Engineering Mechanics

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Module 5
Lecture No 44
Using planar polar, cylindrical and spherical coordinate systems: solved examples

In this lecture, I will solve some problems during principle planar polar coordinates and one problem in spherical polar coordinates to make you familiar with the ideas of how to use this coordinate system.

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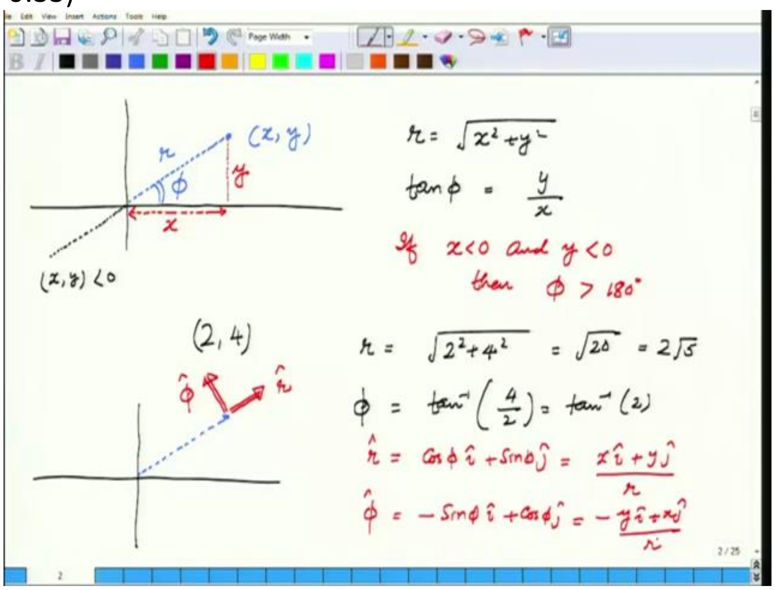

The 1st problem is to give you a visualisation of one transforms between the Cartesian coordinate system and the planar polar coordinate system. The question is given the planar polar coordinates for a particle whose position in the Cartesian coordinates is given as one, 2, 4. 2nd one, 3, 6. 3rd one - 4, - 2. In addition, we would also like to express the unit vectors R and phi at these positions in terms of vectors I and J.

So let us take the 1st one where we have X equals 2 and Y equals. 1st let me make a general statement that when I am dealing with planar polar coordinates, is that the point here, X and Y, the radial coordinate is given by this distance from the origin which we call R and the angular displacement is given by this phi.

So we can easily see, if I make the X and Y distances that this red line horizontally is X and this is Y and therefore I am going to have R is equal to square root of X square + white square always and tangent of phi is equal to Y over X. Keep in mind, if I go to the other side, that is in the 3rd quadrant, both X and Y are less than 0. Therefore tangent phi is going to come out to be positive again.

So we should also write here, if X is less than 0 and Y is also less than 0, then phi is greater than 180 degrees although tangent phi will come out to be positive. So let us now solve the 1st problem where we are given the Cartesian coordinates as X and Y and therefore R is going to be square root of 2 square + 4 square and that is square root of 20 which is 2 root of 5.

And phi is going to be tan inverse 4 over 2 which is tan inverse of 2. How about the unit vectors? Let me again make this system here and this is R at a given XY. The unit vector in R direction is going to be this and perpendicular to it is unit vector phi. And we have seen in the lectures where we set up these coordinate systems that R is nothing but cosine of phi I + sine of phi J which can also be written as XI + YJ over R.

And phi unit vector is - sine of phi I + cosine of phi J which can also be written as - YI + XJ over R.

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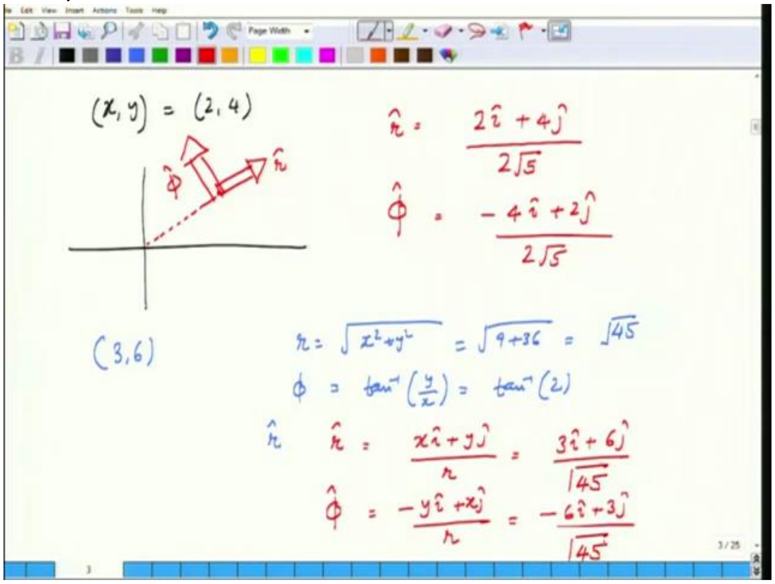

And therefore when I write for X, Y equals 2, 4, the corresponding unit vectors at that position, so I will again make it and here is R unit vector and here is phi unit vector. I have R is equal to 2I + 4J over we have already determined what R is, 2 root 5. And similarly I have, phi unit vector is equal to - 4I + 2J over 2 root 5. That is part 1.

In the 2nd part, we have taken the coordinates to be 3 and 6. Right? Which is both positive and now I can directly write that R is going to be square root of X square + Y square which is square root of 9 + 36 which is square root of 45. + phi is going to be tangent inverse of Y over X which is equal to tangent inverse of 2.

So phi remainS the same as in the previous question but R changes. R unit vector let me write it in red. R unit vector is going to be XI + YJ over R which comes out to be in this case to be 3I + 6J over square root of 45. And phi unit vector is going to be - YI + XJ over R which is equal to - 6I + 3J over square root of 45.

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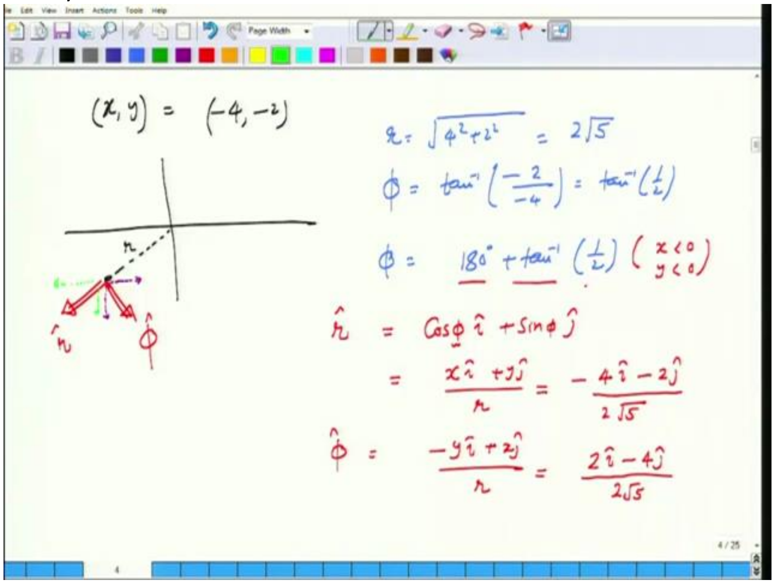

The final question in this X, Y is went to be - 4 and - 2. So this point is actually in the 3rd quadrant out here. And therefore, the distance R is the square root 4 square + 2 square which is again 2 root of 5. Phi is equal to tan inverse of - 2 over - 4 which is tan inverse half and that is going to give you an angle which is less than 90 degrees. In this case phi is actually going to be 180 degrees + tan inverse 1 over 2.

Because let me write in a bracket X is less than 0 and so is Y. How about the unit vectors? R unit vector now is going to be in this direction and phi unit vector in this direction. This is the phi unit vector and R unit vector. This is where now it becomes important that we realise that phi is 180 + tan inverse 1 over 2.

This if I were to write this as cosine of phi I + sine of phi J, then this phi being greater than 180 degrees will be important. On the other hand if I write it directly as XI + YJ over R, then that sign is automatically taken care of and this would turn out to be - 4I - 2J over 2 square root of 5. And phi unit vector is going to be - YY express XJ over R and that is going to be 2I + - 4J over 2 root 5.

You can see that the X component of phi unit vector is in the positive direction and Y component is in the negative direction. And sorry R vector, X component is in the negative direction and so is the Y vector. So we are being all consistent.

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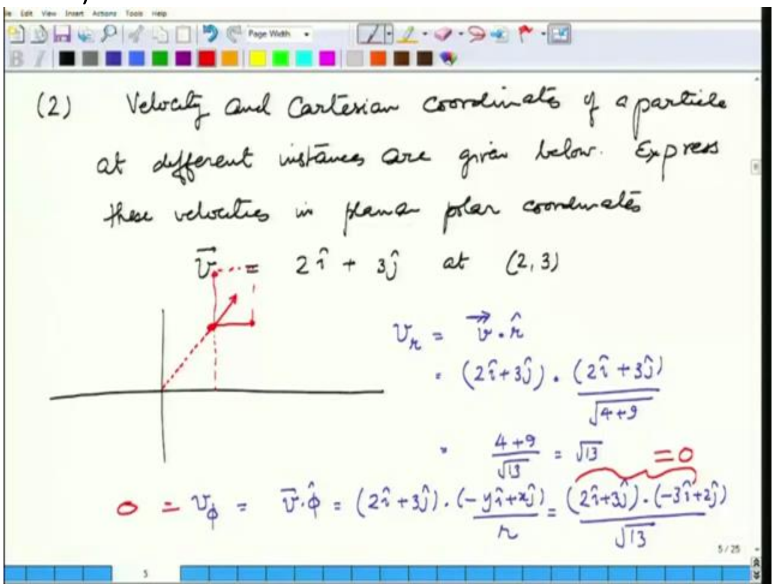

Next problem is, velocity and Cartesian coordinates of a particle at different instances are given below. Express these velocities in planar polar coordinates. So one velocity which is given is V equals 2I + 3J at 2 and 3. So let us see what it means. It means I am at the coordinate 2 and 3 and the velocity X component is proportional to X and Y is also proportional to Y. So this direction is actually going to be same as the R unit vector.

But let us do it systematically. I just gave you a picture where we can actually make out that it is in the R unit direction but in general that will not be the case and that is where the systematics come into the picture. So let us calculate what the radial component of the velocity is. Radial component is going to be V. Oh sorry V vector dot R which is going to be equal to 2I + 3J.

This is the velocity given dot R unit vector. Since the particle is at 2, 3, it is going to be 2I + 3J divided by R which is the square root of 4 + 9. So this is going to be 4 + 9 over root 13 which is root 13. So VR is root 13. How about V phi, the phi component, the tangential component of the velocity?

The tangential component is going to be V dot phi unit vector which is going to be 2I + 3J dotted with the phi unit vector which is - YI + XJ divided by R. And here I can write this as 2R + 3J dotted with - 3I + 2J divided by R which is square root of 13. You can already see that this dot product is 0. And therefore V5 comes out to be 0. This is precisely what we suspected earlier.

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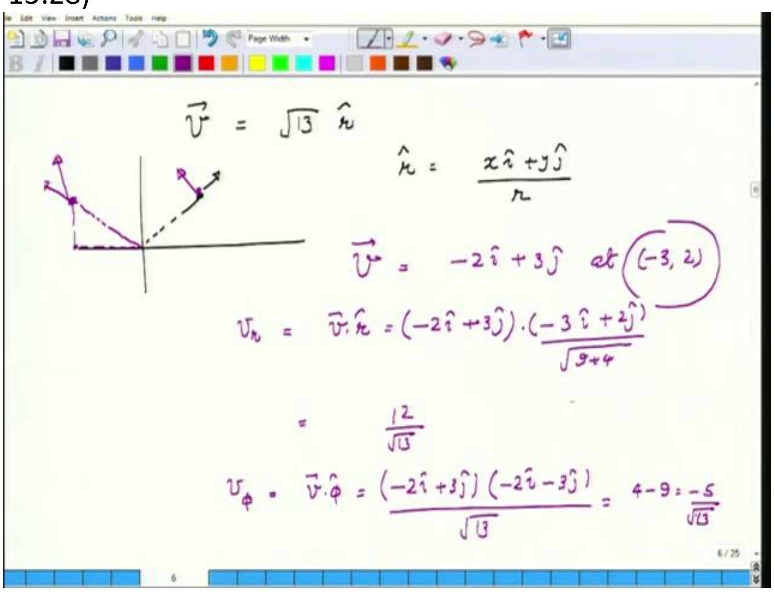

The velocity in this case is radial and therefore it can be written as V equals root 13 R unit vector. Keep in mind that at this instant the particle is moving in the radial direction but the radial unit vector keeps on changing as the particle changes position. I emphasise again that R unit vector is XI + YJ over R and therefore it depends on X and Y. It is not a fixed unit vector in space.

As the 2nd example of this, let me take V equals - 2I + 3J and that means at this point, the particle is moving somewhat like this as shown in purple colour and VR and this is at - 3 and 2. So even the position has changed, I am taking the position - 3 and 2. So somewhere here and the particle is moving - 2 and 3 here, something like this.

So now you see, there is an angle between the radial unit vector and the tangential unit vector and therefore V is going to have both the components. Let us calculate VR which is going to be V dot R unit vector which is - 2I + 3J. This is the velocity dot unit vector is going to be - 3I + 2J divided by R which is square root of 9 + 4. So this comes out to be 2 times 36, 3 times 26, 12 over square root of 13.

So VR is this. How about V phi? V phi is going to be V dot phi which is equal to - 2I + 3J times phi unit vector which is - YI. So - YI is going to be - 2 I + XJ which is - 3J divided by square root of 13. So this is going to be equal to 4 - 9 which is - 5 over root of 13. All right? So let us go to the next page and write it again.

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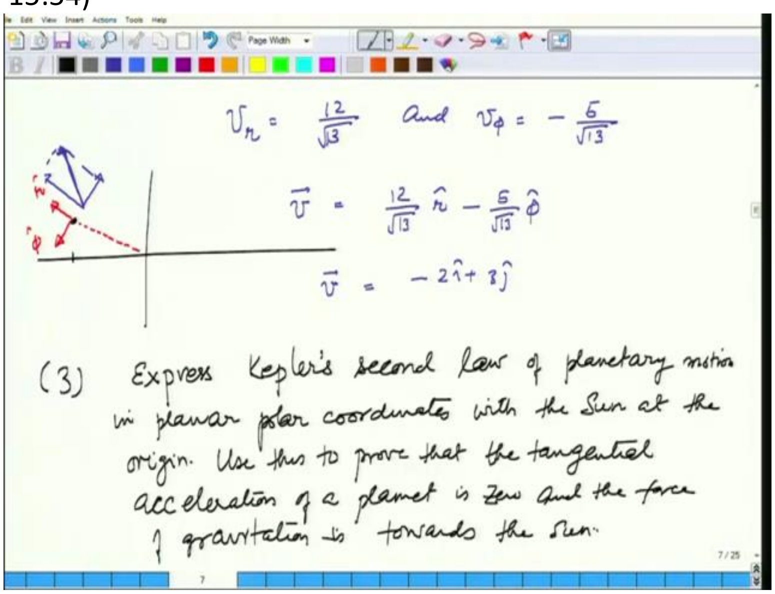

We got VR is equal to 12 over 13, 12 over square root of 13 and V phi is equal to -5 over square root of 13. And the particle is at position - 3 and 2. - 3 and 2. Somewhere here. At this point, let me again show that the unit vectors R unit vector is like this and phi unit vector is perpendicular to this in this direction. What we find is that VR is in the same direction as the R unit vector. It is positive.

V phi however is negative. This is V phi and this is how the particle is moving. So I can read the velocity in planar polar coordinates as 12 over root 13R at this point - 5 over root 13 phi. At this position, as the position has changed, R and phi unit vectors change. Right? If the same velocity, same vector V which is - 2I + 3J was at some other position, VR and V phi will change.

Let me now take the 3rd problem. Express kepler's 2nd law, this is also known as the law of area of planetary motion in planar polar coordinates with the sun at the origin. Use this to prove that the tangential acceleration of a planet is 0 and the force of gravitation to understood between the Sun and the planet is towards the sun. Okay?

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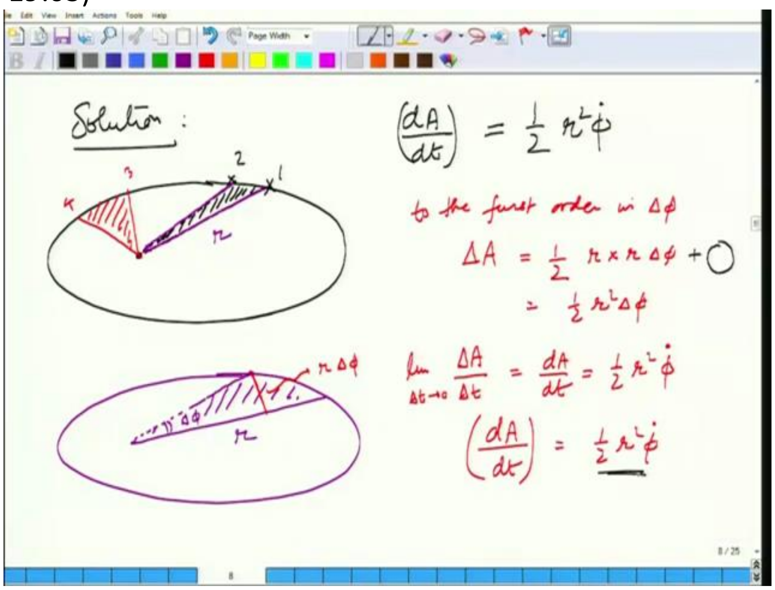

So solution. What is Kepler's 2nd law? Kepler's 2nd law says that when the planets when they move around in an orbit with the sun at the origin, the rate of change the rate of sweeping area sweep is constant. That means at some instant, the planet moves from position 1 to position 2, it sweeps some area in some certain time, or unit time suppose it sweeps this area shown by shaded black region.

And in some other time, it goes from position 3 to position 4 in unit time and covers this much area. These 2 areas covered per unit time are going to be the same. That is Kepler's 2nd law. And let us express this in planar polar coordinates. So how do I express that? You see this distance here is R and if I treat this as a triangle in a very small, short time, this I made for unit time.

But let us take a short time and make it exaggerated. So this is a planet orbit. This is R and in a very short time delta T it comes to this position here. This time is very short. So it covers a very small angle delta phi. If I were to calculate this area of the shaded region, let me drop a perpendicular here, this becomes the base and keep in mind that delta phi is very small, this base distance is going to be R delta phi.

And therefore to the 1st order in delta phi the area delta A is going to be one half of R times R delta phi which is one half R square delta phi. So this is the expression for area, infinitesimal area in planar polar coordinates. So we learn that and therefore delta A over delta T with limit delta T going to 0 is going to be DA DT comes out to be one half R square delta phi by delta T limit delta T going to 0 phi dot.

So DA DT is going to be equal to one half R square phi dot. Let me write that here. So DADT is one half R square phi dot. Now you see why I neglected the 2nd order. If I included the 2nd order here, if I take delta T going to 0, 2nd order is going to vanish because delta phi delta T divide by delta T is going to be finite but delta phi will go to 0. So this is a rate of change of area.

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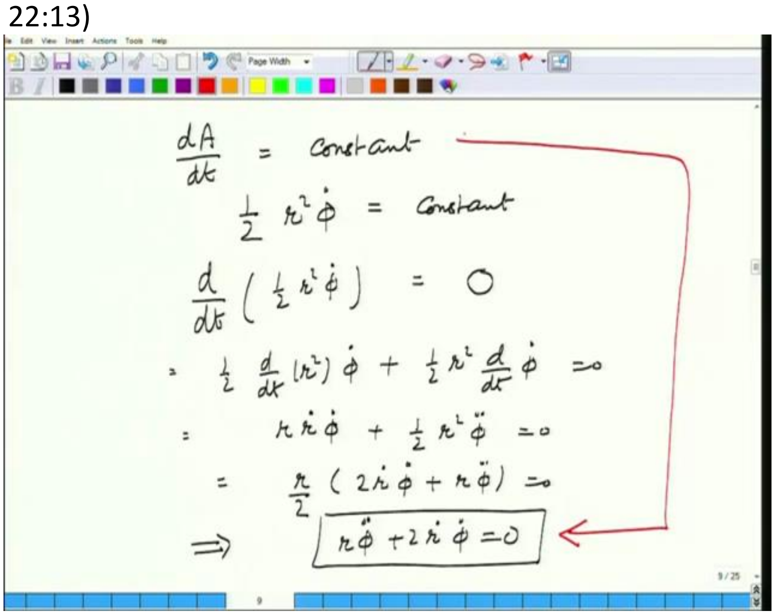

And what Kepler's 2nd law says that DADT is a constant which means that one half R square phi dot is a constant. And therefore if I take the time derivative of this quantity one half R square phi dot, since this is a constant, it is going to be 0. Let us now take the derivative. It is going to be one half D by DT of R square phi dot + one half R square D by DT of phi dot and this is 0 which gives me RR dot phi dot + one half R square phi double dot is equal to 0.

If I take R common, I can write this as R or R by 2 common, 2R dot phi dot + R phi double dot is equal to 0. And as long as R is not zero, this immediately implies that R phi double dot + 2R dot phi dot is equal to 0. Notice that this is the expression for tangential component of the acceleration. So Kepler's 2^{nd} law is equivalent to tangential acceleration being 0.

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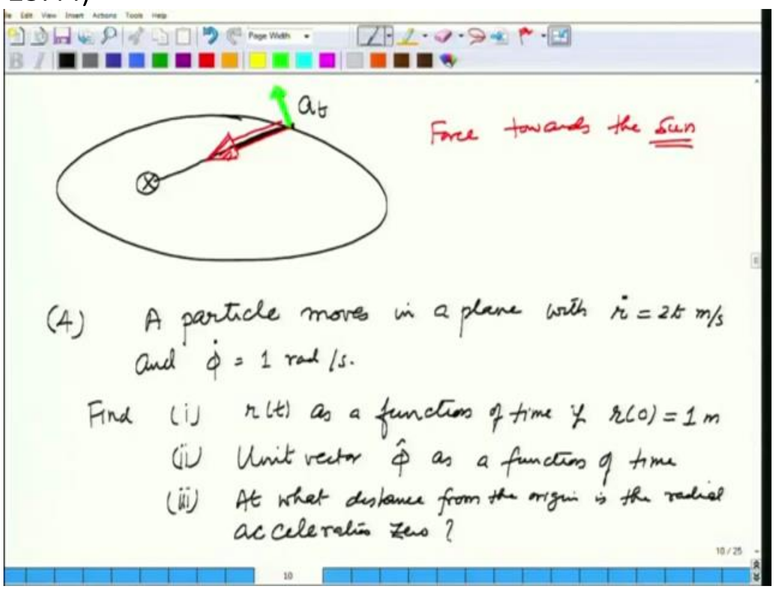

Let us see in figures. This is the planet which is going on. This is the sun. Tangential acceleration means acceleration perpendicular to the radius. Okay? Keep this in mind. This is A tangential. It is not along the orbit. It is perpendicular to the orbit. This isolation is 0 and therefore only acceleration this planet has I show it in red, is going to be in direction towards the sun, towards the radius.

So this immediately shows that the force between the sun and planet is towards the sun. So you see how important Kepler's law is and when we analyse it in using planar polar coordinates how immediately it gives us information. I again emphasise, keep in mind, the tangential acceleration is not along the orbit but perpendicular to the radius.

I have shown it in green here. Okay. Problem number 4. Problem number 4, it says a particle moves in a plane with R dot equals 2T metres per second and phi dot equals 1 radian per second. Find 1, RT as a function of time if R0 that is R time T equal to 0 is 1 metre.

So that means it started at 1 metre. And 2, unit vector phi for the particle position obviously as a function of time. And 3rd, at what distance from the origin is the radial acceleration 0? So let us do this.

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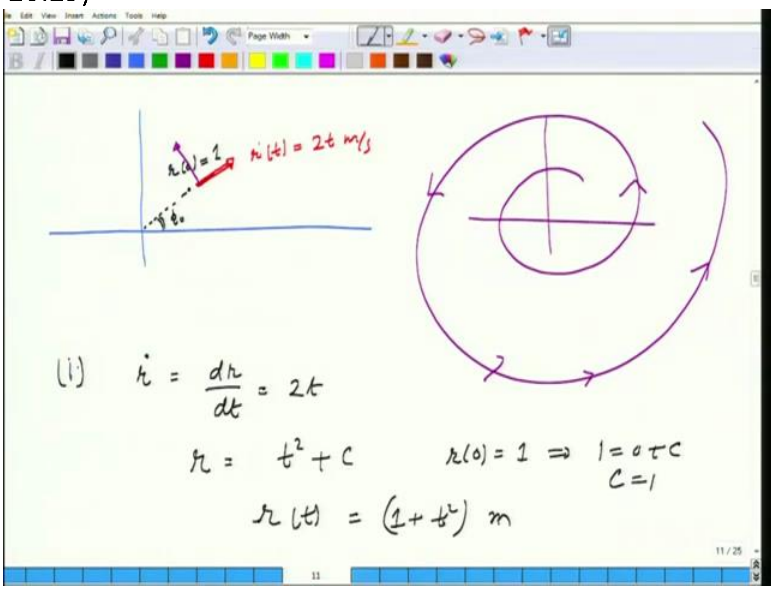

We have been given so let me just 1st make this picture which always tells you a lot of things. A particle is moving out which started at T equal to 0 at some distance R equal to 0 equals 1 metre. It is not given what phi position was. So we will take some arbitrary phi position which is let us say phi 0. And it moves out. Let me make it in red. It moves out at a speed of R dot T equals 2T metres per second.

So as time passes, it moves out faster and faster. At the same time, it is moving in the phi direction with a constant angular speed. And therefore the particle will start going out like this. This will be the trajectory. Now I what I want to find is one, R as a function of time. So I am given R dot which is equal to DR by DT is equal to 2T.

And therefore R is going to be equal to T square + some constant. I am given R0 is equal to 1 which immediately gives me that 1 equals 0 + C or C equals 1. So R as a function of time is given as 1 + T square metres. That is part 1.

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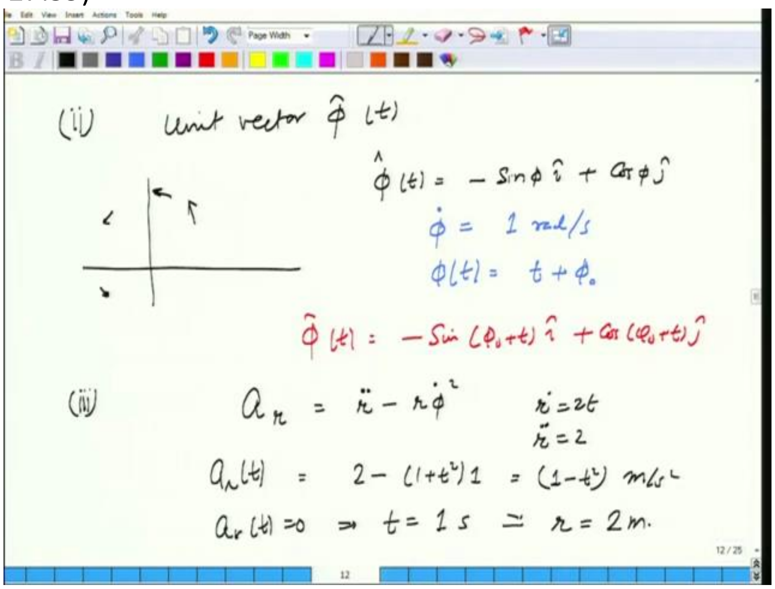

Part 2 says find unit vector phi as a function of time. Remember again, as the particle is moving, it is phi unit vector keeps on changing with its position. And phi unit vector depends only on phi. So let me write this. Phi unit vector at time T is given as - sine of phi I + cosine of phi J. And what I have been given is that phi dot is 1 rad per second.

So phi T is going to be T + sine phi not because the phi at T equal to 0. And therefore phi unit vector as a function of time is going to be - sine of phi 0 + TI + cosine of phi 0 + TJ. 3rd part, I want to find the distance at which the radial acceleration is 0. Now recall that radial acceleration is given as R double dot - R phi dot square.

I am given that R dot is 2T and therefore R double dot is 2. And I can therefore write AR as a function of time is equal to 2 - R is given as 1 + T square which I have already calculated and phi dot square is 1. So this is given as 1 - T square metres per second square. And ART is equal to 0 implies T equals 1 second and therefore R is equal to 2 metres. That is the answer.

So what we have done in this lecture is called some illustrative problems as to show how planar polar coordinates are to be used. Eventually, we do not use I and J but use these planar polar coordinates directly.

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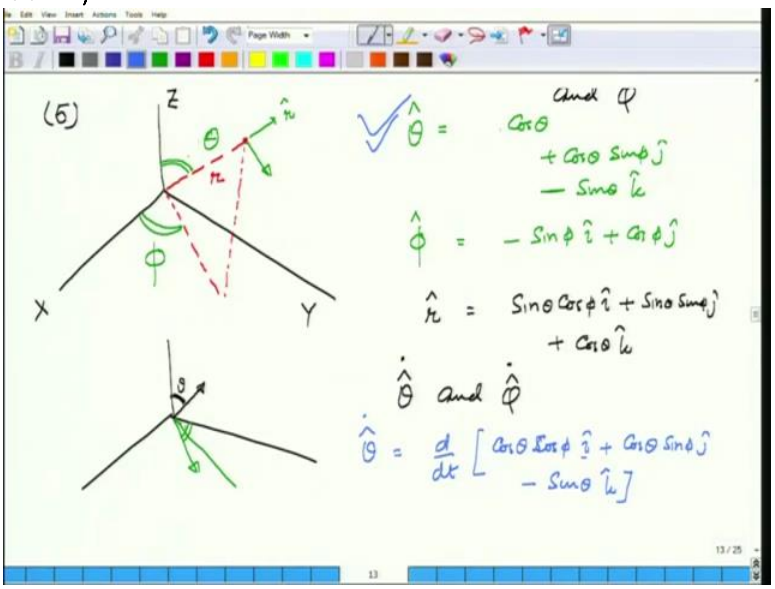

Finally, let me do one problem in spherical polar coordinates and that is, if I take a spherical polar coordinates if we recall from the lecture, what are these? This is X, Y and Z. So if I am given a point, spherical polar coordinates give me distance R from the origin. Then the position of this vector from the Z axis is specified by giving an angle theta and position in the XY plane of this point is given by this angle phi.

The unit vector theta is going to be in the direction perpendicular to unit vector R which is towards the radius and phi is in the XY plane again. So theta unit vector is given as if I were to make it again, going down the plane making an angle theta from the XY plane. So in the XY plane, theta unit vector has a component cosine of theta.

Then its X component is going to be cosine of phi I + cosine of theta. Y component is going to be sine phi J and - sine theta K. Phi unit vector anyway is in the direction in the plane just like in the planar polar coordinates. So it is going to be given as - sine of phi I + cosine of phi J.

And the radial vector that is this vector has a component cosine theta in the Z direction and sine theta in the XY plane. So sine theta cosine phi I + sine theta sine phi J + cosine theta of K. And what I want to do when this problem is find theta unit vector dot and phi unit vector dot. To calculate theta unit vector dot, let me differentiate the expression given here.

I am showing you by taking it. This is going to be D by DT of cosine of theta cosine of phi I, writing these vectors in terms of Cartesian unit vectors because Cartesian unit vectors are fixed and therefore I do not have to differentiate them with time. + cosine of theta sine of phi J - sine of theta K. Let me differentiate them.

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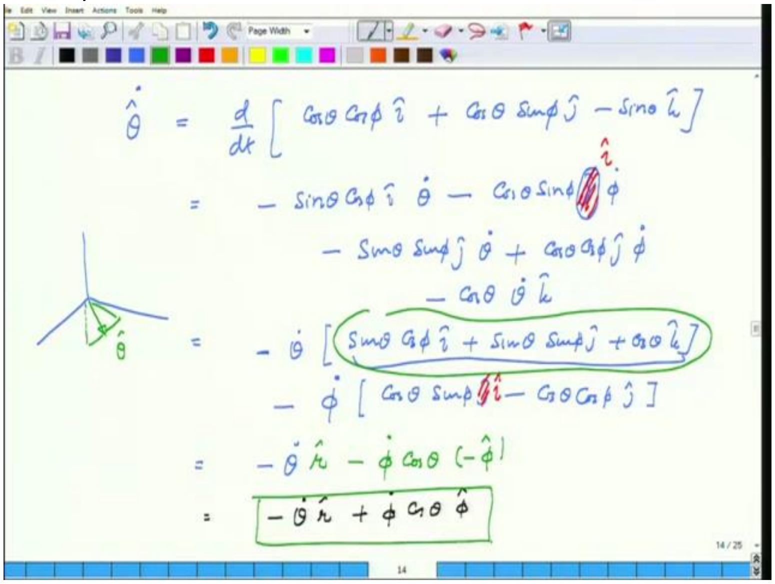

When I do that, I get theta unit vector dot is equal to let me write this again, D by DT of cosine of theta cosine of phi I + cosine of theta sine of phi J - sine of theta K, this comes out to be - sine theta cosine phi I theta dot - cosine theta sine of phi J phi dot - sine of theta sine of phi J theta dot + cosine of theta cosine of phi J phi dot - cosine of theta theta dot K.

If I collect all the theta dot terms, this I can write as theta dot with a - sign, sine theta cosine phi I + sine theta sine phi J + cosine theta K - let me also collect phi dot terms. - phi dot, in the bracket cosine theta sine phi J - cosine theta cosine phi this one mistake here. I am correcting it with red, was I. So this is going to be I.

So what I get is - theta dot and you can recognise this unit vector here, let me encircle this with green, is nothing but I unit vector - phi dot. Cosine theta comes out and sine phi I - cosine phi J is nothing but - phi unit vector. So I can write this whole expression as - theta dot R unit vector + phi dot cosine theta in the phi direction.

Pictorially, if I look at it, this is my theta unit vector. As theta changes, it changes like this keeping the phi the same. So it moves, it has a component in the - R direction and if phi changes, it moves like this. So it has a component in the positive phi direction and this is what this expression reflects. How about phi unit vector?

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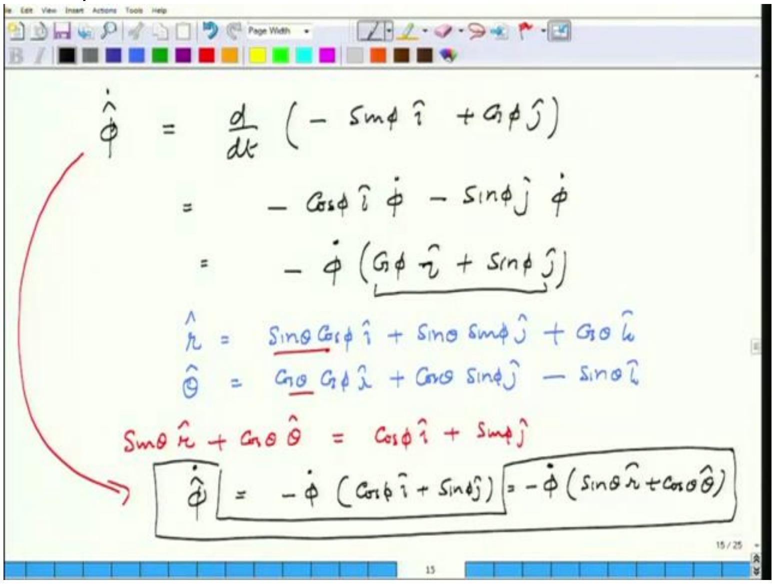

Phi unit vector is nothing but D by DT of - sine phi I + cosine phi J. So this is going to be - cosine of phi I phi dot - sine phi J and phi dot which I can write as - phi dot cosine of phi I + sine of phi J. Now I want to bring this in the form of a combination of R unit vector and theta unit vector.

So recall again R unit vector was sine theta cosine phi I + sine theta sine phi J + cosine theta K. And theta unit vector was cosine of theta cosine of phi I + cosine of theta sine phi J - sine of theta K. Notice, if I take sine of theta, multiply this by R unit vector, add to it cosine of theta, theta unit vector, what will I do? 1st, this term here, I will get sine square theta + cosine square theta.

Cosine phi comes out common. So this will become cosine phi I + 2nd term again will be sine square theta + cosine square theta which is 1 and I get sine phi J and the last term cancels. So therefore using this I can write phi unit vector dot is equal to - phi dot cosine phi I + sine phi J is equal to - phi dot sine theta R + cosine theta theta.

Note is that now we have written all these in terms of R, theta and phi and no reference to Cartesian unit vectors.