Engineering Mechanics
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Module 4
Lecture No 37
Properties of plane surfaces – VIII: second moment and product of an area, solved examples

In this lecture we calculate moments and products of inertia of some simple geometric figures.

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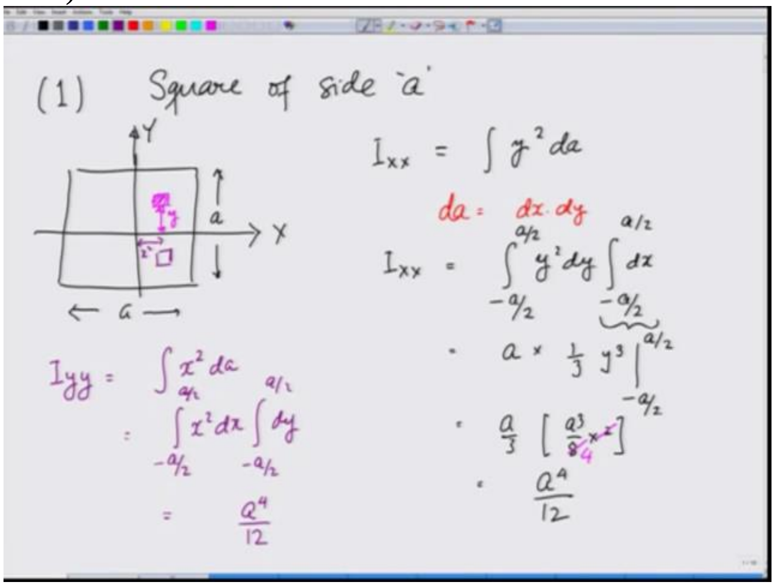

To start with, let us take a square of side A and calculate the moment of area and product of area of the square. Its side is A. Let this be the x-axis and let this be the y-axis. Now you recall that IXX that is the moment of area Y Square DA which indicates that I take a small infinitesimal area here and its distance Y from here multiply by Y Square DA and integrate.

Now in this case, DA happens be very simple, that is infinitesimal area DA is nothing but DX times DY. And therefore IXX is going to be equal to Y Square DY and this varies from - A by 2 to A by 2 times DX which varies from - A by 2 to A by 2 and this comes out to be A. This integral gives you A times 1 over 3 Y cubed varying from - A by 2 to A by 20 which is nothing but A by 3 A cubed by 8 times 2 which is A raised to 4 divided by 12 because this gives you 4 here.

Similarly since this is very symmetrical, IYY let me write in a different colour. IYY will be equal to X square DA. So what I will be doing is taking this area DA and multiplying it by X Square so this will become integral X square DX - A by 2 to A by 2 times DY - A by 2 to A by 2. Exactly the same integral as in IXX. And therefore I get A raised to 4 by 12 here. How about the product?

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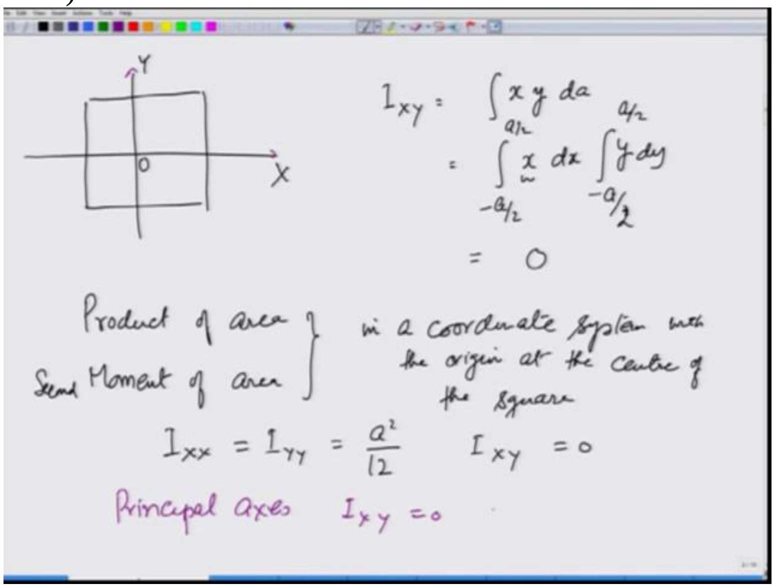

If I were to calculate the product of inertia of this, product of area of this square then IXY is nothing but integral XYDA which is integral XDX YDY and both vary from - A by 2 to A by 2 - A by 2 to A by 2. This being an odd integrand and going from - A by 2 to A by 2 this becomes 0. So for a square if I calculate the product of area, so product of area and moment the second moment of area this is in a coordinate system with the origin at the centre of the square we get IXX equals IYY equals A square over 12 and IXY to be 0.

Recall, in the previous lectures what we have learnt is that in the principle axis, IXY is 0. And therefore these axis shown here are the principle axis for this.

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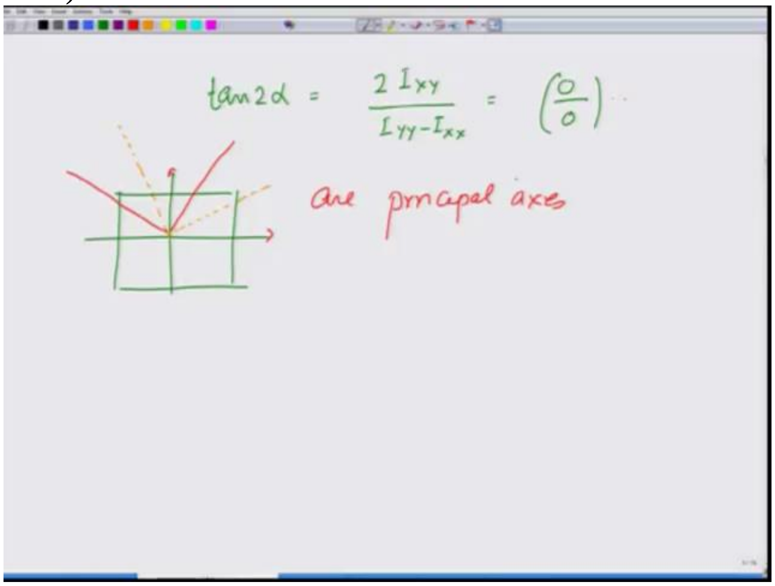

Also notice in this case that you have tangent 2alpha is equal to 2 IXY over IYY - IXX and this in this case comes out to be 0 by 0. That means I could take any alpha and the axis would be principle axis. Therefore in a square, any axis at any angle would be this, would with this, all these are principle axis. If the originally chosen XY axis were the principal axis alone, then this would have come out to be 0.

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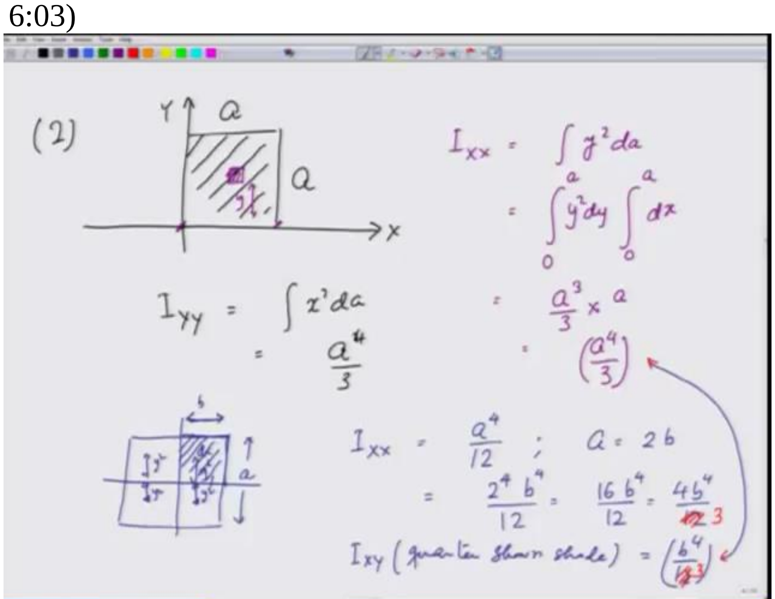

Okay, problem number 2, I wish to calculate the moment of area second moment of area and product of area of a square of side A with one of the corners at the origin. This is X, this is Y. So

I could calculate this very very easily from the result of the previous problem but I will do it explicitly so that you see it how these calculations are done you get some practice. So again, IXX is equal to integral Y Square DA.

DA is this small area and this is the distance Y. So in this case this will become Y Square DY Y varies from 0 to A and DX which varies from 0 to A. The limits are from this point to this point. And this becomes A cubed over 3 times A which is A raised to 4 over 3. Exactly the same distances are involved in calculating IYY.

So IYY which is equal to X square DA would also give you A raised to 4 over 3. How could I have gotten the same result from the previous problem? Just to find the relationship, remember in the previous problem I had a square like this and this side was A and in this case IXX which is actually measuring Y Square times DA was equal to A raised to 4 over 12. If I take this side to be B, then A is 2B.

So this I can also write as 2 raised to 4 times B raised to 4 over 12 which is equal to 16 B raised to 4 over 12 which is 4B raised to 4 over 12. Right? So if the side is B, it is half side is B I get 4 B raised to 4 over 12 as the second moment of area. But notice Y Square is same for all these 4 areas. So all these 4 areas contribute equal to IXX and this implies if I were to calculate IXX only for this, it will be one fourth of whatever I have calculated.

So IXX of the quarter shown shaded, the answer will be B raised to 4 over 12 and that is exactly oh I am sorry this is 3 out here B raised to 4 over 3 and which is the same as the earlier answer. So you could also relate and this also give you some practice as to how to think as these shapes are similar.

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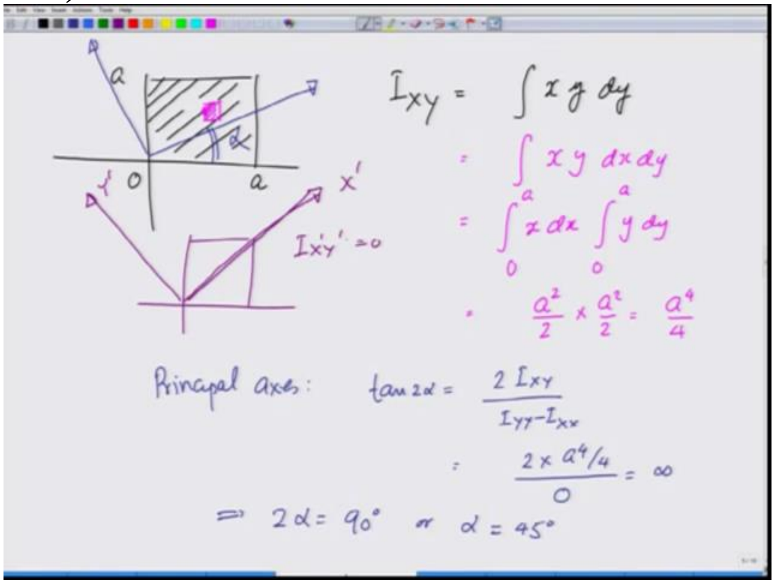

More interesting in this case however would be to calculate the product of area of this square 0 to A A.

Product of area is equal to integral XYDY. So if I take this small area out here, this is integral XYDXDY which I can write as XDX 0 to A integral 0 to AYDY which comes out to be A square over 2 times A square over 2 which is A raised to 4 over 4. Now if I want to find the principle axis, remember we had the formula tangent 2 alpha equals 2IXY over IYY - IXX where tangent 2alpha where the tangent of this angle alpha through which I have 2 rotate the axis to find the principle axis.

So this comes out to be 2 times A raised to 4 over 4 divided by 0 which is Infinity. And this implies 2 alpha is 90 degrees or alpha is 45 degrees. So we find that the principle axis in this case is going to be at 45 degrees to the x-axis and therefore this is the principle axis. And this IXY in this new set of axis, let us call it X prime, Y prime, I X prime Y prime will be 0.

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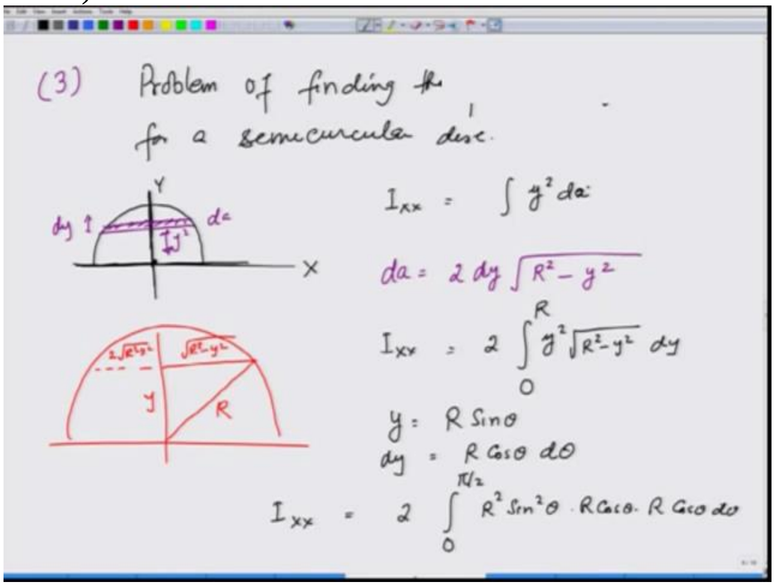

Next I solve the problem of finding the product of area for a semicircular disc. I will take this disc with the centre at the origin. Here is the centre and the diameter is along the x-axis. Either diameter is along the y-axis. So let us calculate this. IXX is going to be integral Y Square D area. Since Y is the same for this entire strip out here which I am showing by purple colour, I will take this area as DA, multiply by this distance Y Square and then integrate.

So let us calculate the area DA. DA is nothing but this height of this strip which is DY times the length. The length is going to be square root of R square - Y Square times 2. So this is the area. Right? So let me make it again to show it clearly. This is the semicircular thing. This distance is R. This is Y and therefore this is our Square - Y Square.

This entire distance becomes 2 times Square root of R square - Y Square. And therefore IXX is going to be 2 integral Y Square R square - Y Square DY and Y varies from 0 to radius R. This integral is easy to perform. I take Y equals R sine theta. DY becomes R cosine of theta D theta. And therefore IXX becomes equal to 2 integral lower limit theta would be 0, upper limit theta will be pi by 2.

Y Square will be R square sine Square theta. Square root of R Square - Y Square will be R cosine of theta and then you have DY which is R cosine of theta D theta.

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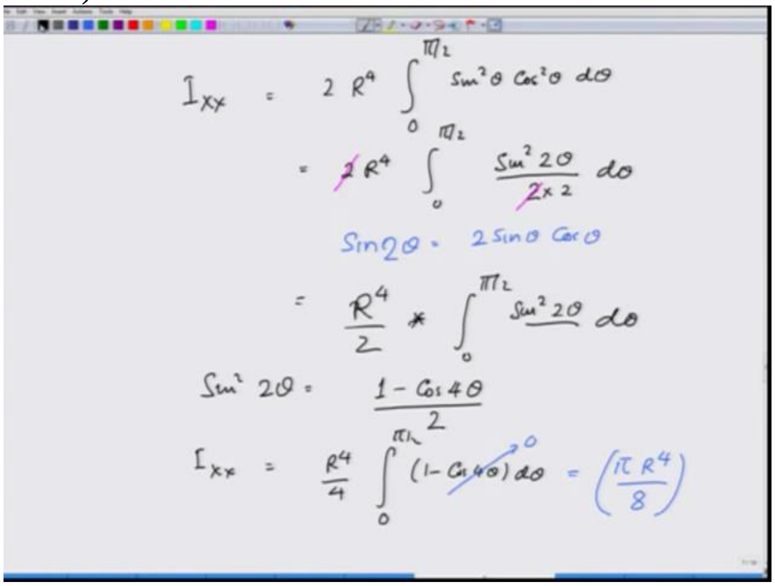

Therefore I can write this whole thing as IXX is equal to 2 R raised to 4 integral 0 to pi by 2 sine square theta cosine square theta D theta which I can write as 2R raised to 4 integral 0 to pi by 2 sine square 2theta divided by 2 times D theta. I made use of the fact that sine 2theta is equal to 2 sine theta cosine theta and therefore IXX becomes R raised to divided by 2 because this cancels 0 to pi by 2 times 0 to pi by 2 sine square 2theta D theta.

Sine square 2theta I can write as 1 - cosine of 4theta divided by 2. So this becomes IXX becomes R raised to 4 divided by 4 integral 1 - cosine of 4 theta D theta 0 to pi by 2. The second integral vanishes, this one vanishes. So you get pi R raised to 4 by 8. Okay, that is the answer. How about about the Y axis?

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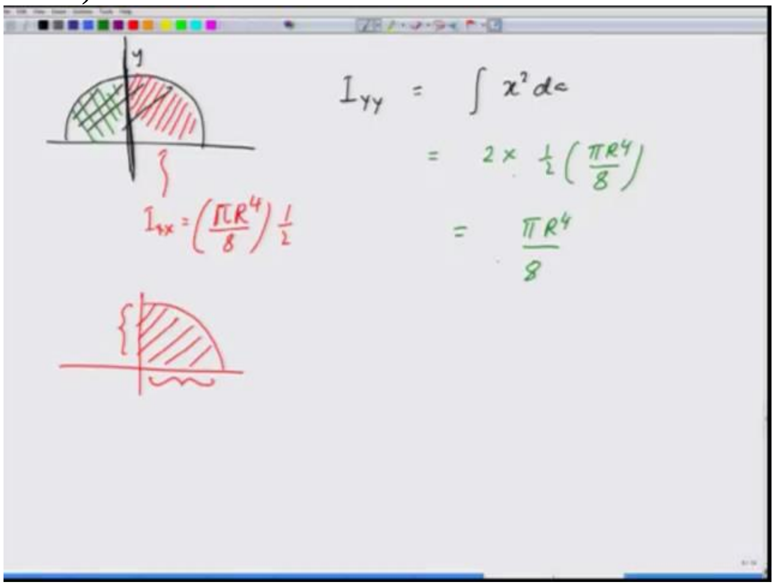

If I were to calculate this about this axis, Y axis, IYY will be equal to integral X square DA. You could do it explicitly or immediately use a trick that this half of this whole thing, half of the semicircle has moment of inertia is equal to pi R raised to 4 over 8 one half of it. And so does the other one. But for a quarter whether I take the moment of area about this axis or this axis, is the same.

And therefore both the quarters, the one shown with orange and the other shown with green has the same moment of area about the Y axis and therefore this will become 2 times one half pi R raised to 4 by 8. The same answer is pi R raised to 4 by 8.

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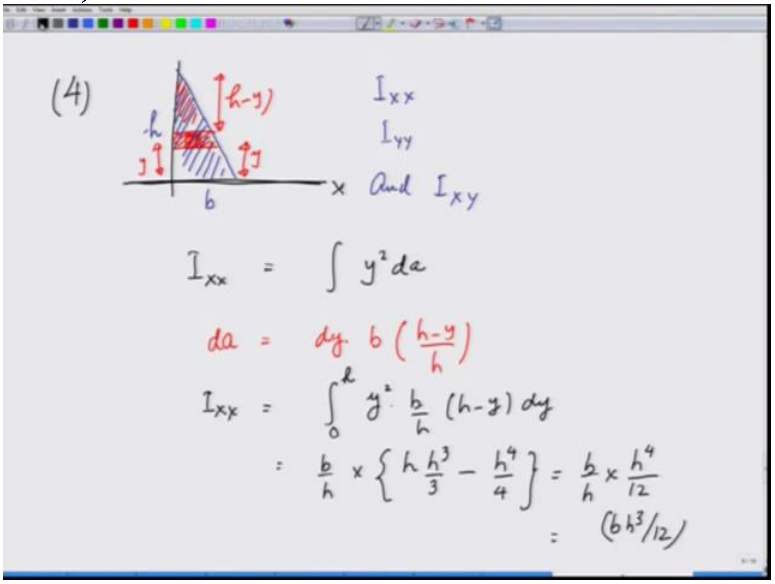

As the final example, I am now going to do a triangle and to keep things simple initially, I am going to take a right angle triangle of height H and base B and calculate IXX, IYY and IXY for this. To calculate IXX I again do integral Y Square DA and this small area out here is this orange area. All right? I can make things simpler by taking rather than this area, the entire strip out here because Y is the same for this entire strip.

All right? Now DA for this entire strip is going to be DY times it is width and the width is going to be at height Y is going to be B times H - Y over H by similarity of triangles because this height is Y, this height therefore becomes H - Y and this triangle is similar to the entire triangle. Therefore IXX is equal to integral Y Square times B over H H - Y DY which is B over H times H and Y is varying from 0 to H.

So H H cube over 3 - Y cubed integral which will be H raised to 4 over 4 and this comes out to be B over H times H raised to 4 over 12 which is nothing but B H cubed over 12. So IXX that is the second movement of area about the x-axis for this triangle is BH cube over 12.

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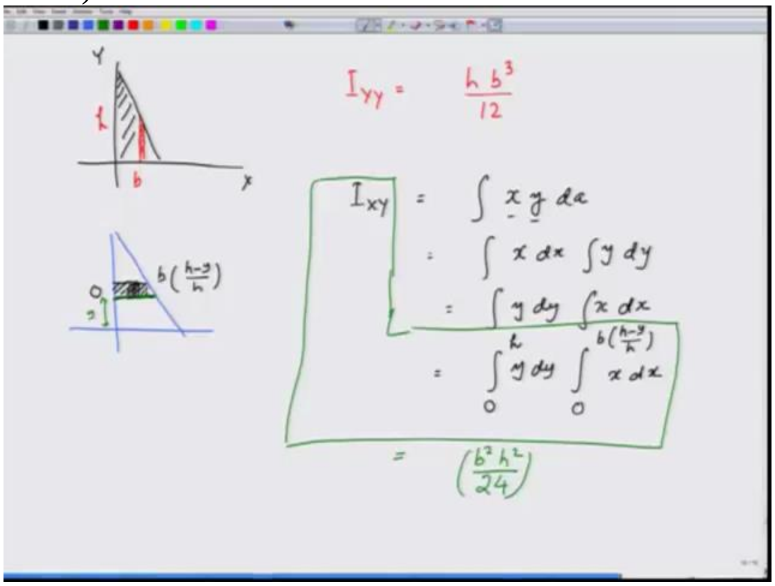

The same calculation can be repeated for calculation about the Y axis except in this case we will take the area to be like this and I leave that exercise for you to show that IYY would beautifully come out to be H B cube over 12. This is exactly the same exercise and finally let us calculate IXY which is nothing but integral XY DA.

In this case I have to be a little careful because this integrand involves both X and Y. I will take a small area here and write this as integral XDX YDY or equivalently YDY XDX. You can perform one of the integrals to start with. Now X integral if I were to perform, notice that X integral over X in this strip is going to be from 0 to B H - Y over H to this distance.

And therefore this integral becomes YDY Y varies from 0 to height H, integral X varies from 0 to B H - Y over HXDX. That is the integral that we have perform. So let me box this. So for a given Y, X varies from here to here which is 0 to BH - Y over H. And then Y varies from 0 to H. And you calculate this. I will let you do the integral.

This comes out to be B square and square over 24.

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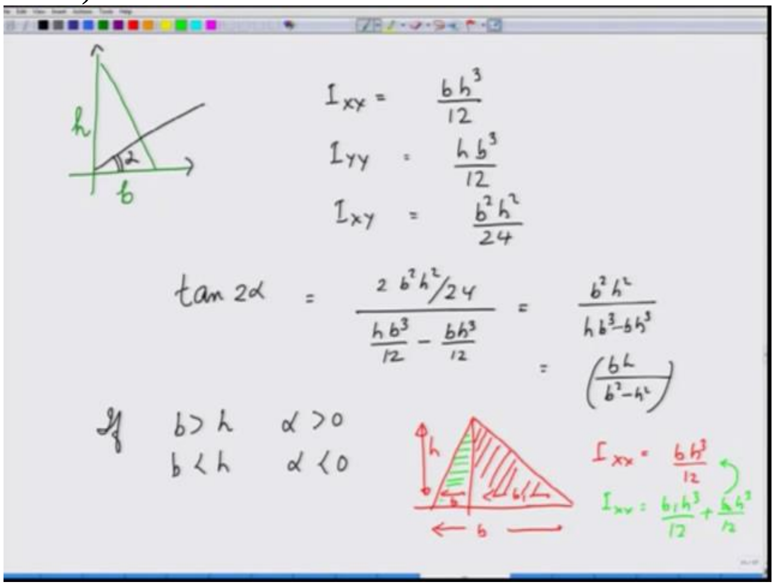

So for a right angle triangle, we just calculated if its base is B and height is H, we have calculated that IXX comes out to be BH cubed over 12, IYY comes out to the H B cubed over 12 and IXY comes out to be B square H square over 24. So obviously, X and Y axis in this case are not the principal axis.

If I want to find the principal axis, I would do a rotation by an angle alpha such that tangent 2 alpha is 2 B square H square over 24 over IYY which is HB cubed over 12 - IXX which is BH cubed over 12. This gives me B square H square HB cubed - B H cube which comes out to be BH over B square - H square.

So if B is greater than H alpha comes out to be greater than 0. If B is less than H, alpha comes out to be less than 0. These are the principal axis. One more comment is if the triangle was not right angle but something like this with total base size B.

This being B1, this being B2 and height H, IXX will still come out to be BH cubed over 12 because both these triangles, one shaded and the other shaded by green are right angle and each one of them has the IXX which is B1 H cube over 12. And the other one is B to H cube over 12 and you add them and get this answer. Thank you.