Engineering Mechanics
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Module 5
Lecture No 41
Motion of a particle in a plane in terms of planar polar coordinates

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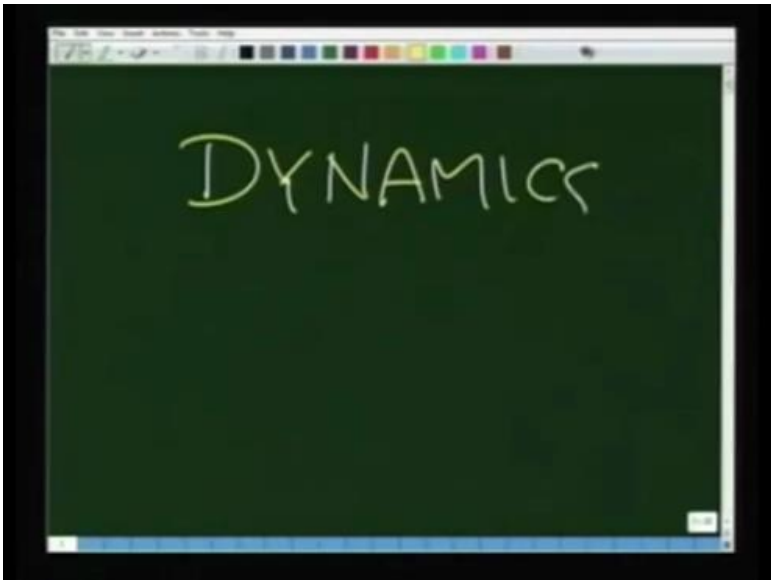

We have been dealing with statics. This is the 1st lecture in dynamics. So we start with how to describe the motion of particle using different coordinate systems.

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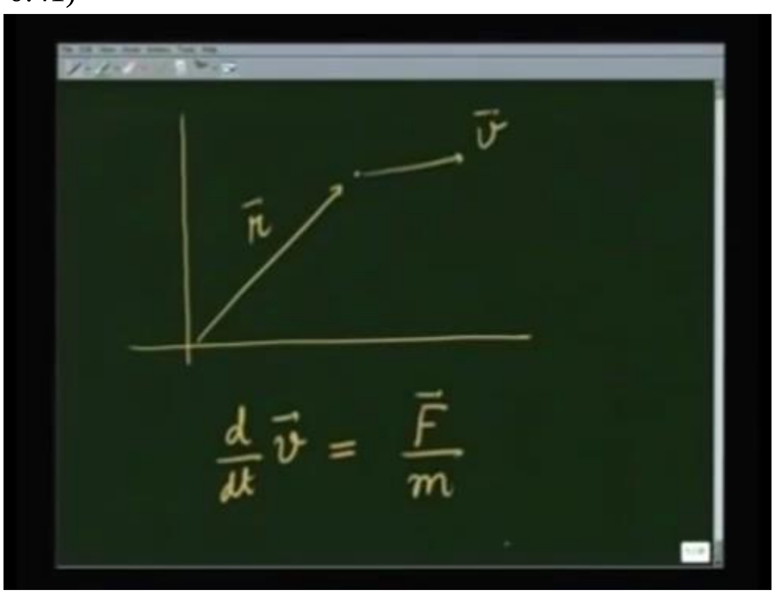

As you well know in mechanics, if I want to describe the motion of a particle, it is prescribed using its position given by vector R and its velocity. With time, the velocity changes and the velocity change D over DT is related to the force applied F divided by mass of the particle by Newton's 2nd law. So to describe the vectors, I need to fix a coordinate system with respect to which I describe the motion.

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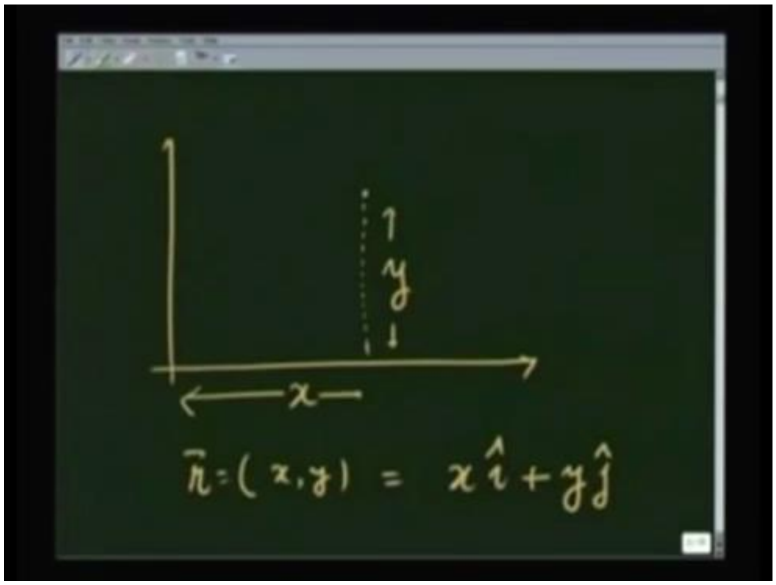

For example, you are well familiar with the Cartesian coordinate system where in a plane, the position of a particle is given by X coordinate and the Y coordinate. So that I would write that R in this plane is given by X, Y or the vector X unit vector I in X direction + Y unit vector J in Y direction.

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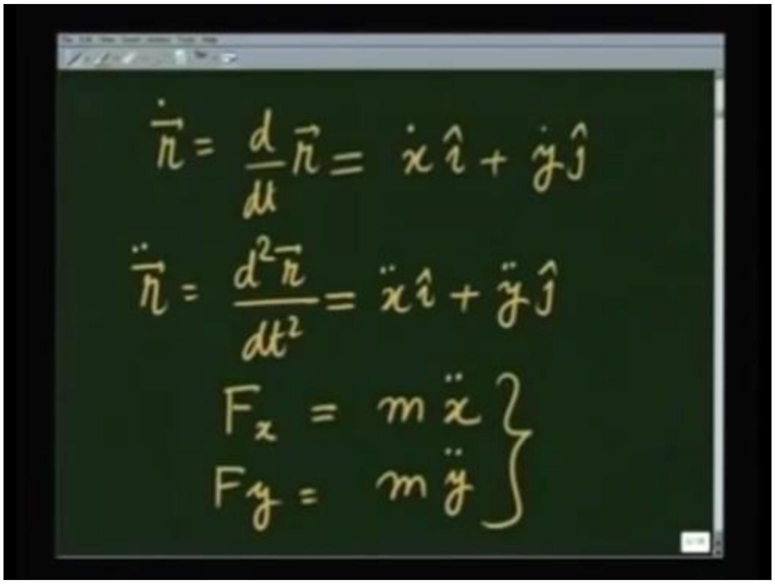

The velocity accordingly is given as R dot, this dot is given to indicate the time derivative with respect to the derivative with respect to time which is equal to X dot I + Y dot J. Similarly the acceleration R double dot is given as D2 R over DT Square which is equal to X double dot I + Y double dot J.

And the Newton's 2^{nd} law then takes the form that I equate the components along each direction with respect to forces so that Fx becomes equal to MX double dot, FY becomes equal to MY double dot. And I can integrate these 2 equations of motion to get how X and Y change with time. However, there may be situations where describing the motion in X and Y coordinates may not be as easy as we think.

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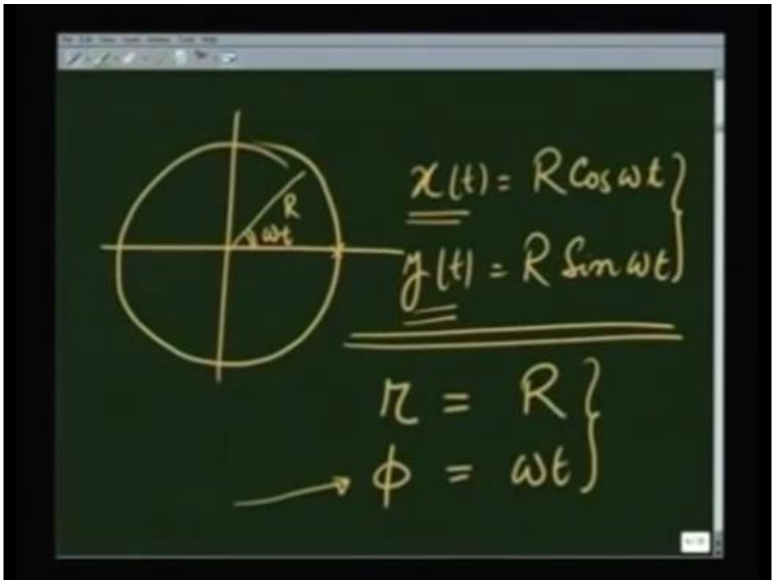

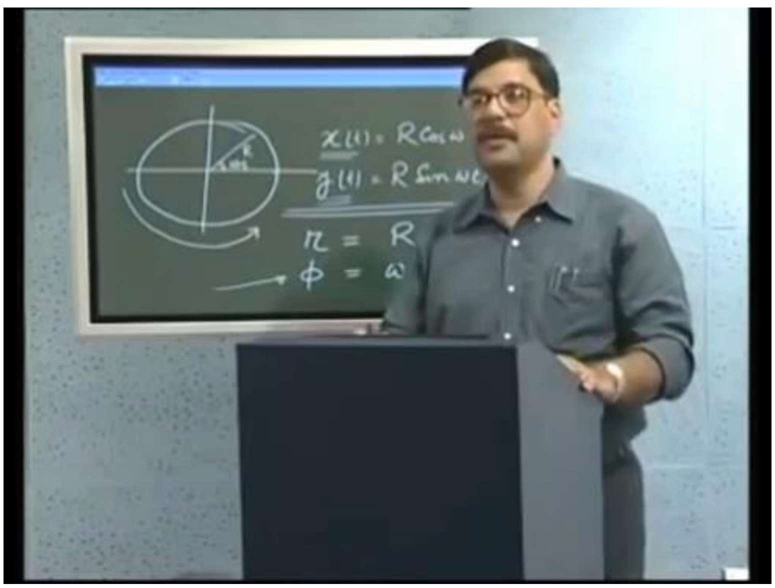

For example, if there is a particle moving in a circle, let us say of radius R with angular velocity omega so that this angle is omega T if the particle started from here, then it is X coordinate with time would be given as R cosine of omega T and the Y coordinate will be given as R sine of omega T. Both are functions of time. On the other hand, I could very well write this position as the radius of the circle being constant with time equal to R and the angle of the radius vector to be equal to omega T.

You appreciate the difference between the 2 descriptions. One in terms of X and Y coordinates and the other in terms of R and phi coordinates. In this case, both X and Y are functions of time. Whereas in this case, only phi is the function of time. So that effectively the change of coordinate with time is in only one coordinate. This is known as planar polar coordinate system and we are going to explore it further. As you can well imagine, this is most useful system and a particle is making rotatory motion or moving around in a circular manner.

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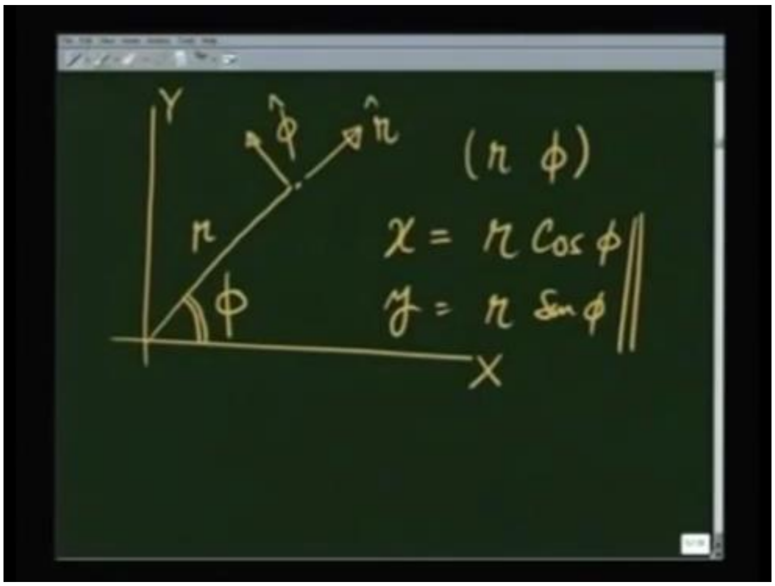

So let us start with the description of planar polar coordinate system. In this system, if a particle is given at a point, distance R from the origin and the radius vector R is making angle phi from the X axis, I am going to describe this as position R and phi. As you can see, I am using 2 coordinates in a plane. Obviously, X coordinate is given as R cosine of phi and the Y coordinate is given as R sine of phi. I also need to give the unit vectors in this coordinate system if I want to describe their vector quantity.

The unit vectors are given as R unit vector in this direction and the other unit vector is in increasing phi direction perpendicular to R. So it is in this direction which I will call unit vector phi.

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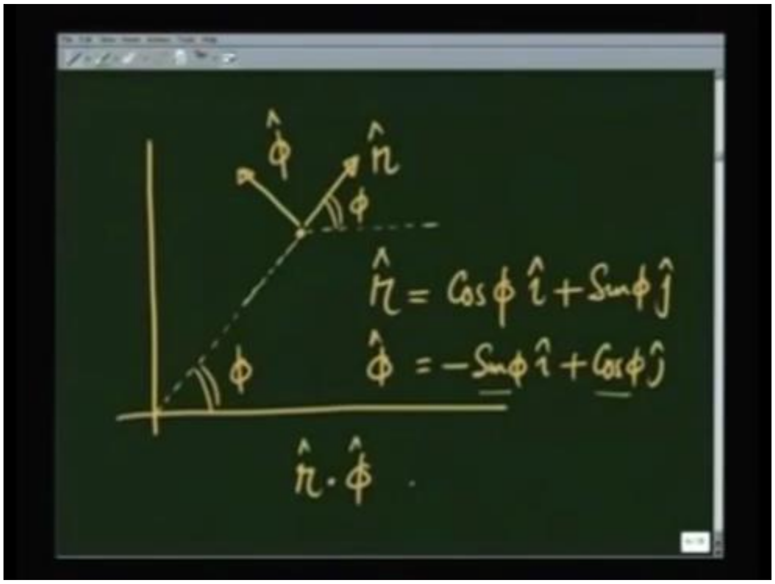

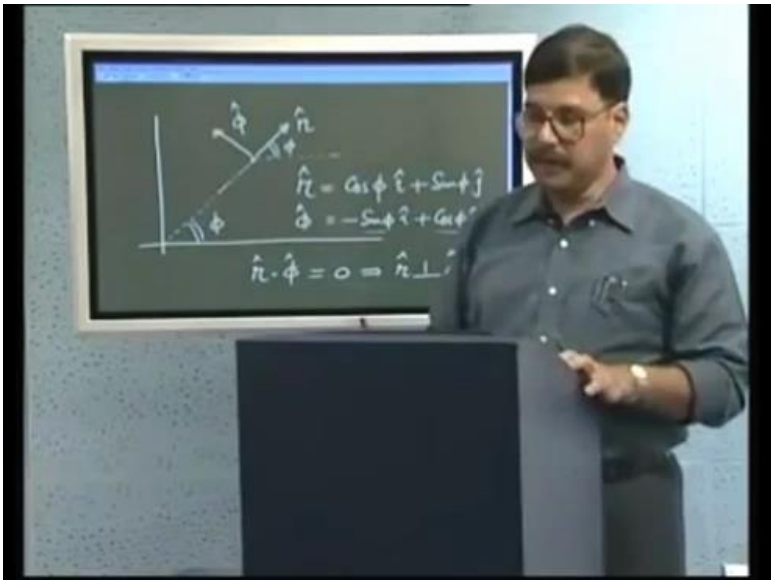

To make it more clear, let me redraw it and show you these vectors more clearly. So for a particle here, I have the unit vector R and unit vector phi. This angle is phi and so is this. You can already see that R unit vector is equal to cosine of phi unit vector I + sine of phi unit vector J. Since cosine square phi + sine square phi is 1, the magnitude of unit vector is obviously 1.

Similarly unit vector phi is going to be equal to - sine of phi I + cosine of phi J. Again, the magnitude of phi unit vector is sine square phi + cosine square phi which is 1. You can also see right away that R dot phi that is their dot product is equal to 0 which implies that the 2 unit vectors are orthogonal to each other. It is clear from their definition.

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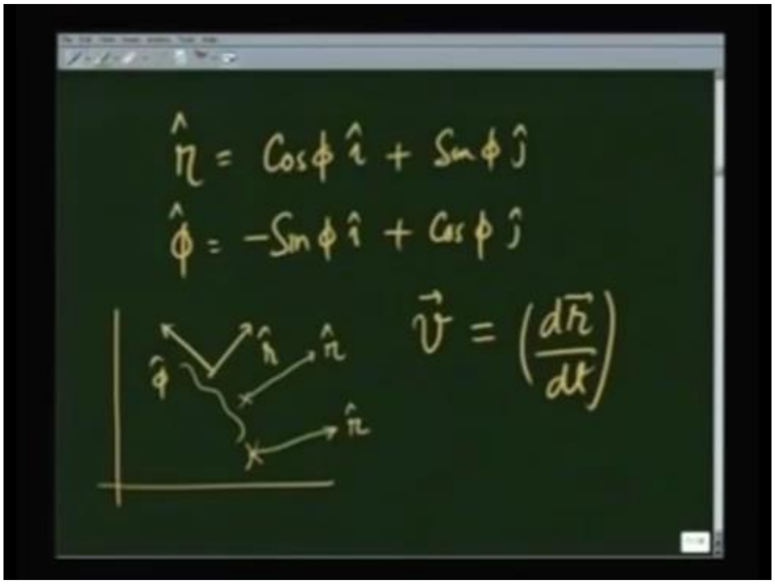

And let me write them separately now R is equal to cosine phi I + sine of phi J. Phi is equal to - sine of phi Y + cosine of phi J. Just to remind you again, for a given position, this is my R unit vector and this is my phi unit vector. So it is clear from their definition that the unit vectors R and phi depend on what phi is.

And what that means is, as I move around, depending on the position of the particle, the unit vectors are going to change. For example, at this position, the unit vector R is going to base this direction. On the other hand, at this position of the unit vector R is in this direction. So it changes as the particle moves around. And in calculating the velocities and accelerations in planar polar coordinates I need to take this change into account.

For example if I calculate the velocity which is given by DR by DT, as the particle moves around, both the magnitude of R and its directions are changing and in the next few minutes they will see how we take that into account.

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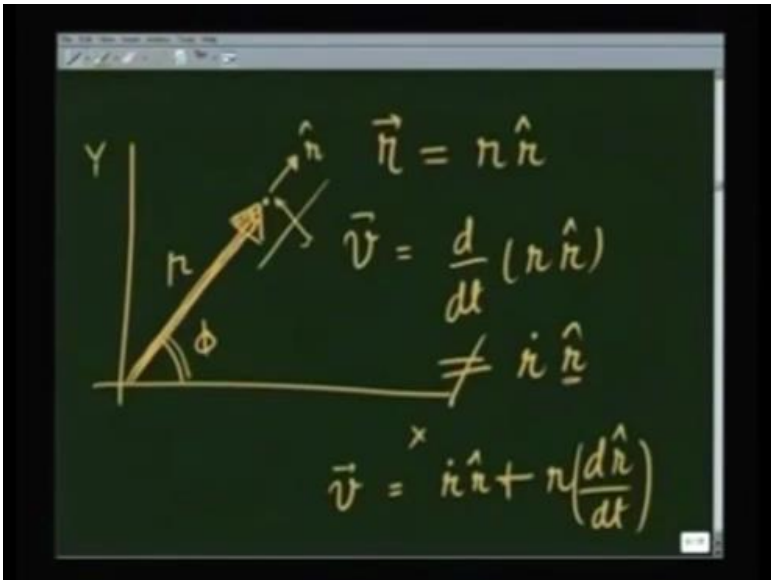

To start with, let us start with the position of the particle. So a particle is given here at a distance R from the origin making an angle phi from the x-axis. So position vector R of the particle is going to be given as R R unit vector because this is the displacement vector of the particle and this is unit vector R. Velocity is going to be D over DT R R unit vector and this is not equal to just R dot R because particle is not just moving in this direction.

It is also moving in this direction. So and also other way R and unit vector R depend on phi, depend on the position of the particle. So the velocity really should be written as R dot R + RDRDT that is the rate of change of the unit vector R itself. Let us see how much it is.

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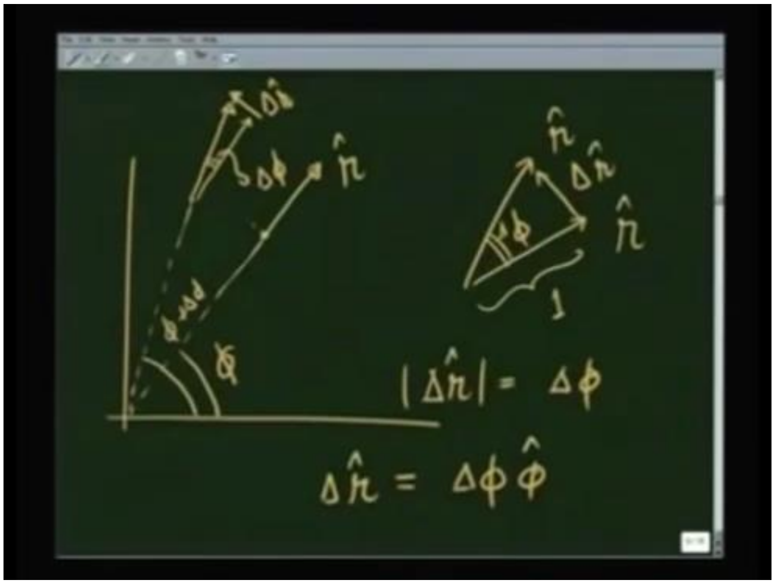

So if I were to make the unit vector R again, here is the particle. Here is unit vector R. As the particle moves, let us say in this direction so that phi changes from phi to phi + delta phi, the unit vector has changed by this much. Initially if I moved this parallely, the unit vector was in this direction. So this is the change in unit vector R. How much is this angle? This angle is delta phi.

If I redraw it, so initially the vector was like this. The final unit vector is like this. This is the change and this angle is phi. Since this magnitude is 1, you can see that delta R magnitude is going to be this is delta phi, is going to be delta phi in the direction it is in the direction of increasing phi. The way I have made it. Therefore delta R is going to be equal to delta phi phi. Let us see it from a mathematical point of view.

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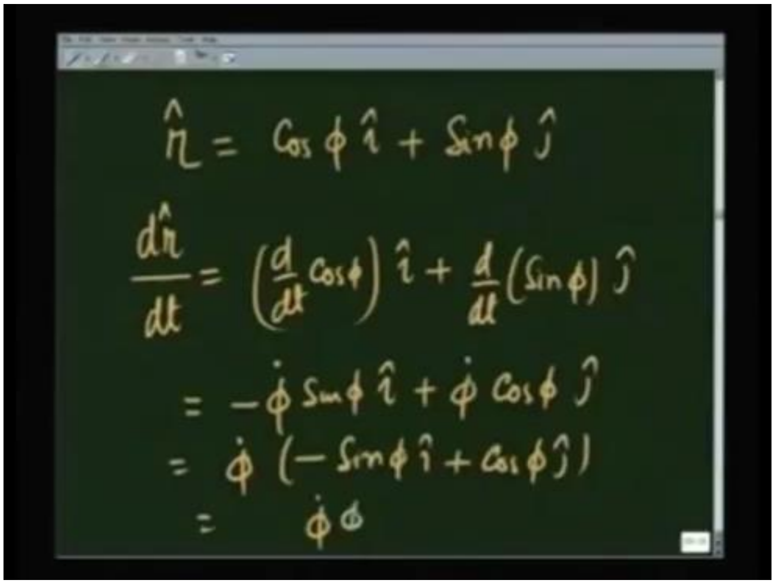

Recall from the earlier slide that the unit vector R was written as cosine of phi Y + sine of phi J and therefore rate of change of the unit vector DRDT is going to be equal to D over DT of cosine of phi I unit vector is a fixed unit vector. So it is constant + D over DT of sine of phi J and this comes out to be - phi dot sine of phi I + phid dot cosine of phi J which is equal to phi dot - sine phi I + cosine phi J which is phi dot phi.

So we see from the definition also that the rate of change of unit vector R is phi dot phi. It is good to see a derivation mathematically as well as geometrically.

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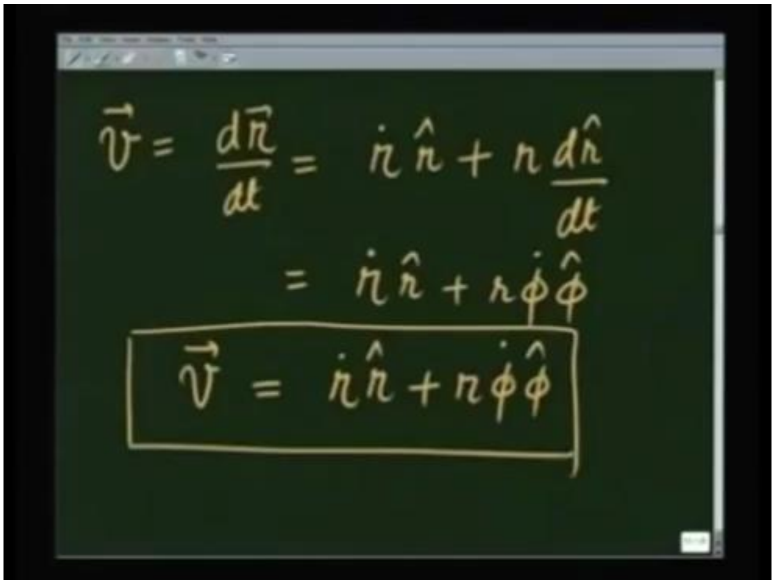

So if we write the velocity V as DRDT, this is equal to R dot R + RDRDT which is going to be equal to R dot R + R phi dot phi unit vector. So the velocity of a particle in planar polar coordinates I want to remind you again we are talking about motion in a plane, is equal to R dot R unit vector + R phi dot phi unit vector. Let us look at it geometrically.

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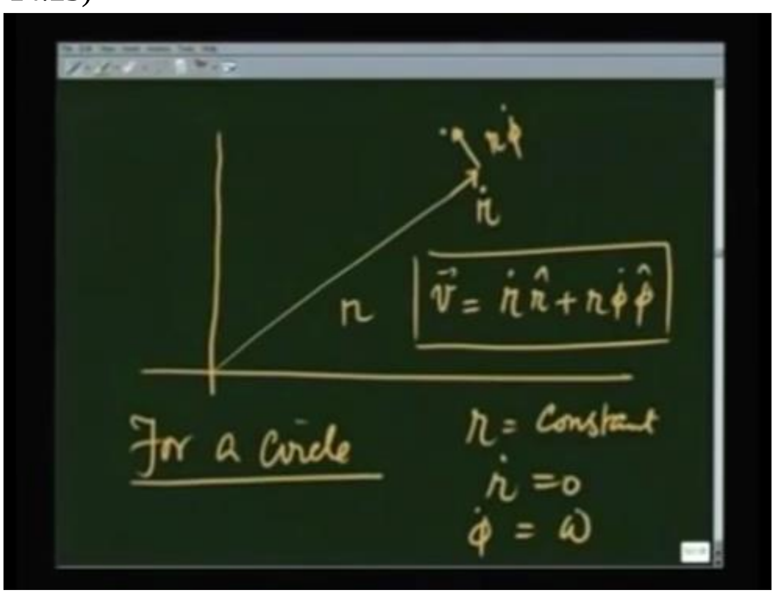

So a particle is moving along at a distance R. As moves to its new position, it has moved this way at a rate of R dot and it has moved this way at a rate of R phi dot so that its net velocity is a combination of R dot in R direction + R phi dot in phi direction.

To make it look more familiar, let us look at a particle which is rolling around in a circle. For a circle that is ever particle is moving in a circle, R is a constant and therefore R dot is 0 and phi dot is the angular speed.

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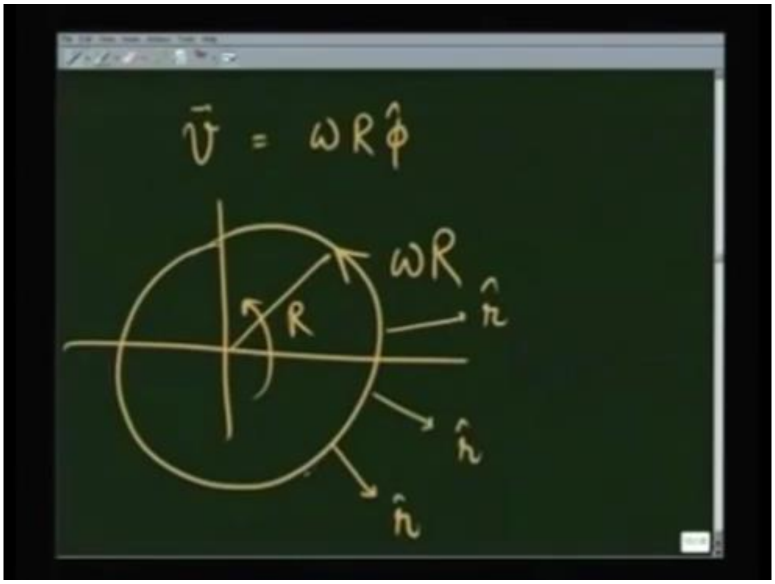

And therefore it is velocity is going to be given as V is equal to omega R phi. That is if a particle is moving around in a circle, it is moving in direction phi with the speed omega R where R is the radius of the circle, omega is at which, the angular speed of the particle. You see, I am describing the same motion except now I am using planar polar coordinates.

And it is important to keep in mind that when I am considering this motion, the R unit vector keeps on changing depending on the position of the particle. If R unit vector changes, so does the phi unit vector. And that is going to be important when we derive the expression for acceleration in planar polar coordinates.

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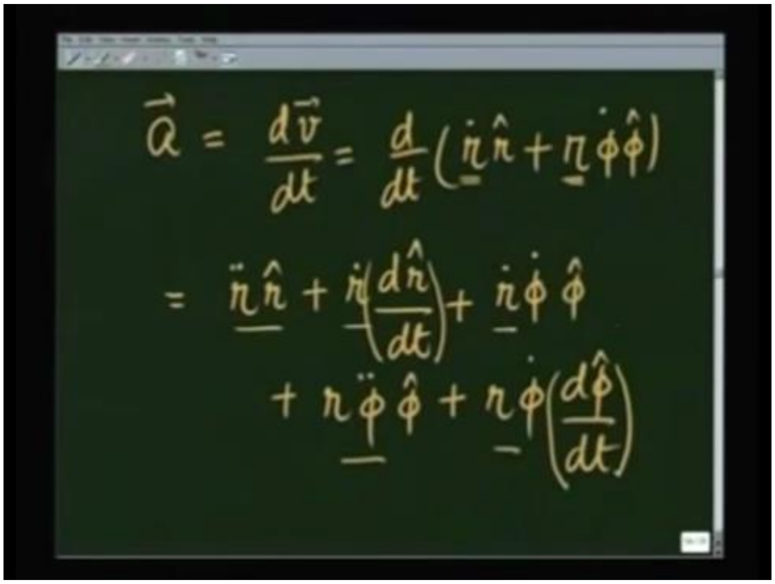

To derive the acceleration, I just differentiate the velocity of the particle DVDT and this is going to be equal to D over DT of the velocity in planar polar coordinates R dot R + R phi dot phi which is equal to R double dot R that is I differentiated this term + R dot DRDT + R dot phi dot I differentiated this term phi unit vector + R phi double dot phi unit vector + R phi dot D phi DT. So I have 1, 2, 3, 4 and 5 terms.

DRDT I have already calculated. The new term now is D phi DT and this is what I am going to calculate next. Collect all the terms and see what the expression for acceleration looks like.

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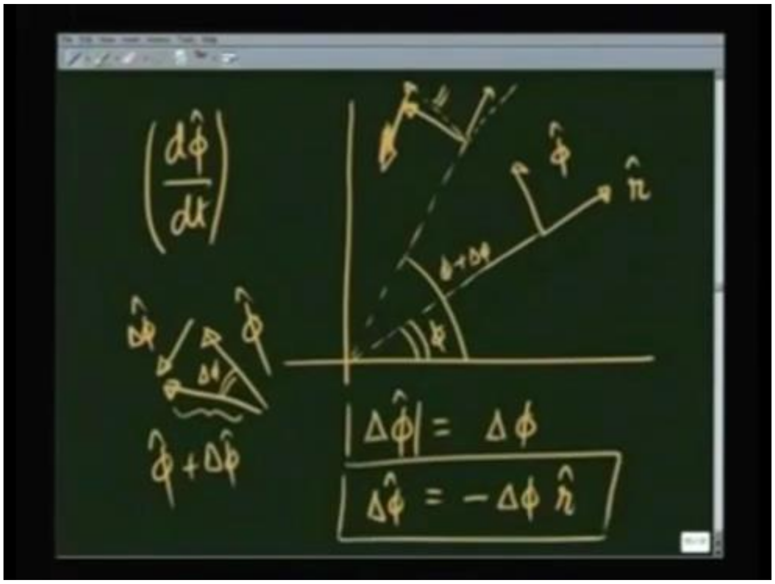

To get D phi by DT I will 1st again look at it geometrically and then derive it from the definition. If you look at it geometrically, this was my R unit vector and this is the phi unit vector. When I moved to a new position, this was phi, this was phi + delta phi. Both unit vectors have also moved from their original direction by angle delta phi.

This was my original phi. So original phi was something like this which is this vector and the new phi is like this. And this is the change, delta phi in the unit vector. You can already sense that this is in the radial direction but pointing the other way. So delta phi is going to be equal to, this angle is delta phi, this magnitude is 1.

So this is going to be equal to, the magnitude is going to be equal to delta phi itself. And the direction is in the direction of R but in the opposite direction. So - delta phi R.

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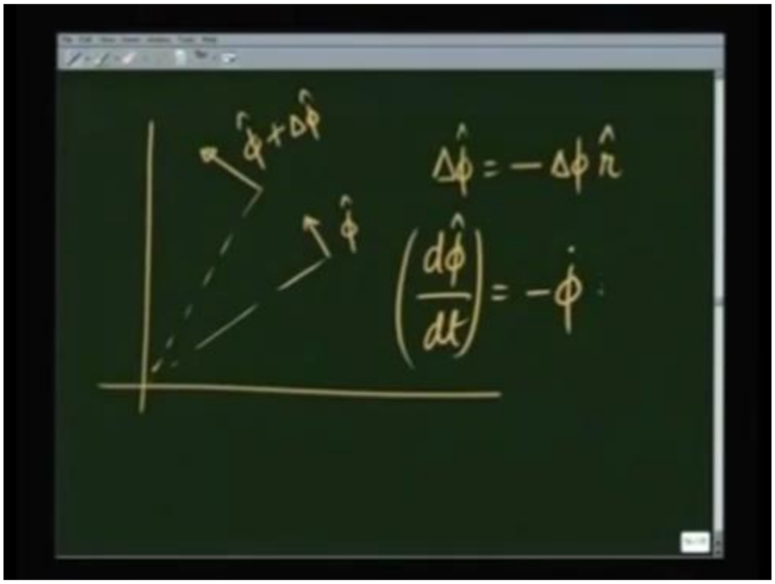

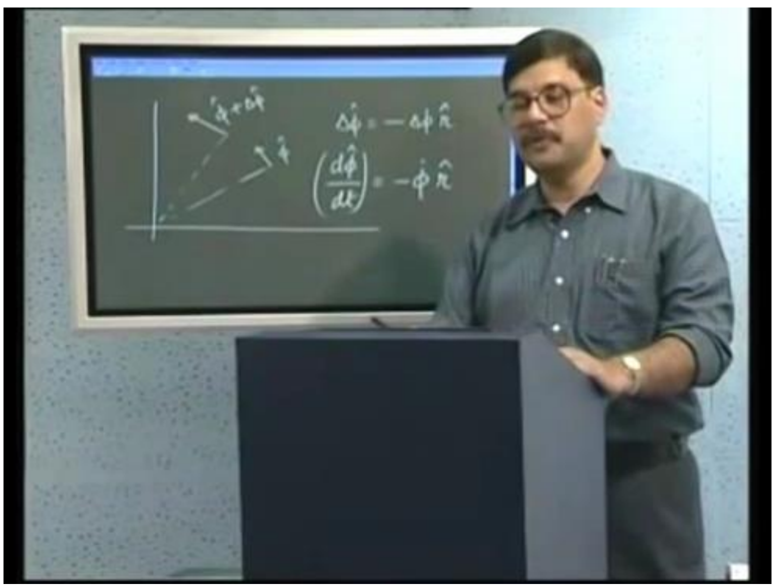

To show it again, this was my original phi. After the particle moved, phi has become this. So let me write this as phi + delta phi. And delta phi comes out to be - delta phi R unit vector and therefore D phi over DT is nothing but - phi dot R. I have shown it purely on geometric basis.

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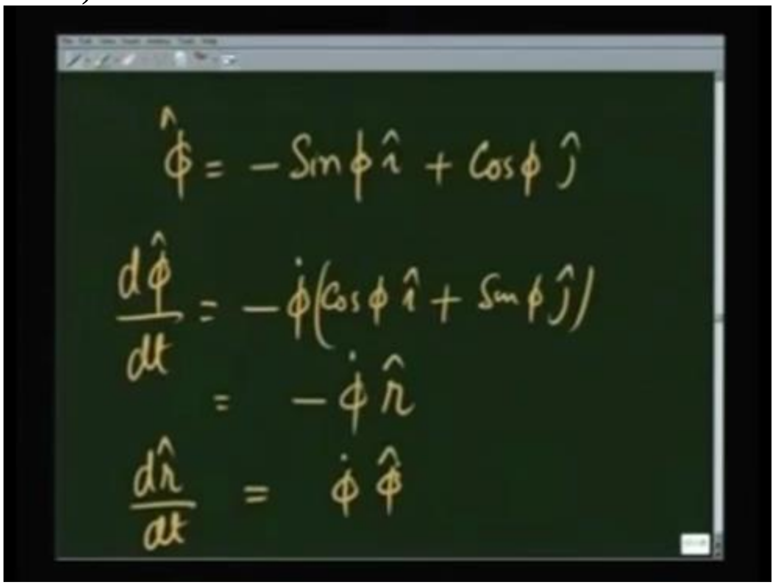

I will go to the definition of phi unit vector which was - sine phi I + cosine of phi J and differentiate to get - phi dot cosine phi I + sine phi J. I left the steps in between because I have already worked them out and this is - phi dot R unit vector, gives the same answer as by geometrical argument. So now we have both DRDT and D phi DT. DRDT you recall was equal to phi dot in phi direction. And therefore I can write an expression for the acceleration in planar polar coordinates.

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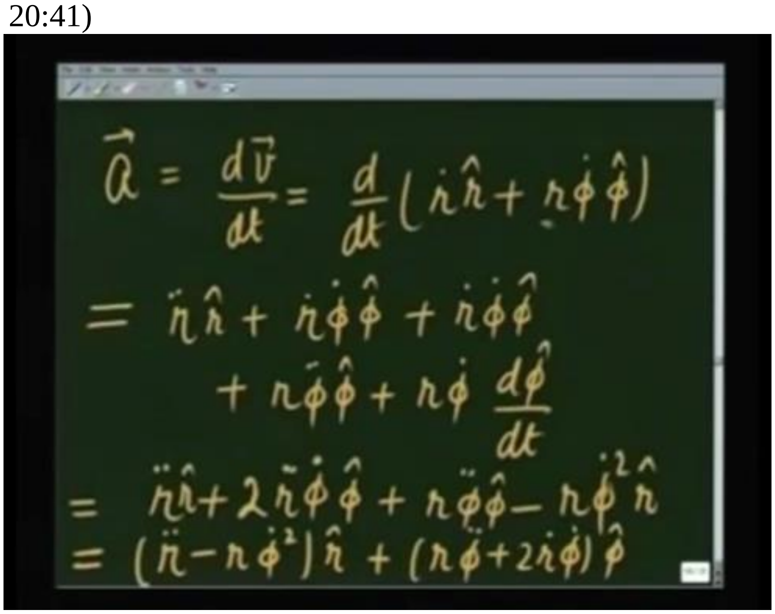

Let me write it again. A was equal to DVDT which is equal to D over DT of R dot R + R phi dot phi which is equal to R double dot R + R dot DRDT which is nothing but phi dot in phi direction + R dot from here phi dot in phi direction + R phi double dot in phi direction + R phi dot D phi DT which I have just calculated.

Collecting all terms together, you will see this comes out to be R double dot + 2R dot phi dot. This is R double dot in phi direction. R dot phi dot in phi direction + R phi double dot in phi direction and D phi by DT recall is - phi dot R. So this becomes - R phi dot square in R direction which therefore can be written as R double dot - R phi dot square in R direction + R phi double dot + 2 R dot phi dot in phi direction.

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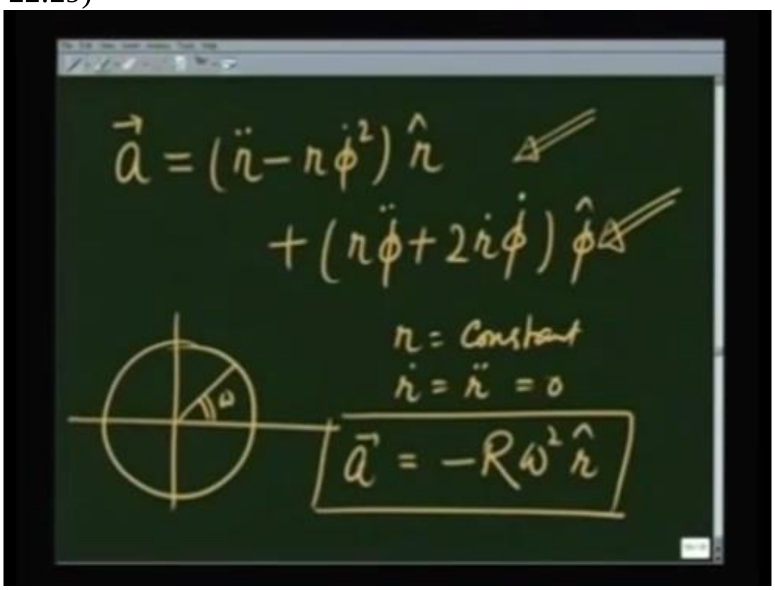

Let me rewrite it for you. Again this is equal to R double dot - R phi dot square in R direction + R phi double dot + 2R dot phi dot in phi direction. This relationship looks quite complicated. So let me give you again a familiar example of a particle moving in a circle for which R is constant. And therefore R dot is equal to R double dot is equal to 0 and if it is moving with constant angular velocity omega then A comes out to be - the radius of the circle omega Square R which is your familiar centripetal acceleration.

So you can see when the particle is moving around in a circle use of planar polar coordinates makes things slightly easier although the expressions may look complicated. Now to apply it, you have to practice it.