Engineering Mechanics
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Module 5
Lecture No 45
Motion with constraints, constraint forces and free body diagram

In the previous lecture, we talked about 2 different coordinate systems that we used to describe the motion of particle moving under the influence of a force. In this lecture we start with how to solve a question of motion when the particle is going under constraints.

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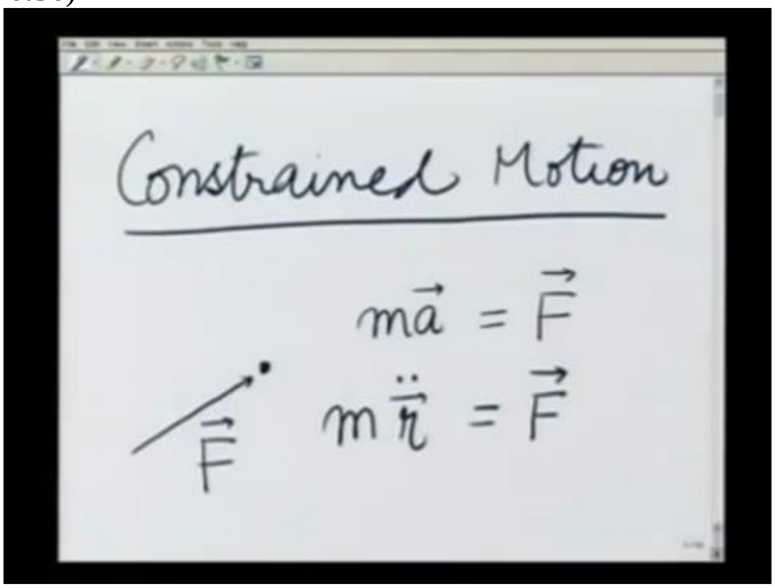

So what we are going to talk about is the constrained motion and I will explain to you what we mean by that and how do we go about solving this. Suppose there is a particle that is moving under the influence of a force F, then its motion is described by equation of motion that is given as MA equals the applied force where M is the mass and A is its acceleration.

Written in terms of its coordinates, I can write this as MR double dot equals F where double dot denotes 2^{nd} derivative with respect to time. Of course, if the particle is free to move, that means it can go anywhere. Then I solve this equation and get R as a function of time given a force.

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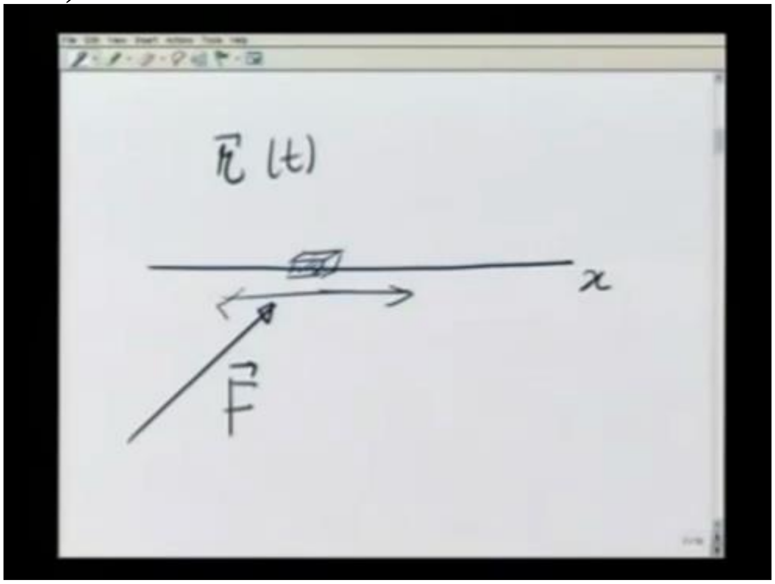

On the other hand if I restrict its motion and I will talk in a minute about what I mean by restricting its motion. The motion can be quite different. For example, I could take a bead moving on a straight wire and the bead can move only along this wire. And let us say this wire is in X direction. Then no matter what force I apply on this bead, the bead is going to move only along the X direction. Therefore its motion is restricted.

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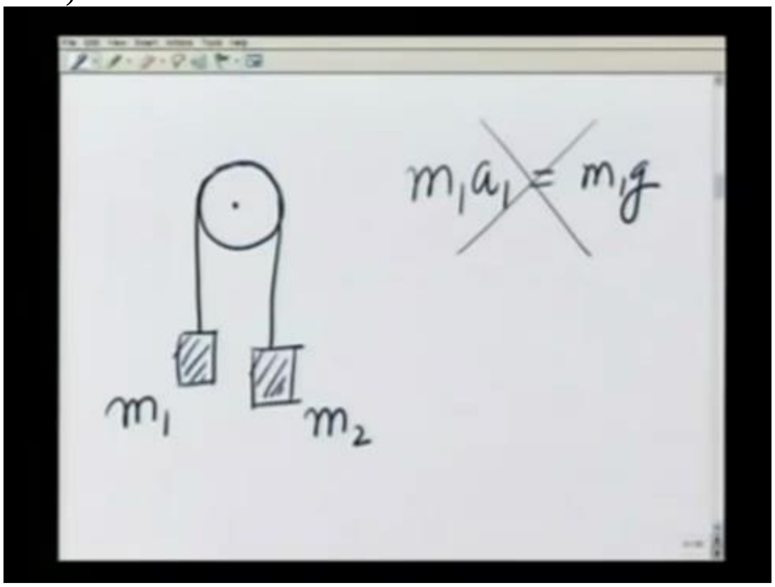

As another example which is quite well-known and also known as Atwood's machine is two masses moving on up only when they are tied by the string. In this case, the masses are no longer free to move. That means if I have this mass M1 and the other mass M2, M1's motion is not determined by this equation alone. That is, its motion is affected, this is not correct. Its motion is affected by the presence of the other mass.

Similarly motion of M2 is also affected by the presence of mass M1. And this constraint comes about because of this rope that keeps them moving together. These are 2 examples of constrained motion and how to deal with this when the motion becomes quite complicated or number of masses increase or number of constraints increase is going to be the topic of discussion in today's lecture.

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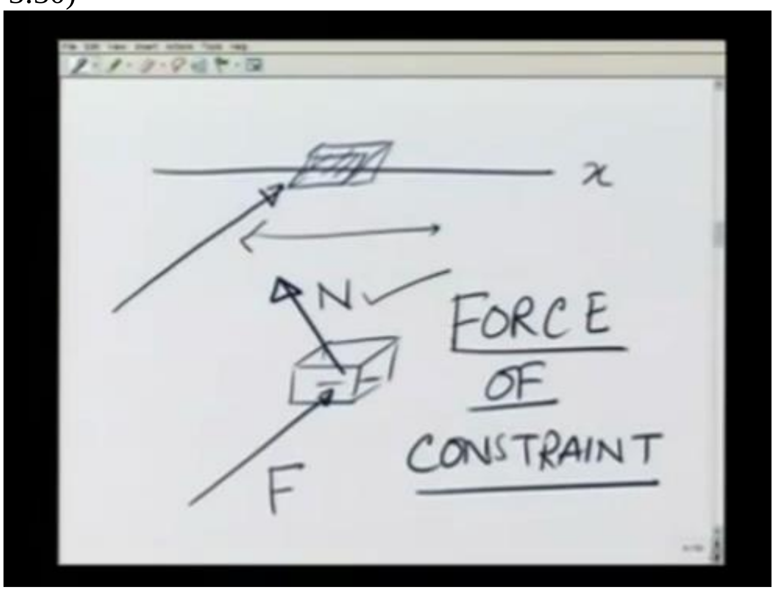

So let us start with our 1st example that is a mass or a bead moving on a straight wire as I said in X direction let us say. Through this example, I would like to introduce terminologies that are going to be in use when we talk about constrained motion. So if I apply a force on this, what does the wire do? The wire applies a normal reaction on this body. Why does the wire apply a normal reaction on this body?

Because it does not want it to move anywhere except along the wire. So it is this normal reaction N that is really restricting the motion of the particle along the wire. This is called the force of constraint.

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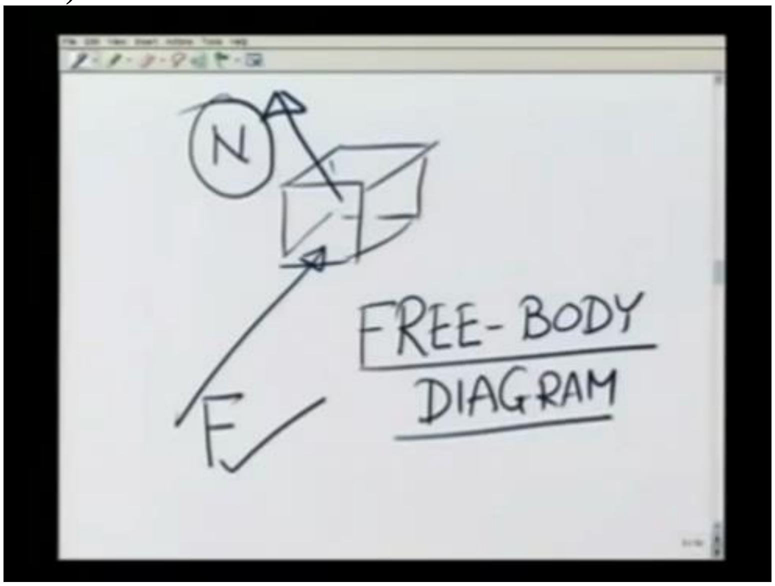

To recap it, I will again say that the effect of this wire on this body, anything is slightly bigger now, is just to produce a force N or a force of constraint. If you look at this body, I have replaced the wire by this force of constraint. This is known as the free body diagram of a particle or a body under motion. So what do we mean by free body diagram?

By that, I mean I will make the body by itself, nothing else and represent all the forces applied externally or applied by the constraints on it and that I will call free body diagram. Let us see how do I write equations of motion for this body.

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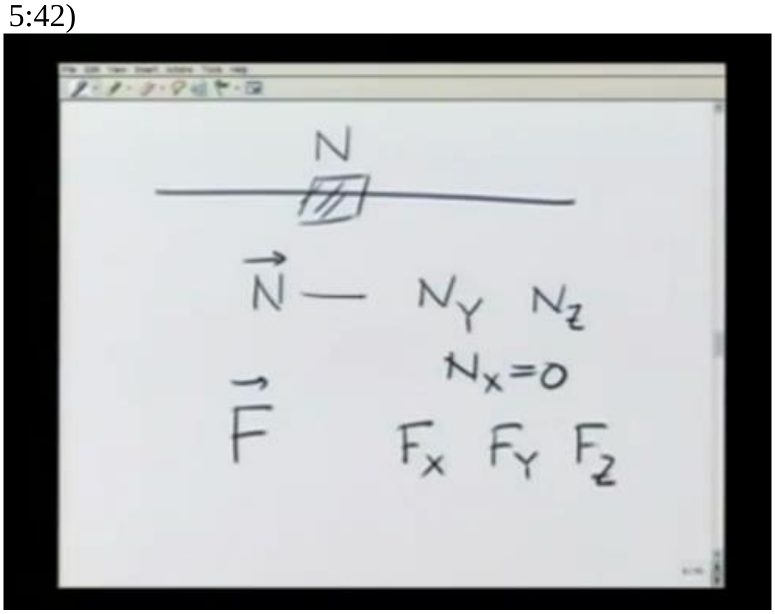

Since the normal force is going to be perpendicular to the wire, so N is going to have only 2 components in Y direction and in Z direction and X is always going to be 0. The force F that I have applied or I should write a vector here, is going to have all 3 components FX, FY and FZ.

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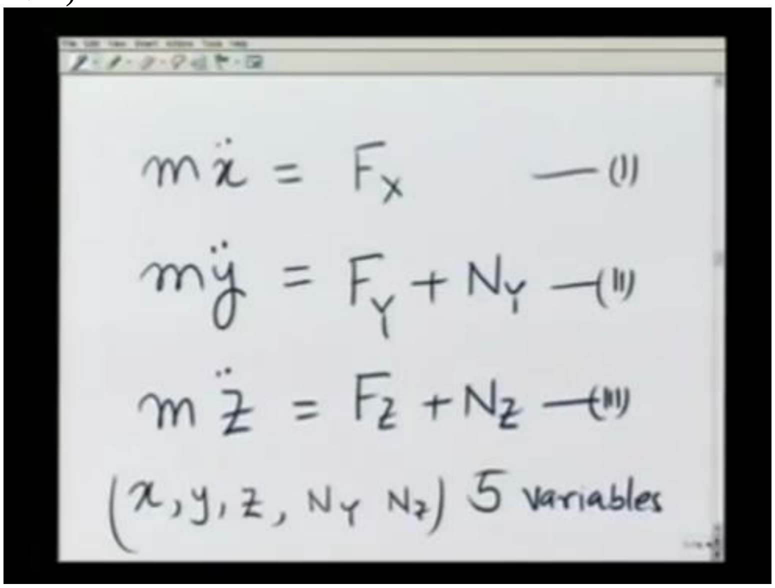

With these components of the forces, the equations **are** of motion of this particle are going to be MX double dot equals Fx because NX is 0, M by double dot equals FY + NY and MZ double dot equals FZ + NZ. These are 3 equations of motions that I get from F equals MA. Let us see how many number of variables there are. There are X, Y, Z, NY and NZ, 5 variables. On the other hand, I have only 3 equations.

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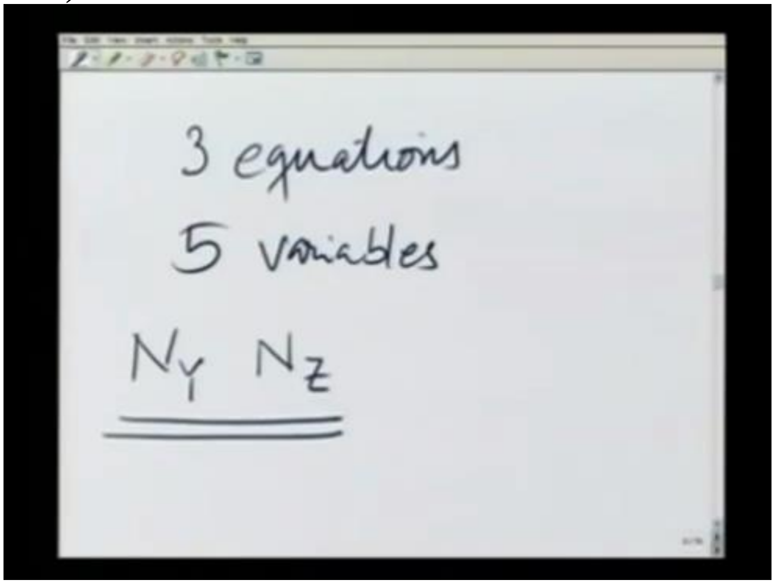

So I have 3 equations and 5 variables. Obviously, with this I cannot solve the problem. Where do I get the 2 other equations from? And this is where I start thinking the 2 of these variables NY and NZ came because I imposed a constraint. So the equations should also come from the constraints themselves and that is precisely where they come from.

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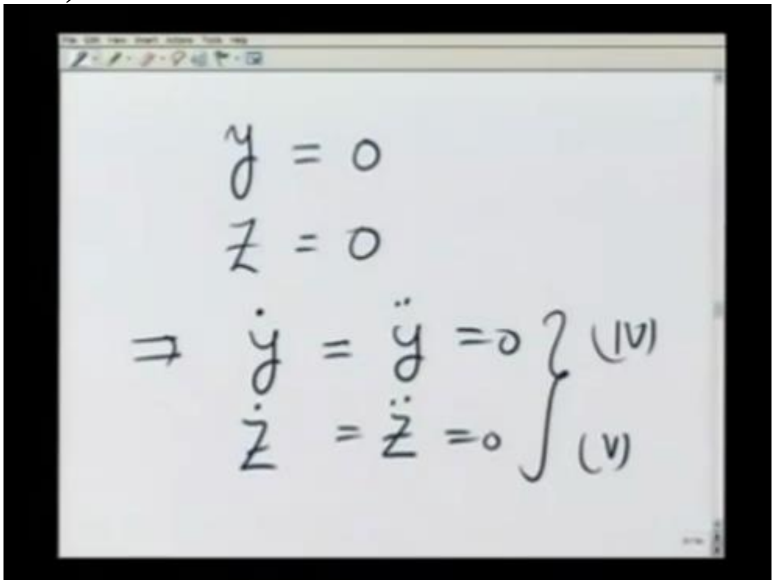

The equations of constraints are that Y is equal to 0 and Z is also equal to 0 and this implies Y dot or Y double dot, both are 0. And so is Z dot and Z double dot. So I have 2 more equations, total 5 equations and 5 unknowns and I can solve the problem. Let us now at the same example of constrained motion that I considered earlier.

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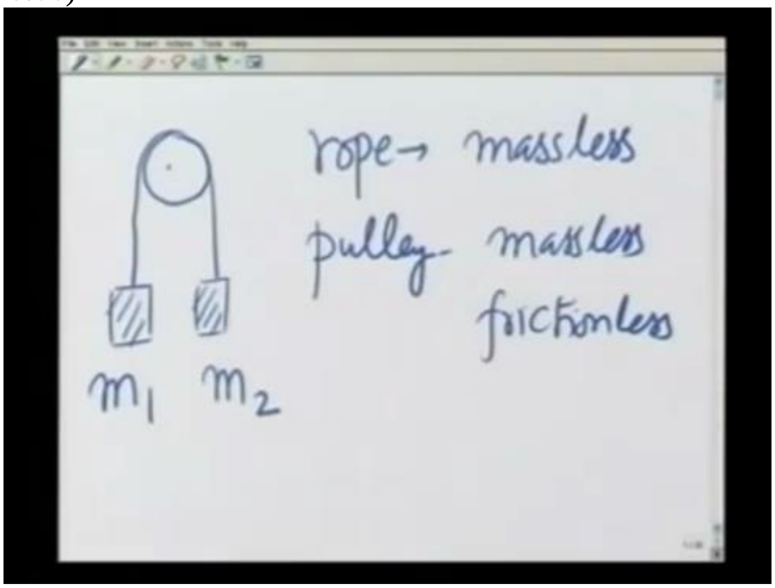

And that was the Atwood's machine in which 2 masses are tied on a string that is passing over a pulley that is massless and frictionless and so is the string. So let us write again, this is mass M1, this is mass M2. Rope or the string is massless for simplicity and pulley is massless as well as frictionless. You may have solve this problem time and again in your previous courses but what I want to focus on here is how to really deal with it from the constrained motion point of view.

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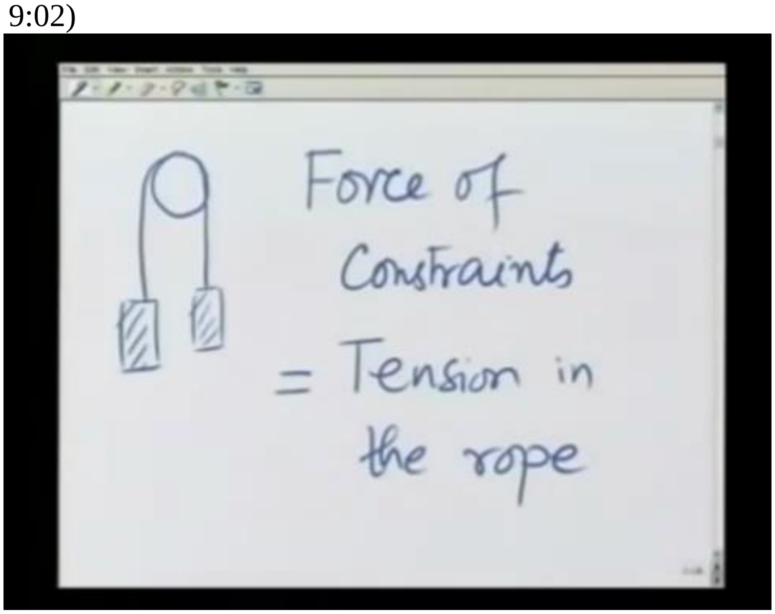

And therefore, 1st thing I am going to do is see what is causing the constraint. The constraint is caused by this rope that forces the 2 masses to move together. What is the force of constraint here? The force of constraint that really makes the masses to move together is nothing but the tension in the rope. So tension in the rope is the one that keeps these masses together.

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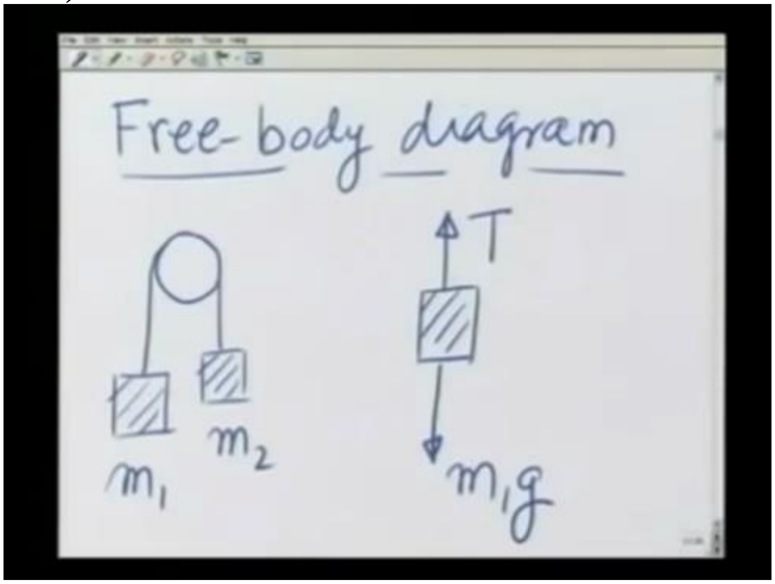

Next if I were to make the free body diagram for the 2 masses, let me remind you again what a free body diagram is. A free body diagram is where I replaced the agency causing the constraint by the force of constraint that is applied. So if I were to write, make the free body diagram for mass M1, it will be something like this that there is a force pulling it down M1G the gravitational force and there is a tension T which is the force of constraint pulling it up. And that is the free body diagram for mass M1.

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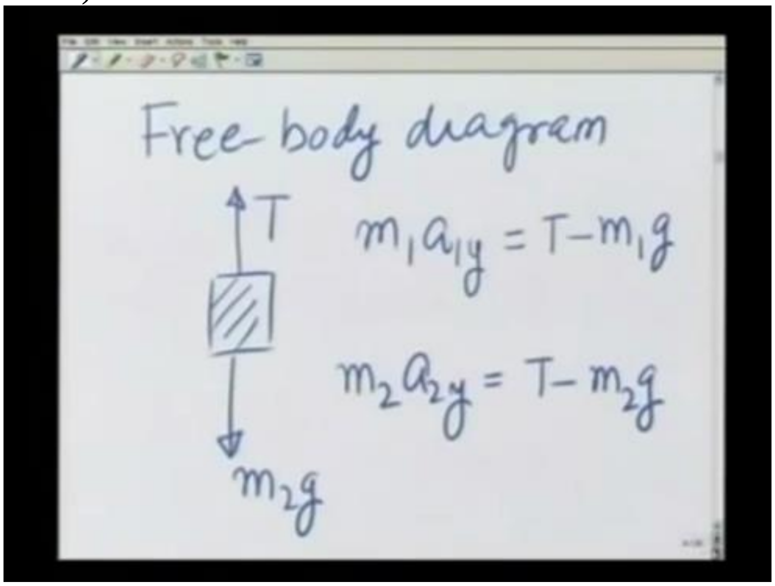

Similarly I make free body diagram for mass M2 that will also be similar M2G pulling the body down and tension T pulling it up. You may ask me at this point why did I keep the tension the same? The tension remains the same because the pulley is massless, frictionless and so and the rope is also massless and therefore it does not require any force to move that.

And that is what I said earlier, I had taken massless rope, massless pulley and fiction less pulley to keep things simple because I want focus on the constraint part of it. Now let us read the equations of motion. If you recall from the previous slide, the M1G and M2G are acting down and T is acting up on both the bodies.

So equation of motion is going to be M1A1 in Y direction is equal to T which is pulling it up - M1G. M2A2 in Y direction is equal to T - M2G. You may ask I am writing A1Y, A2Y. Which way is my Y and things like those? So let us be more precise.

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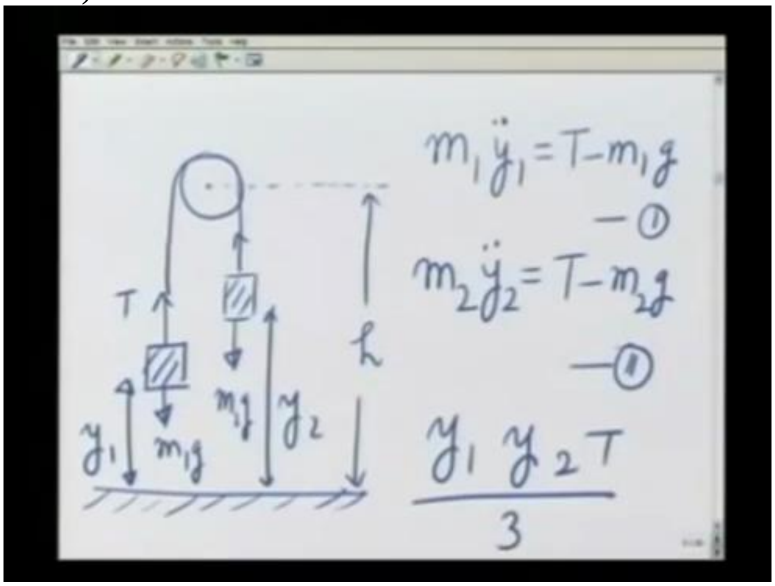

Let me measure my distances from the ground and be more precise because that will also help me later to write my constraint equation. Let this be the ground. Let the distance of the mass M1 be Y1 from here, let the distance of mass M2 be Y2 from here, let the Centre of this text fully be at height H from the ground. And now I can write, this is tension T pulling it up, M1G pulling it down.

Tension T pulling it up, - M2G pulling it down and therefore I can write M1Y1 double dot is equal to T - M1 G. Notice that all the directions and everything is correct. I must take care of that because we are dealing with vector quantities although I am writing this in one dimension. Similarly - M2Y2 double dot is going to be equal to T - M2G. I have gotten my 2 equations of motion but again, the number of variables are Y1, Y2, T, 3.

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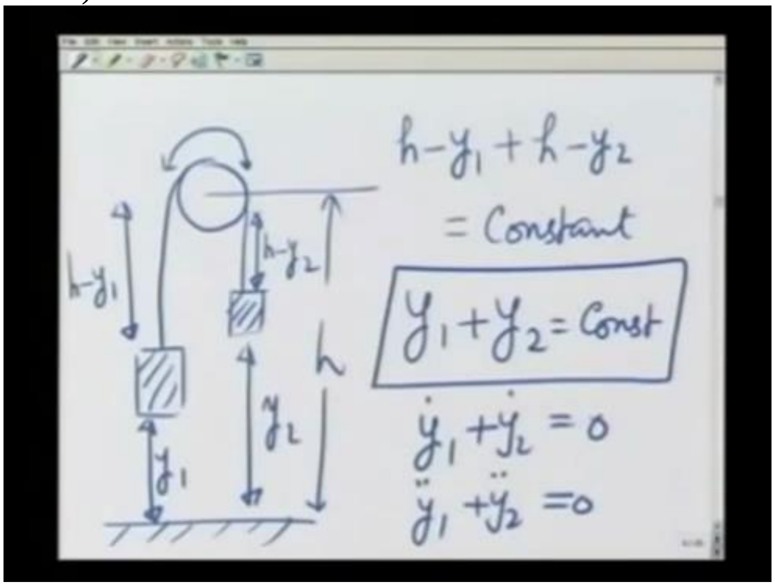

So one equation that is missing must be provided by the constraint itself and that is where this picture comes in handy. Recall again that the constraint that the 2 masses move together is provided by the fact that the rope is of fixed length. And I can express this mathematically as this part which is H - Y1 + this part which is H - Y2 + this part is of constant length. Since this part is always of constant length, I can write my constraint equation as H - Y1 + H - Y2 is equal to a constant.

Or shuffling terms around, I can write this as Y1 + Y2 is equal to a constant. This is my 3rd equation. To bring it to usable form in equations of motion, I differentiate it and then get Y1 dot + Y2 dot equals 0 and also Y1 double dot + Y2 double dot is equal to 0. So let us write the equations once more.

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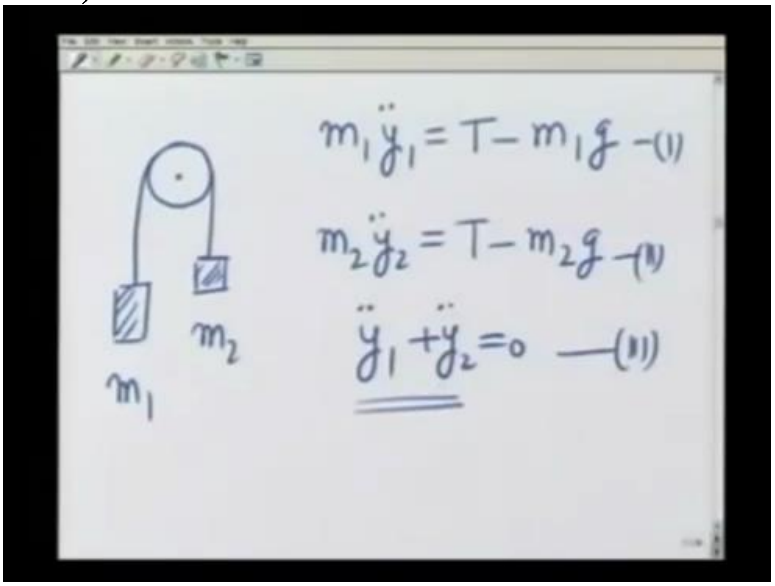

Again, I am solving this problem of 2 masses attached to a string which is passing over a fixed pulley. I have M1 Y1 double dot equals T - M1G, M2Y2 double dot equals T - M2G and Y1 double dot + Y2 double dot equals 0. I have 3 equations and 3 unknowns and I can solve my problem. Notice, this constraint equations just tells you that when one particle is moving up, the other one is going down. When one particle is accelerating up, the other particle is accelerating down. Let us now solve these equations. Let us eliminate T 1^{st}. So for that, I subtract equation 2 from equation 1.

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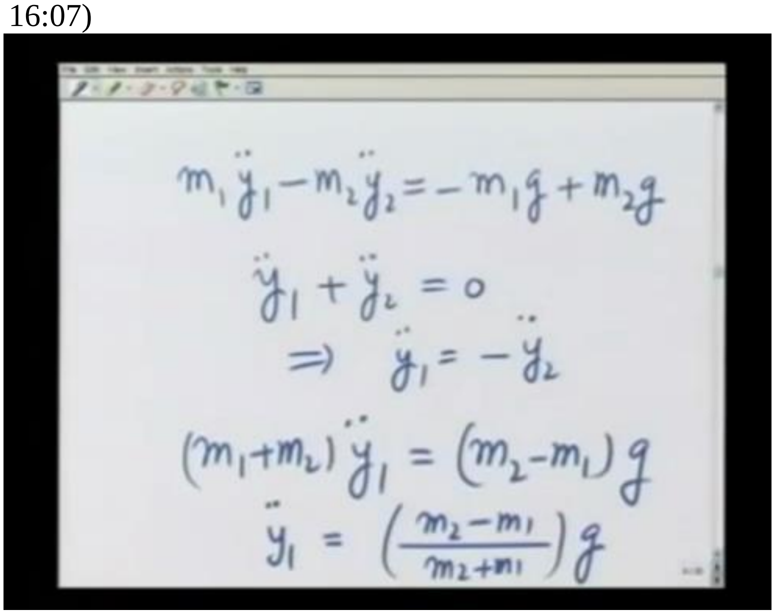

In doing so, I get M1Y1 double dot M2Y2 double dot is equal to M1 - M1G + M2G. Since I know that Y1 double dot + Y2 double dot is equal to 0, this implies, Y1 double dot equals - Y2 double dot. If I substitute, I get M1 + M2 Y1 double dot equals M2 - M1G or Y1 double dot equals M2 - M1 over M2 + M1 G, a result we already know.

If M2 is heavier than M1, then Y1 double dot is up and that is precisely what I am getting. You see, I did not make any assumption about M1 or M2 being lighter or heavier and things like those. Results automatically popped up. Through these examples, what I have tried to demonstrate to you that constraint in a particle or a body's motion can because either by external agencies as we did in the case of a particle moving on a wire.

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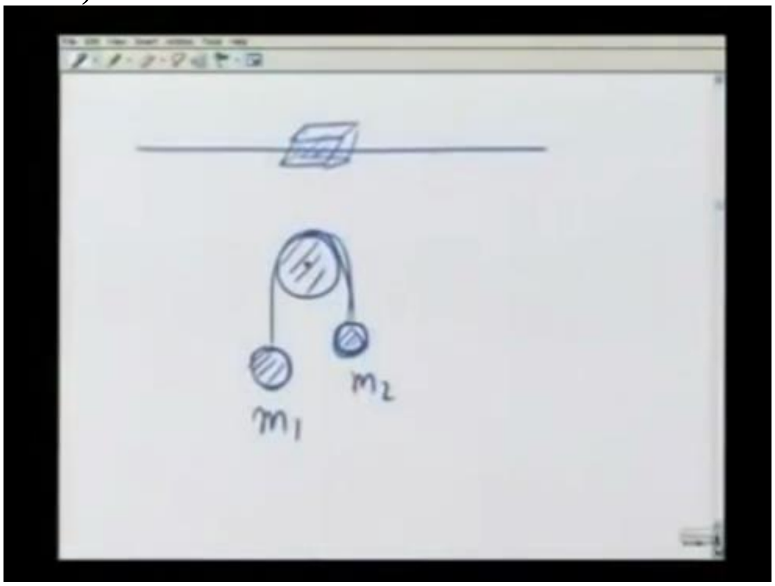

The constraint was caused by this wire. Or by other body itself. That is in this case, this mass M1 and M2 when they were connected by this rope this was providing the tension but the presence of one mass restricted the motion of the other mass. And also when the solve these 2 simple problems that you already knew the answers of, we also try to develop step wise strategies to solve motions of different bodies when they are moving under constraints. Let us try to write them.

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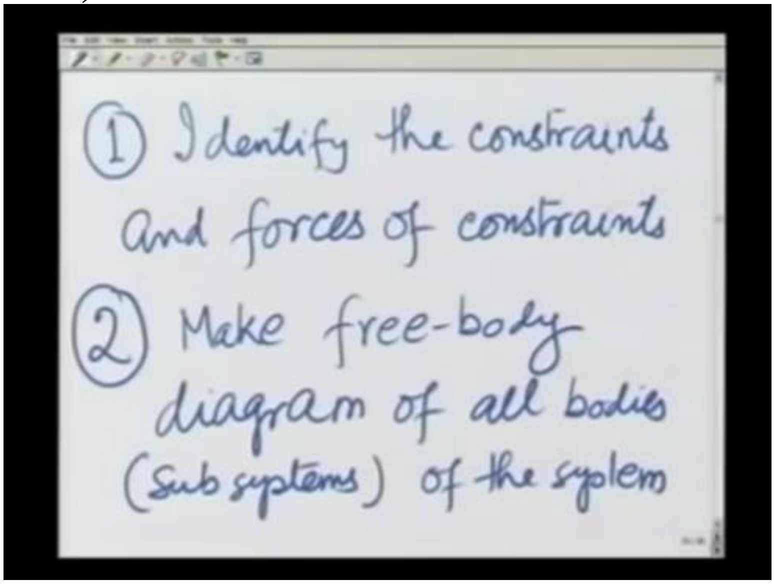

So 1st step in solving a problem is going to be, identify the constraints and forces of constraints. Once I have identified the constraints and forces of constraints, the 2nd step is make free body diagram of all bodies and these bodies which are treated separately or also called subsystems of the system. Let me remind you what a free body diagram is.

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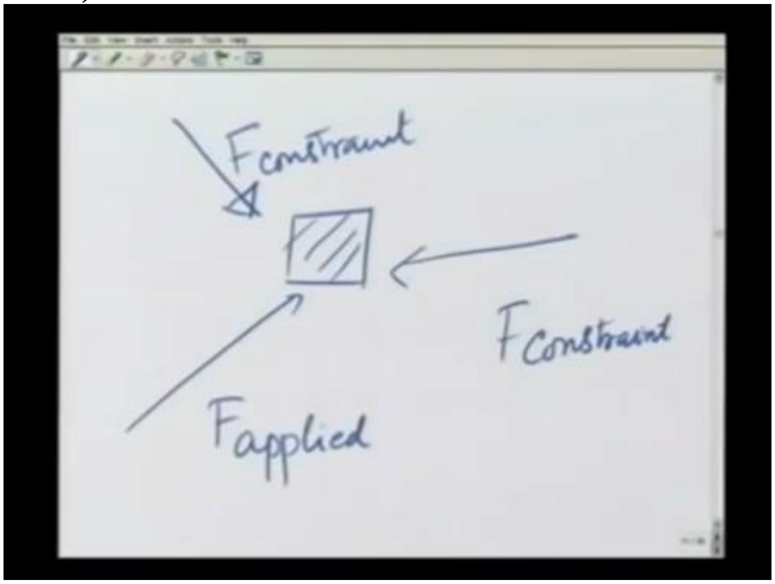

When I make a free body diagram for a body, I make that body freely and consider all the forces that are being applied from outside and the constrained forces on it. There may be more than one constrained force. And that is my free body diagram.

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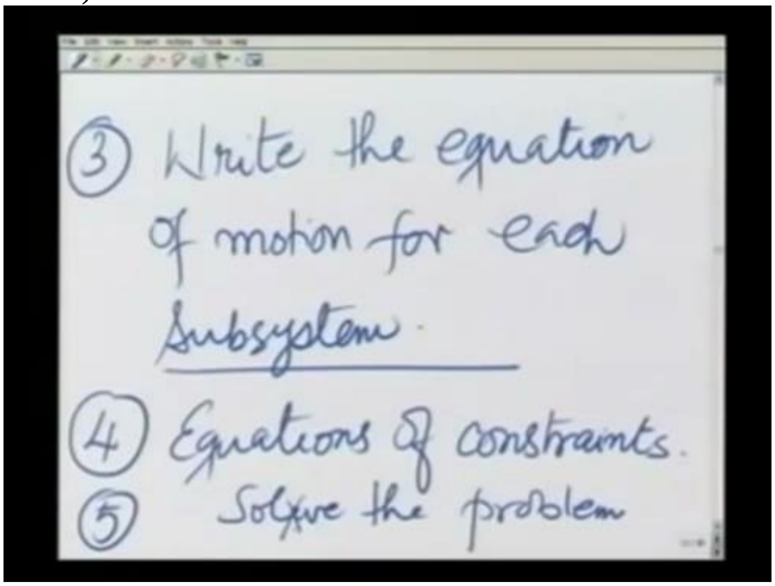

The 3rd step is going to be then, write the equations of motion for each subsystem. At this stage, you will see that the number of variables in the problem is more than the number of equations that you have written. And therefore, there are some missing equations and these equations are going to be provided when you write the equations of constraint constraints because there may be more than one.

Still if you have the equations do not match, either you have missed something or you have misformulated the problem, go back and do it all over again. And once the equations and number of variables match, the last step is, solve the problem.