by one.

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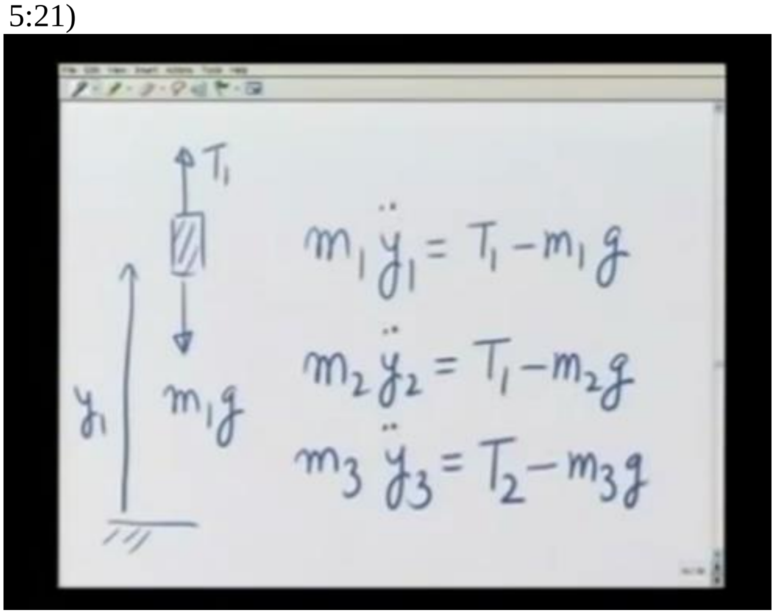

If I go to mass M1, it had tension T1 acting this way, mass M1G pulling it down and its height is measured from the ground as Y1 and therefore the equation of motion for this is going to be M1Y1 double dot equals the 1 - M1G.I can similarly write the equation of motion for mass M2 which is going to be M2 Y2 double dot equal to T1 - M2G.

Same thing for M3. M3 Y3 double dot is going to be equal to T2 because it is the tension in the 2nd rope that is pulling it up - M3G. The 4th moving part is the pulley itself.

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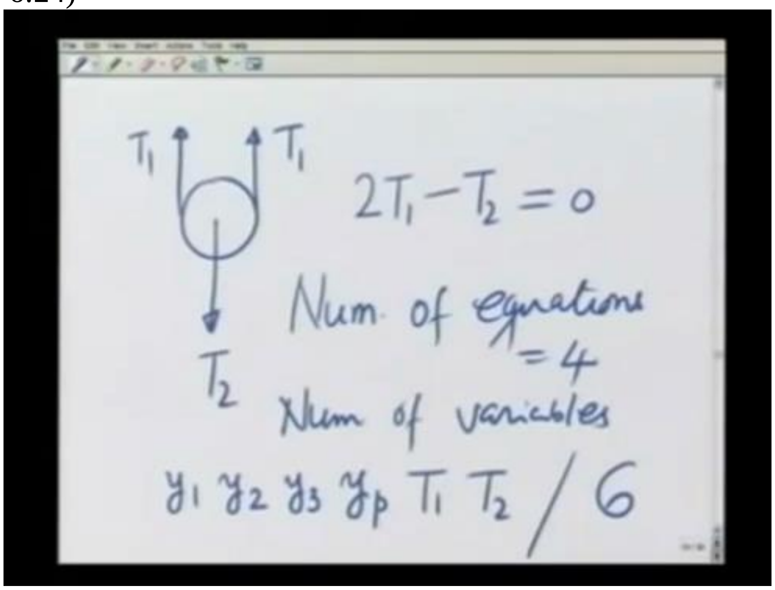

And for that the forces pulling it up are T1, T1 on both the sides and tension T2 in the middle. Since the mass of the pulley is 0, therefore I am going to have 2T1 - T2 which is the net force that must be equal to 0. You see I have gotten 4 equations, 4 equations of motion. So number of equations is 4. How about the number of variables?

This is Y1, Y2, the heights of masses 1 and 2, Y3, height of mass 3, YP, the height of the pulley, T1, tension in rope 1 and T2, tension in rope 2. Total number of variables is 6. As I had said earlier, at this stage when you write the equations, the number of equations are going to be less than the number of variables.

The missing equations are provided by the equations of constraints and that is what we got to write next. That should give us 2 equations of constraints and that will make the number of equations to be 6 which is equal to the number of constraints. Let us do that.

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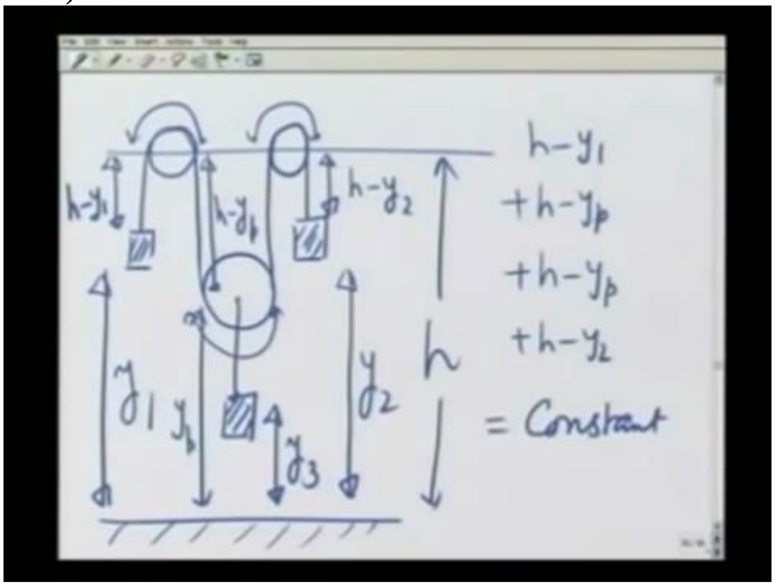

For that again we go back to our diagram where I will now take the constraints into account that the lengths of these ropes, both the ropes are fixed. So this is Y3. These were at height H from the ground. This is at Y2, this is at Y1. So let us see what are the different distances involved. This is going to be H - Y1, this going to be H - Y2. Oh, this is YP. I need that also.

This is going to be H - YP and so is this. Therefore, the length of the rope is H - Y1 + H - YP. H - Y1 + H - YP + another H - YP + H - Y2. And these distances are also fixed because the radius of pulley is fixed. So this is going to be equal to the constant.

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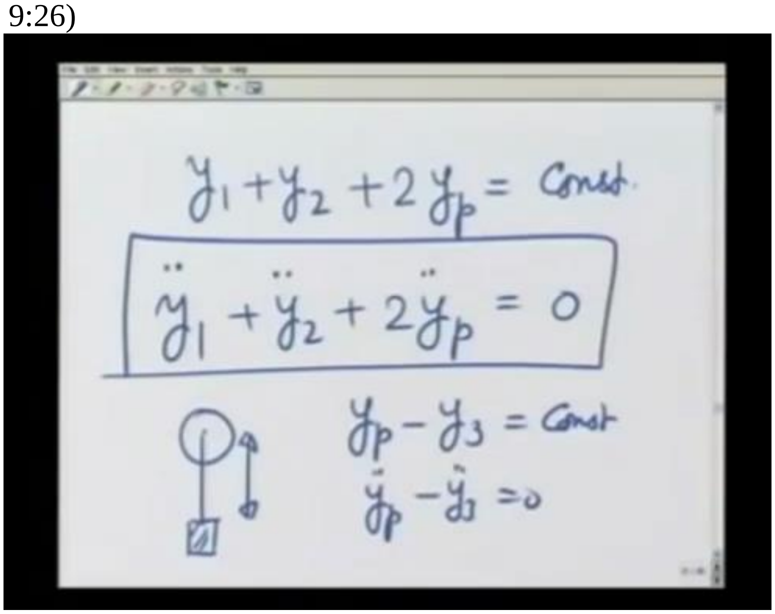

In other words, I can write, if I reshuffle the variables Y1 + Y2 + 2YP equals a constant. To again bring them to usable form, I differentiate them and write Y1 double dot + Y2 double dot + 2 YP double dot equal to 0. That is one of the equations that I need. I have now gotten 5 equations. Sixth equation is very easy to get.

Realising that the length of this rope is fixed and therefore YP - Y3 is constant for YP dot - Y3 dot is 0 and so is YP double dot and Y3 double dot. Basically the pulley and the mass are moving up and down together. So let me rewrite all the equations once more. I have now gotten 6 equations and there are 6 variables in the problem.

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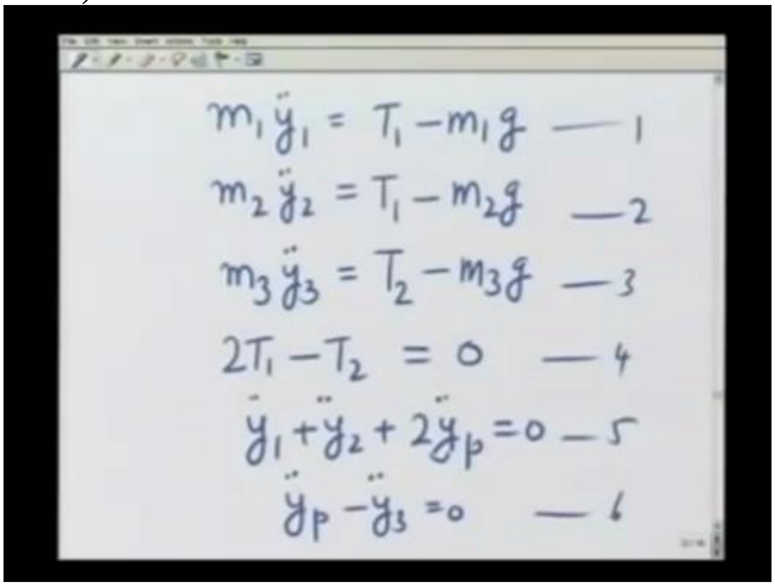

The equations are M1 Y1 double dot is equal to T1 - M1G. M2 Y2 double dot equals T1 - M2G. M3 Y3 double dot equals T2 - M3G. 2T1 - T2 is equal to 0. Y1 double dot + Y2 double dot + 2 Y pulley double dot equals 0. And Y pulley - Y3 double dot equal to 0. Equation 1, 2, 3, 4, 5, 6 and 6 variables.

I can solve them and get my desired answer. I am not going to solve it for you here but I will give you partial answer and you can try to get that after solving these equations at home.

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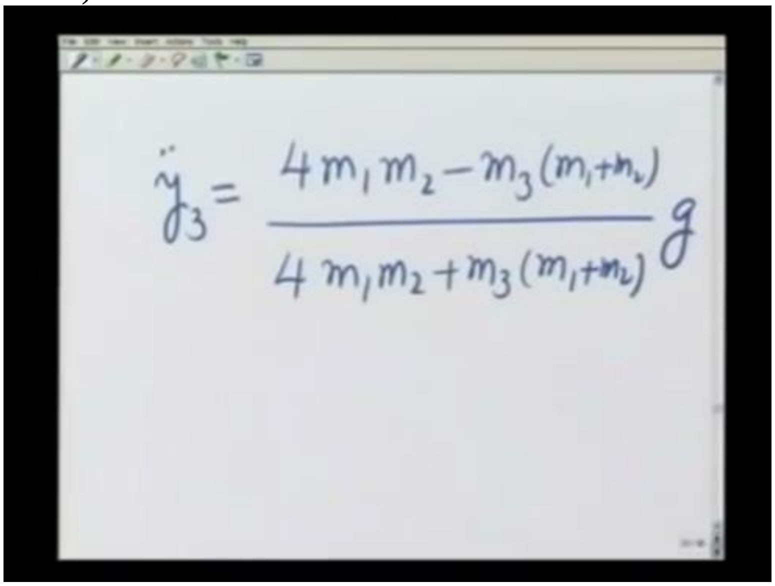

The answers for Y3 double dot comes out to be 4 M1 M2 - M3 M1 + M2 divided by 4 M1 M2 + M3 M1 + M2 G. I will change it slightly and give this as an exercise to you.

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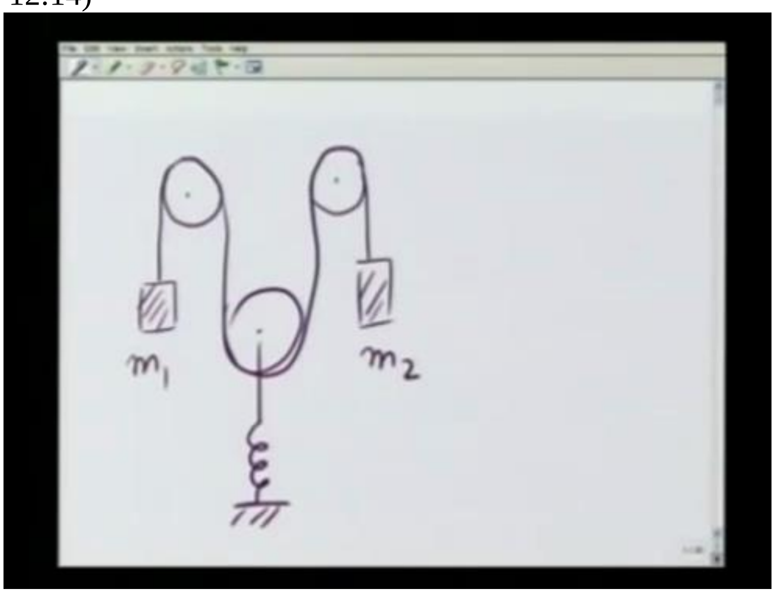

Let me again attached two masses with a pulley. Let it go like this. Except that now I will attach a spring to this mass. I would like to set up the equations for this. I would like you to setup the equations for this and solve them using the methods that we have learned so far.

I will do two more examples of the methods that we have learnt to demonstrate how effectively we can solve problems using the method of making free body diagrams, taking subsystems using constraints, constraint equations and equations of motion.

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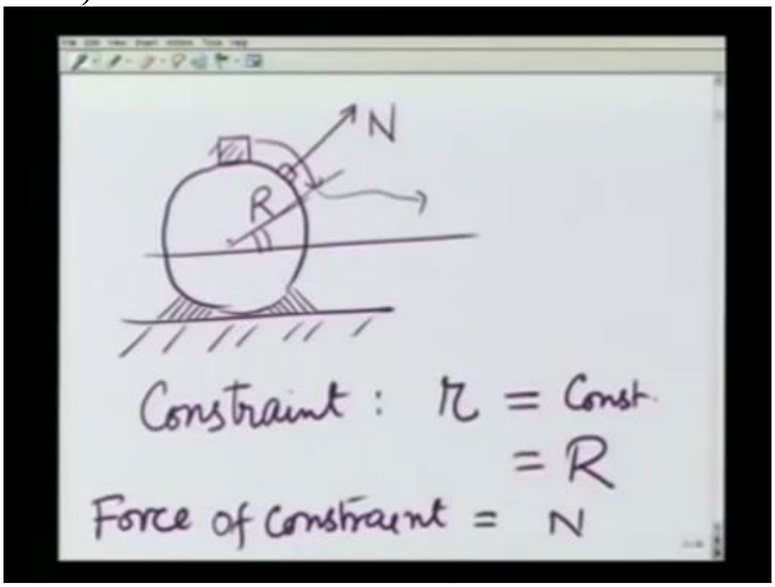

As an example, again let me take a block that is sliding on the surface offered drum which is fixed on the ground. The block starts from the top, slides down this way and the problem that we ask is at what angle does this drum, at what angle does this block fly off the drum? The radius of the drum is R. I am choosing this particular problem because this will also give you some practice on planar polar using planar polar coordinates.

So to start with again, we identify what the constraints and forces of constraints are. So constraint on the moving block here is that the radius R is a constant and this is equal to the radius of the drum R. The force of constraint then obviously is the normal reaction that the drum applies on the block.

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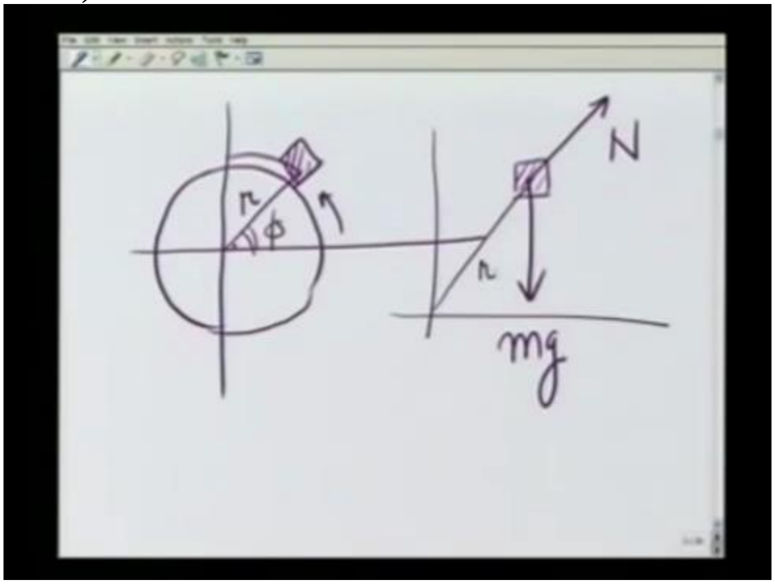

Since the motion is circular because before leaving the drum, the block is moving in a circular path here I am going to use planar polar coordinates taking this angle as phi, increasing this way and R obviously is the radius. My 2nd job is to make the free body diagram for the block. So if I take the block here, the free body diagram that involves only the normal reaction or the force of constraint and the external force is going to be MG of the block acting downwards and the normal reaction acting this way.

And after making this free body diagram, I write the equations of motion for the block except that this time I want to use planar polar coordinates. So at this position, the block is at distance R.

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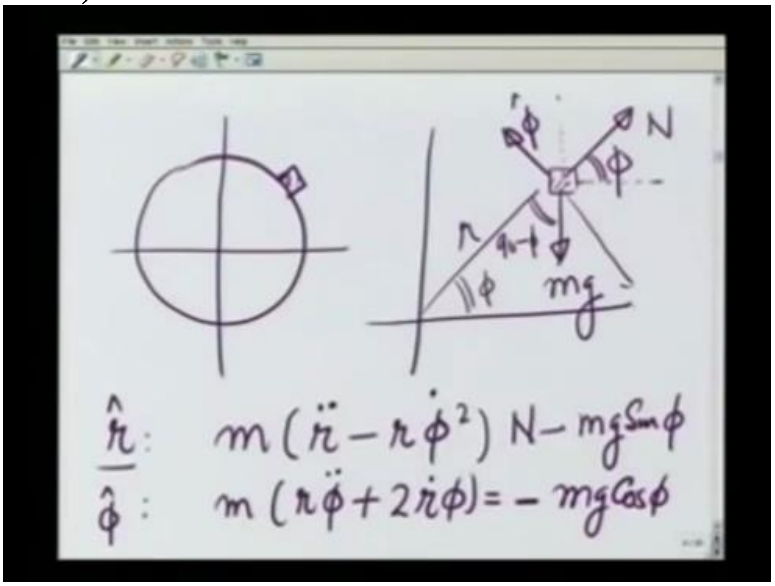

The block I am taking to be here. So at this position, it is a distance R at an angle phi. This is the force MG acting this way. This is the normal reaction acting this way. Since this is phi, this angle is also phi. And therefore, in R direction if I want to write the equation of motion, it is going to be M times the acceleration in the R direction which if you recall from the previous lecture is R double dot - R phi dot square.

And this is going to be the force in R direction is N in the positive R direction and since this angle is 90 - phi - MG sine of phi. Similarly in phi direction the equation of motion is going to be MR phi double dot + 2R dot phi dot equals MG cosine phi but with a - sign because the forces acting in negative phi direction. Remember that phi is increasing this way and the component of MG is in this direction.

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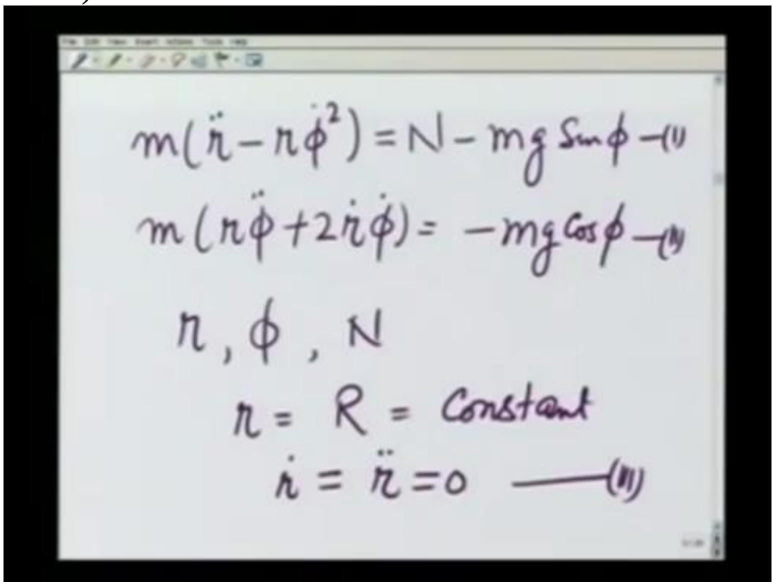

So the equations of motion that I have gotten are MR double dot - R phi dot square is equal to N - MG sine of phi and MR phi double dot + 2R dot phi dot is equal to - MG cosine of phi. These are 2 equations of motion and number of variables involved are R, phi and N. As I had said earlier, at this stage, the number of equations would generally be less than the number of variables in the problem.

And that is because so far I have not taken into account the equation of constraint which in this case happens to be r is equal to R which is a constant and therefore R dot is equal to R double dot is equal to 0. And that is my 3rd equation which I can solve. So now I am going to substitute 3rd equation in number 1 and number 2 and solve the resulting equations to get my answers.

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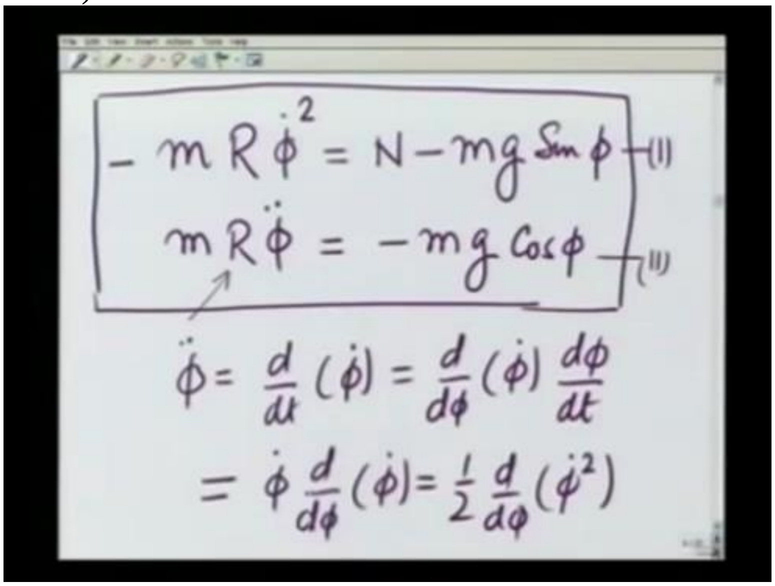

When I do that, I get MR phi dot square with a - sign in front is equal to N - MG sine of phi. And the other equation comes out to be MR phi double dot is equal to - MG cosine of phi. It is these 2 equations that are going to give me N and rate of change of phi with respect to time. R anyway is fixed by 3rd constraint equation. To solve this equation, I 1st solve the 2nd equation and I use a trick for this.

Because you see, I cannot really integrate with respect to time. So I write phi double dot which is the time derivative of phi dot as the time derivative of phi phi dot times D phi DT using the chain rule for differentiation. And that gives me equals phi dot D over D phi as of phi dot which is nothing but one half D over D phi of phi dot square.

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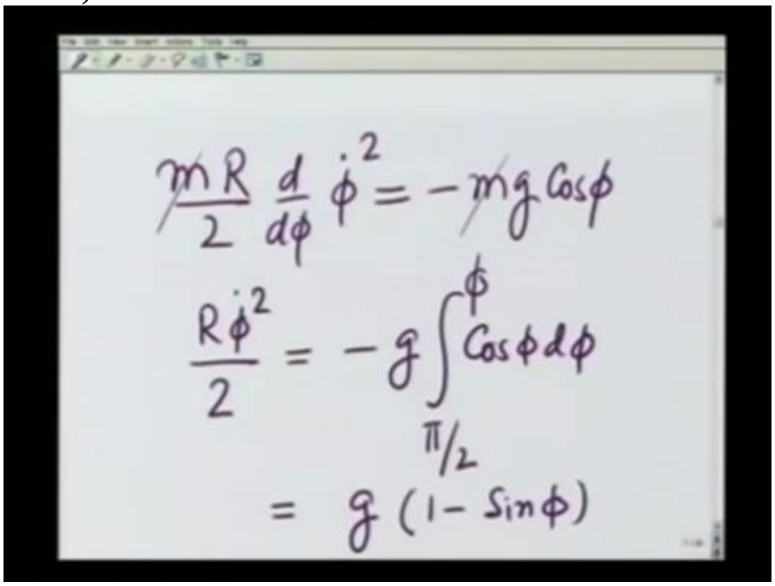

And therefore my 2nd equation now reads MR over 2D over D phi phi dot square is equal to - MG cosine of phi. M cancels and therefore I have R phi dot square over 2 is equal to - G cosine of phi D phi. And remember, the particle is starting from the top. That means, it is starting from pi over 2 and coming up to some angle, phi. This gives me, this is equal to G 1 - sine of phi.

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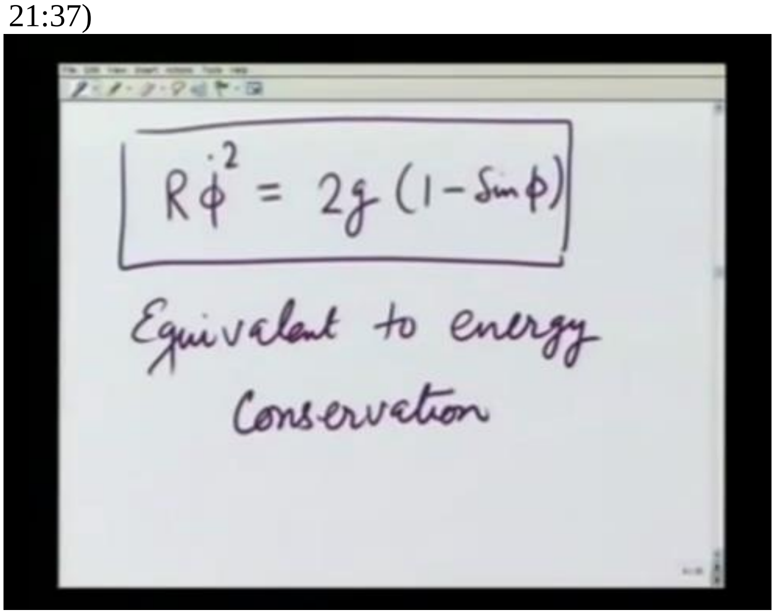

And therefore the result that I have is R phi dot square is equal to 2G 1 - sine of phi. I leave this as an exercise for you to show that this is equivalent to energy conservation. Because the lefthand side is related to the kinetic energy and right-hand side is related to the change in potential energy. Using this, I can now solve equation 1 for the normal reaction.

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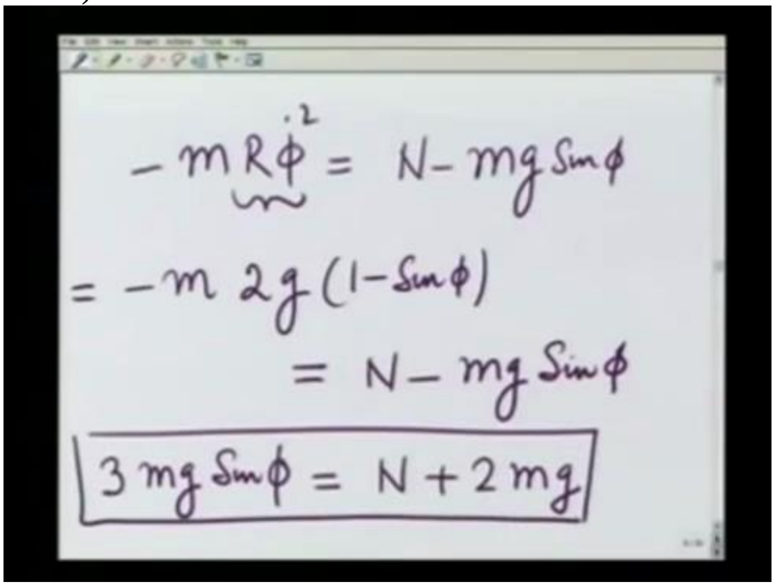

Recall that equation 1 is nothing but MR phi dot square with a - sign here is equal to N - MG sine phi. I substitute for R phi dot square that I just derived and I get - M2 G1 - sign of phi is equal to N - MG sine of phi which gives me after reshuffling 3 MG sine of phi is equal to N + 2 MG. This gives me the normal reaction as a function of phi. Recall that this, the normal reaction that is enforcing the constraint that is the particle moving, moves with a constant radius.

When it flies off the cylinder, I had that point there is no contact between the block and the particle.

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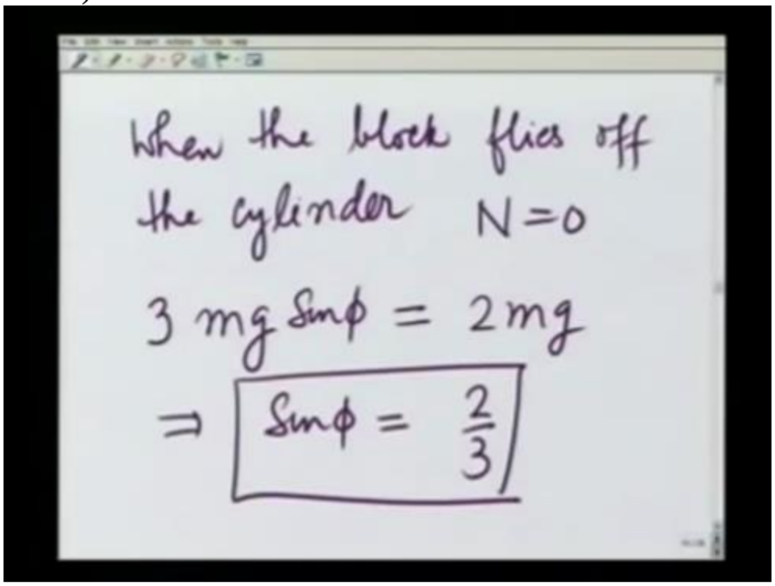

And therefore when the block flies off the cylinder, at that point N would become 0. And therefore, 3 MG sine phi at that point would become equal to 2 MG which implies that at angle phi from the horizontal such that sine phi is two thirds, the block will go off the cylinder. This is one problem that I have solved using the steps that I have outlined and polar coordinates.

Next I am going to solve a problem that involves 2 particles and constraints on their motion due to their contact. In our next example, we take 2 bodies that are in contact and apply constraint on their motions because of that.

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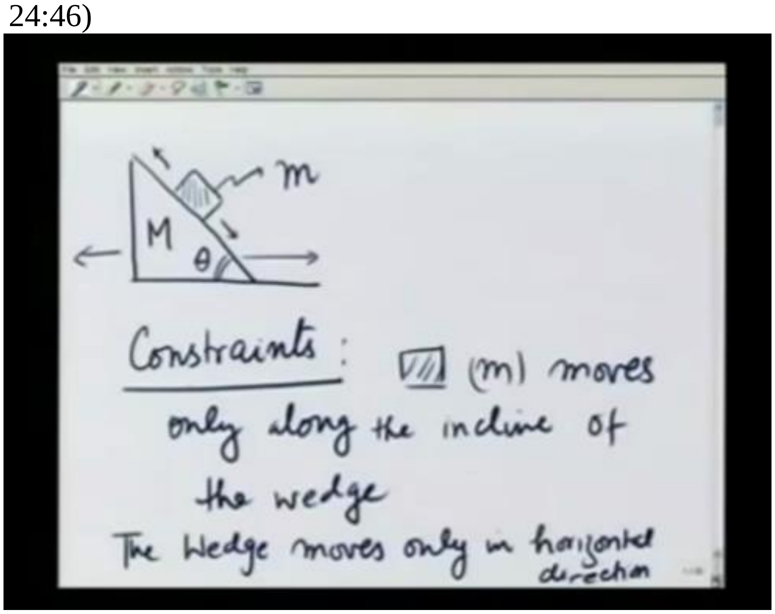

That I am taking a wedge with an angle theta here and there is a block that slides on it. The wedge has mass M. This block has mass small M and all the surfaces are frictionless. Therefore this wedge is free to move in this direction and the block is free to move along the inclined plane. This is a constrained motion because the block of mass small M free to move only along the incline.

Therefore it is Y and X displacements are going to be related. Let us see then what are the constraints. Constraints of the motion are that this block of mass M moves only along the incline of the wedge. And the other constraint is that the wedge moves only in horizontal direction.

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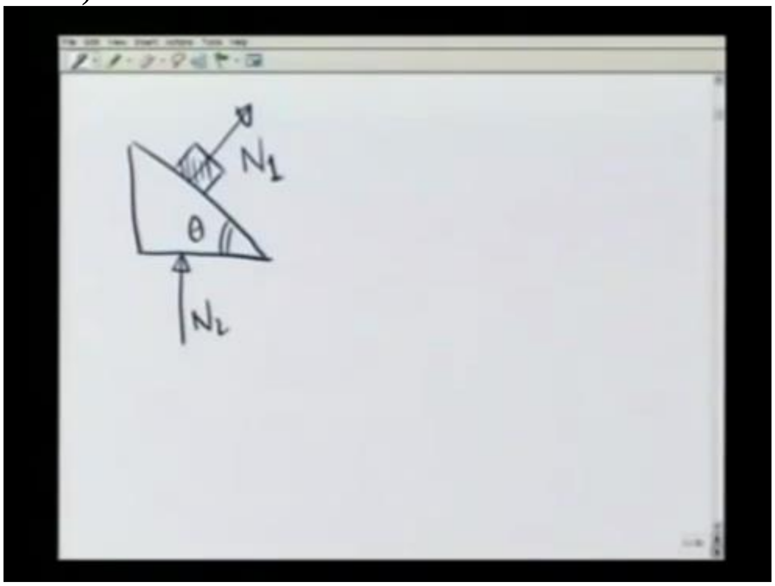

So let us see what the forces of constraints are going to be. Obviously, the force between these 2 is the normal reaction on the block. Let me call it N1. And the force that keeps the wedge on the ground is the normal reaction on it N2. The next step in our strategy is to make free body diagram for all the moving parts of the system. There are clearly 2 subsystems here.

One, the block here, the other the wedge here. For the block, the free body diagram would give a force N1 acting in this direction and its own weight pulling it down. I remind you once more that the free body diagram means I isolate the system and represent all the constraints by the forces of constraint. The free body diagram for the wedge is going to be by Newton's 3rd law, a force N1 in this direction, its own weight MG pulling it down and the normal reaction N2 by the ground.

I would like you to notice that I have not involved the mass small MG in making the free body diagram of the wedge. That is because it is taken care of by normal reaction N1. After making free body diagrams, I am going to write the equations of motion. To write my equations of motion, let me 1st fix my coordinate system.

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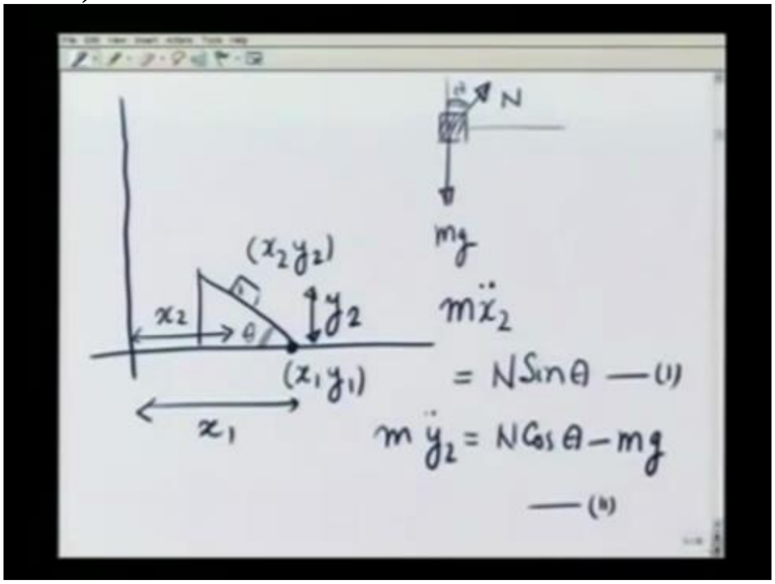

And for that what I am going to do is make the wedge here. This is the block. For convenience, I will choose this point to represent the coordinates for the wedge. Let this be X1, Y1. Notice, since I am putting this constraint that the wedge moves only in the horizontal direction, I am explicitly writing Y1 and later I am going to eliminate this using the constraint equation so that this distance right now is X1.

Similarly, the coordinates for the mass M, let them be X2, Y2 so that this distance is X2 and this distance is Y2. Looking at the free body diagram where mass small M has a normal reaction like this. This angle is theta and a force MG pulling it down. If this angle is theta, this angle is going to be theta. And therefore, I am going to have the equation of motion as MX2 double dot is equal to N sine theta. That is it.

And similarly for the Y coordinate, I am going to have, this is my equation number 1. MY2 double dot is equal to N cosine of theta - MG. That is my equation number 2. 3rd and 4th equations are going to be for the wedge.

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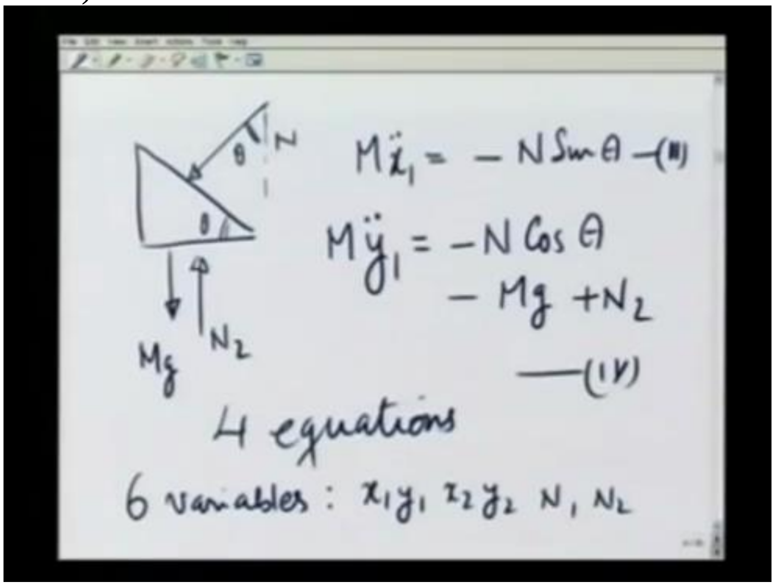

For the wedge, the forces are N1 in this direction with this angle being theta. MG in this direction and N2 in this direction. So the equations are going to be MX1 double dot is equal to - N sine of theta equation number 3 and MY1 double dot is going to be equal to - N cosine of theta - MG + N2 equation number 4.

I have 4 equations and 6 variables which are X1Y1, X to Y2, N1 and N2 as expected because I have not yet taken the constraints into account. The remaining 2 equations are given rise to by the constraint equations. What are the constraint equations?

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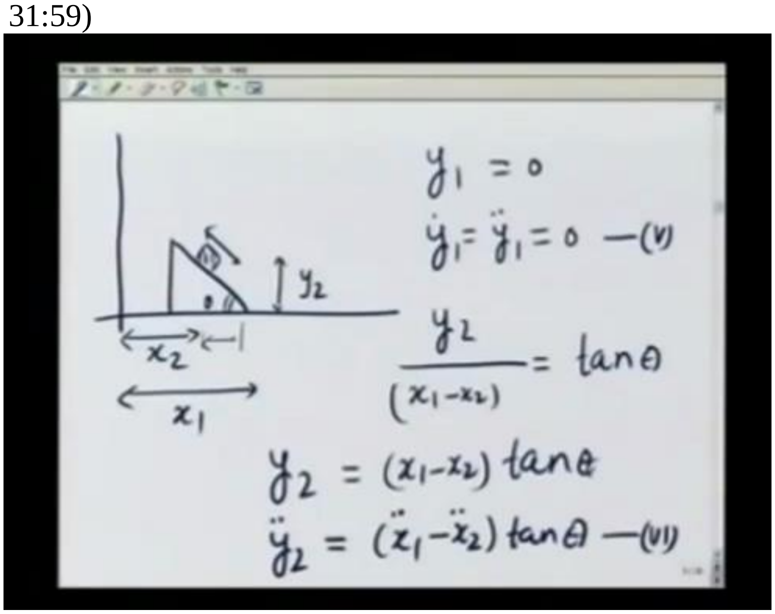

The constraint equations are given by, let us draw this again and we will see. This is theta, this is the block, this is Y2, this is X2, this is X1. Number 1 that the wedge moves only in horizontal plane and therefore Y1 is always 0 and therefore Y1 dot is equal to Y2 or Y1 double dot is equal to 0. That is my equation number 5. And the block always moves on this inclined plane.

And that means, Y2 divided by this distance which is X1 - X2 is fixed is equal to tangent of theta. And therefore Y2 is equal to X1 - X2 tangent of theta or Y2 double dot is equal to X1 double dot - X2 double dot tangent of theta. That is my equation number 6. I now have 6 equations and 6 variables and I can solve the equations. I will solve it again partially to get the value of N1. Rest follows very easily.

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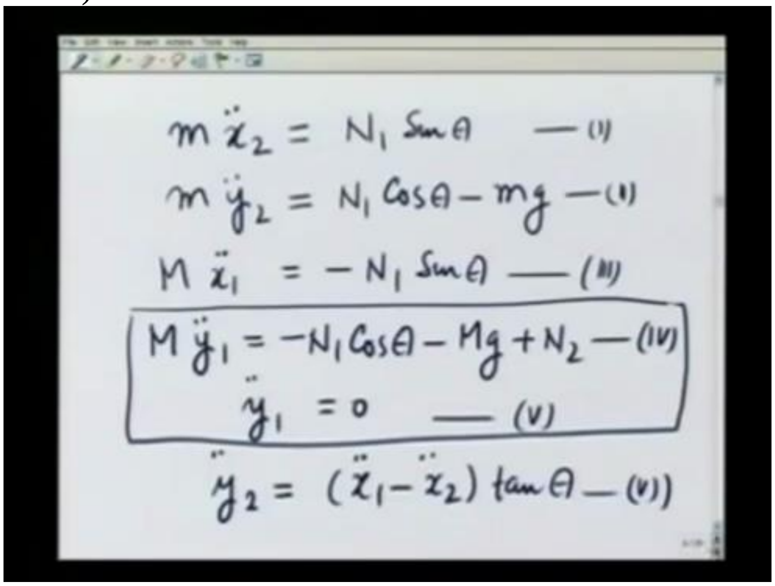

Let me rewrite the equations once more. So all the equations are MX2 double dot is equal to N1 sine theta, MY2 double dot is equal to N1 cosine of theta - MG. MX1 dot is equal to - N1 sine of theta equation number 3. MY1, sorry this was X1 double dot. Y1 double dot is equal to - N1 cosine of theta - MG + N2 equation number 4. Then Y1 double dot is equal to 0, equation number 5.

And Y2 double dot is equal to X1 double dot - X2 double dot tangent of theta, equation number 6. So now I have 6 equations and 6 unknown which can be solved very easily.

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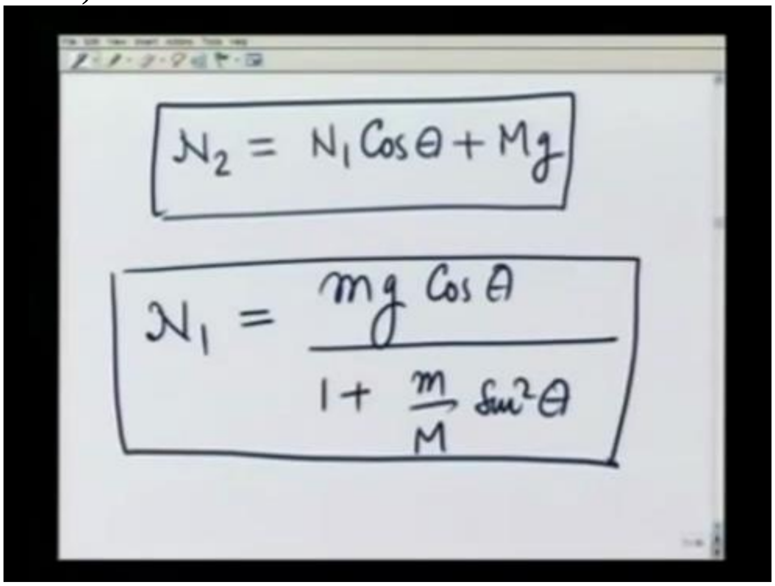

These 2 equations together really give you nothing new but that N2 is equal to N1 cosine of theta + MG which tells you that M2 balances the all the vertical forces acting on the wedge. The other equations when substituted for, would give you a result for N1. I will leave the solution and substitution and all those things for you.

Give you the final answer which gives you N1 is equal to MG cosine of theta divided by 1 + M over M sine square theta. Once I know N1 which is a constant quantity here, I can calculate all the other quantities, X1 double dot, X2 double dot, Y2 double dot and their integrated forms, velocities and displacements easily.

That brings us to the conclusion of this lecture in which through simple examples we saw how we take constraints into account when writing equations of motion and solving for them in cases where the motion is not completely free. It can be restricted either by external constraints or by 2 or 3 moving particles themselves that impose restrictions on the motion of each other.