Egineering Mechanics
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Module 4
Lecture No 36
Properties of plane surfaces – VII: transformation of second moment and product of an area under rotation of coordinate axis

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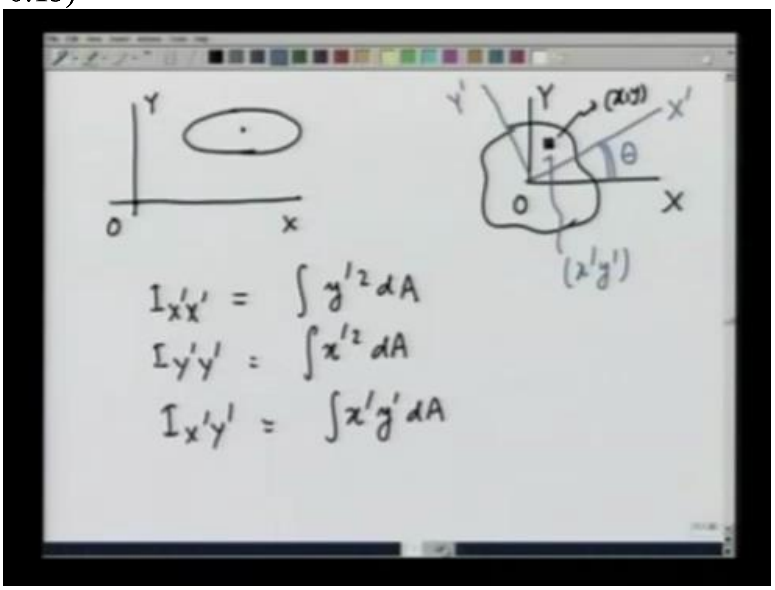

So far what we have considered in the transfer theorem is the product and second moment of area when the centroid is displaced with respect to the origin of a given system. Now we want to look at another transformation where given an area and its second moment of inertia and product of inertia about an set of axis XY we wish to calculate it about another set X prime, Y prime which is rotated with respect to the first set by an angle theta.

Let us see what happens in this case. So if I were to calculate IX prime Y X prime X prime in the rotated set, this is going to be equal to integral Y prime square DA. IY prime Y prime is going to be equal to X prime Square DA. And I X prime Y prime in the second frame is going to be X prime Y prime DA where we choose a small area DA whose coordinates in the original system are X and Y and in the new system are X prime, Y prime.

We can find out the relationship of IXX prime with those similar quantities in the unrotated frame by simple transformation laws of X and Y coordinates. So let us do that now.

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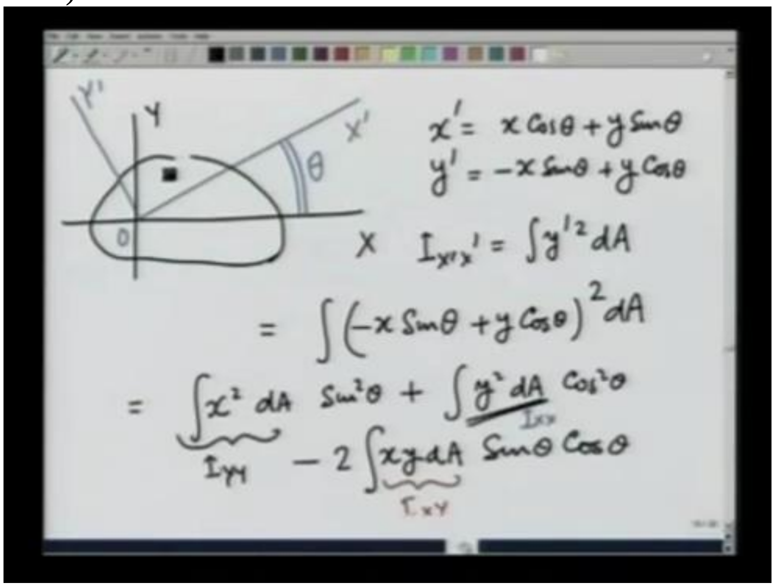

So what we are given is an area and we wish to calculate its second moment of area and product of area with respect to a set of axis X prime, Y prime when they are given in X and Y. We know from our previous lectures that X prime for a given point is equal to X cosine of theta + Y sine of theta. Similarly Y prime is equal to - X sine of theta + Y cosine of theta. Using these let us find what IX prime X prime is.

From the previous slide we know this is equal to Y prime square DA where DA is a small area chosen. Y prime square is going to be equal to integral - X sine theta + Y cosine of theta square DA which I can write as X square DA integral sine square theta + integral Y Square DA cosine square theta - 2XYDA sine theta cosine of theta. But X square DA is nothing but IYY, Y Square DA is nothing but IXX and XYDA is nothing but IXY.

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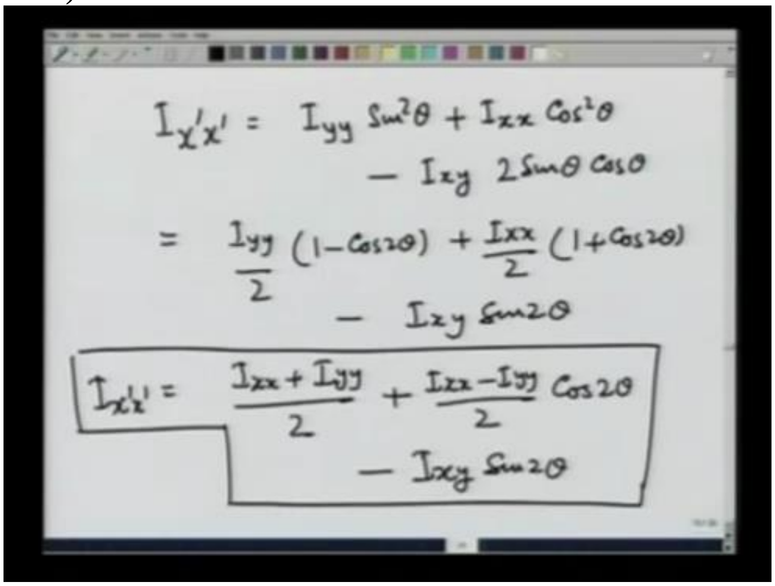

Therefore I can write this quantity as let us go to the next page. I X prime X prime as IYY sine square theta + IXX cosine square theta - IXY 2 sine theta cosine theta which can be written as IYY divided by 2 1 - cosine of 2 theta + IXX divided by 2 1 + cosine of 2theta - IXY sine of 2theta which is nothing but IXX + IYY divided by 2 + IXX - IYY divided by 2 cosine of 2theta - IXY sine of 2theta. Thus if I know the second moment of area and product of area in one frame, I can calculate it in the rotated frame.

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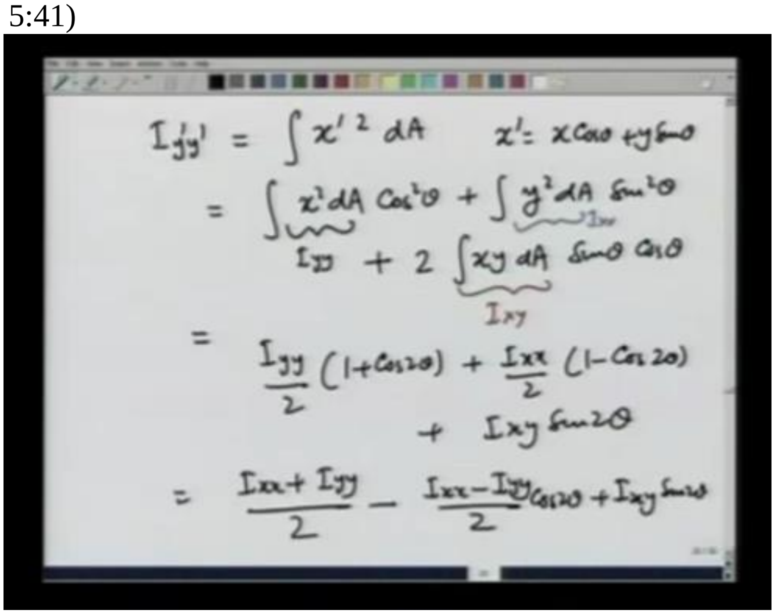

Let us do the same exercise for IYY prime Y prime Y prime which is going to be equal to X prime Square DA but I know X prime is equal to X cosine theta + Y sine of theta and therefore I can write this as integral X square DA cosine square theta + integral Y Square DA sine square theta + 2 integral XYDA sine theta cosine of theta which is this is nothing but IYY, this is nothing but IXX and this is nothing but IXY.

So this whole thing can be written as IYY over 2 1 + cosine 2theta + IXX over 2 1 - cosine 2theta + IXY sine of 2theta which is nothing but IXX + IYY divided by 2 - IXX - IYY divided by 2 cosine 2theta + IXY sine of 2theta.

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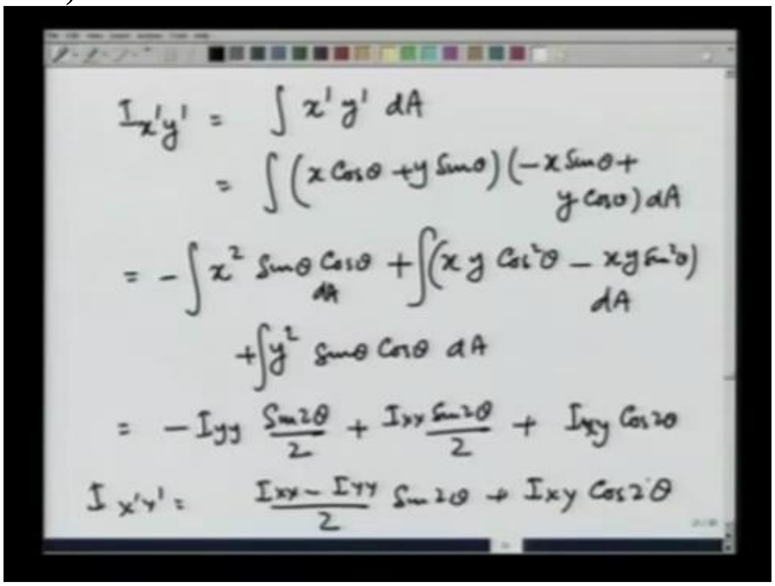

Thus we calculate IX prime Y prime which is nothing but X prime Y prime DA which is equal to integral X cosine theta + Y sine of theta times - X sine theta + Y cosine theta DA which comes out to be integral - X square sine theta cosine theta + XY cosine square theta - XY sine square theta DA + Y Square sine theta cosine theta DA. This is also DA.

X square DA is nothing but - IYY. This can be written as sine 2theta divided by 2 + Y Square DA is IXX sine 2theta divided by 2. And XYDA is nothing but IXY cosine square theta - sine square theta is cosine 2theta. So I X prime Y prime is nothing but IXX - IYY divided by 2 sine 2theta + IXY cosine of 2theta. Let us summarise.

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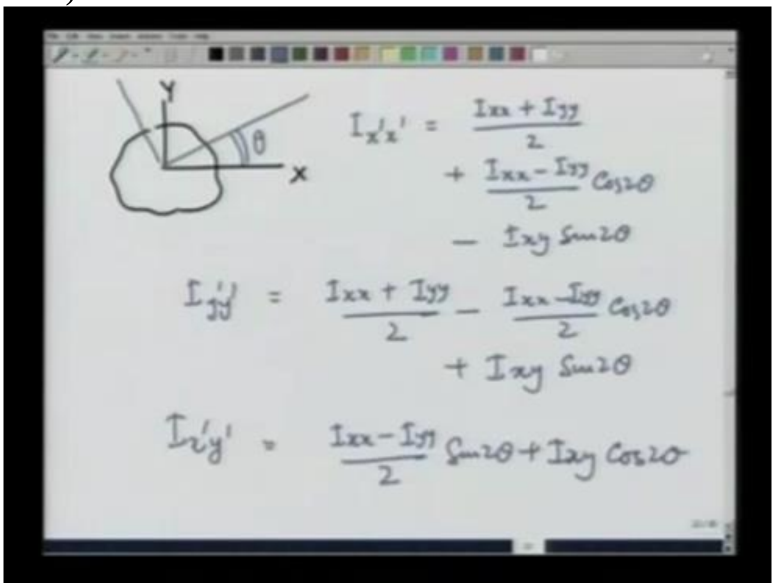

What we are looking for is if we know for a body the products and second moments of inertia in one particular frame, how about its values in the rotated frame. In the rotated frame, let me now write it in blue. IX prime X prime is nothing but IXX + IYY divided by 2 + IXX - IYY divided by 2 cosine 2theta - IXY sine of 2theta. Similarly IY prime Y prime is going to be IXX + IYY divided by 2 - IXX - IYY divided by 2 cosine 2theta + IXY sine of 2theta.

And I X prime Y prime is going to be equal to IXX - IYY divided by 2 sine of 2theta + IXY cosine of 2theta. What these transformation laws give me is if I am given the second moment and product of area about a set of axis, I can calculate about any other set of axis which is rotated with respect to the first set of axis. Let me just illustrate this thing by a couple of examples which are very interesting.

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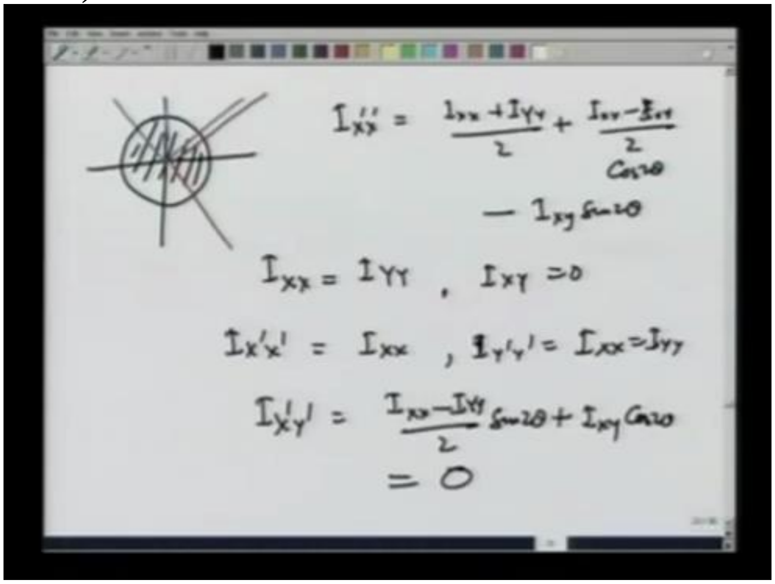

Suppose I take a circular area. For a circular area no matter how I choose my rotated set of axis, just take a third one like this, the circle always looks the same. And therefore IXX and IYY should always come out to be the same no matter what cosine theta or sine theta is and IXY should always come out to be 0. Let us see if that happens.

So IXX we saw already is I X prime X prime is IXX + IYY divided by 2 + IXX - IXY divided by 2 cosine of 2theta - IXY sine of 2theta. Now for a circular area, IXX is equal to IYY and IXY is 0. And therefore I X prime X prime is going to be equal to IXX and IY prime Y prime is also going to be equal to IXX equals IYY. And I X prime Y prime which is equal to IXX - IYY divided by 2 sine of 2theta + IXY - IXY cosine of 2theta is also going to be 0.

This is expected for a circle. What is very interesting that is the same thing comes out to be true for a square.

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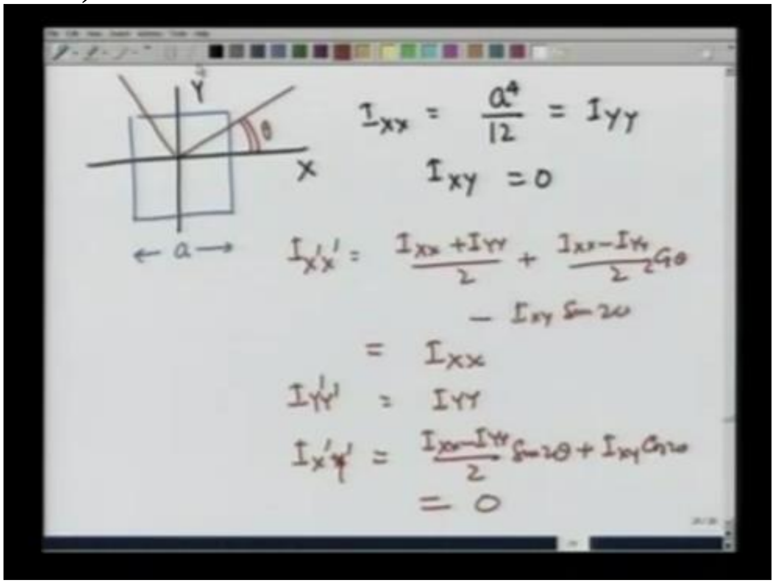

Let us look at that case. So if I take a square of side A, we have already calculated that IXX for such a square is A raised to 4 over 12 and so is IYY where these are the X and Y axis and IXY is 0. The fact that IXX and IYY are equal and IXY are 0 make these quantities the same no matter which other frame we look at. So let me write it in red, I X prime X prime which is equal to IXX + IYY divided by 2 + IXX - IYY divided by 2 cosine 2theta - IXY sine of 2theta is going to be equal to IXX again.

Similarly IY prime Y prime is going to be equal to IYY. And I X prime Y prime is equal to IXX - IYY divided by 2 sine 2theta + IXY cosine 2theta. This is always going to come out to be 0 no matter how much you rotate the axis by. So for a square, about any set of axis I X prime X prime is always equal to A raised to 4 divided by 12. I Y prime Y prime is always equal to A raised to 4 divided by 12 and I X prime Y prime is always 0.

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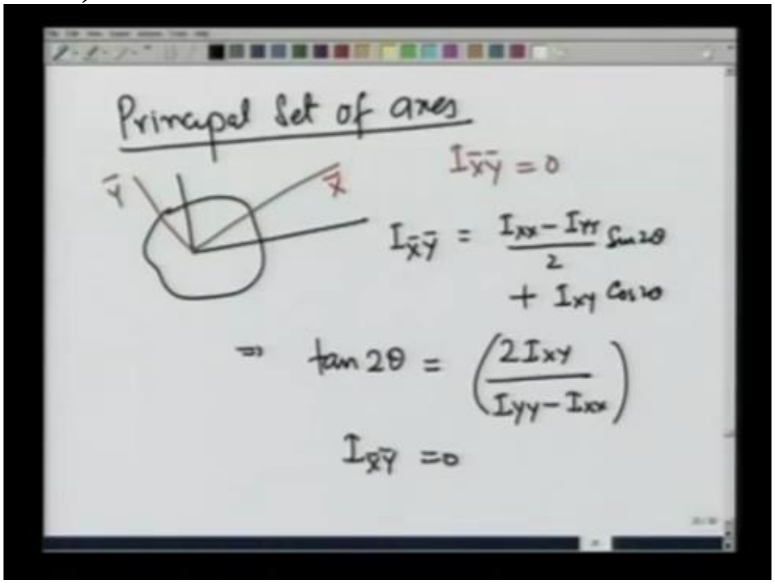

Having given these 2 examples, I use these transformations to define something called the principle set of axis. So given an area, I look for those set of axis let me call them X and Y so that IXY is 0. Let us us give them a special name, I X bar Y bar so that I X bar Y bar is 0. How do we accomplish that?

Since we already know that IX bar Y bar is going to be equal to IXX - IYY divided by 2 sine of 2theta + IXY cosine of 2theta this implies that if I choose rotate the new set of axis such that tangent 2theta is equal to 2 IXY divided by IYY - IXX X, I will get new IX bar Y bar is equal to 0.

Such a set of axis where the product of area vanishes is known as the principle set of axis. And you can see from the construction that you can always find one set of axis because tangent 2theta is varies from - infinity to + infinity. You can always find a set of axis where the product of area would be 0.

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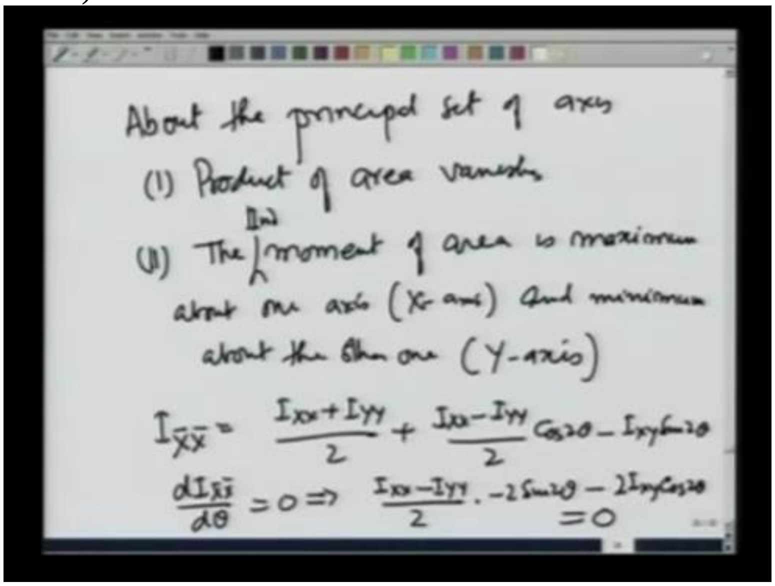

An interesting fact about the principle set of axis is that about the principle set of axis the one product of area vanishes and two, the moment or second moment of area is maximum about one axis, say the x-axis and minimum about the other one. If it is maximum about the x-axis, the other one is going to be y-axis. Let us see how does that come about.

So let us look at I X bar X bar which is IYY divided by 2 + IXX - IYY divided by 2 cosine 2theta - IXY sine 2theta. And ask for a new frame such that I X bar X bar is the maximum. So for that I go to do I X bar X bar over D theta is equal to 0. And when I do that here, this implies that IXX - IYY divided by 2 times - 2 sine 2theta - 2 IXY cosine 2theta is equal to 0.

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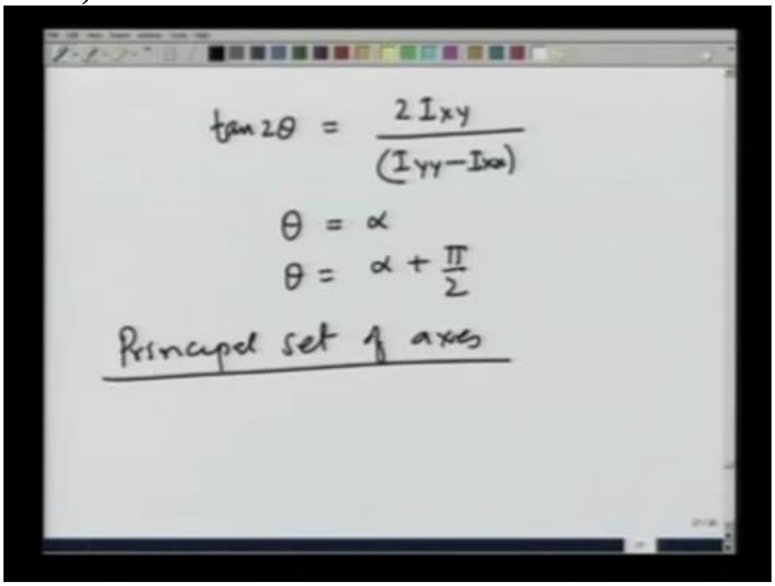

And that immediately gives me that tangent of 2 theta is equal to IXY times 2 divided by IYY - IXX. So when I accomplish by this rotation, the fact that the product of area vanishes, at the same time it maximises or minimises the moment of area. This equation has 2 solutions. Suppose one of the solutions is theta equals alpha then theta equals alpha + pi by 2 is also a solution.

So by rotating it by angle alpha I maximise or minimise the moment of inertia about that particular axis. We can show that about the axis at alpha + pi by 2 it will be the other way. If it maximises at alpha, at alpha + pi by 2, it will minimise and vice versa. So we have found a set of axis, principle stresses that not only the product of area vanishes the second moment of area is also either maximum or minimum.

If it is maximum about the X axis, about the other Y axis it becomes minimum. If it is minimum about the X axis, it becomes maximum about the other axis, the Y axis.

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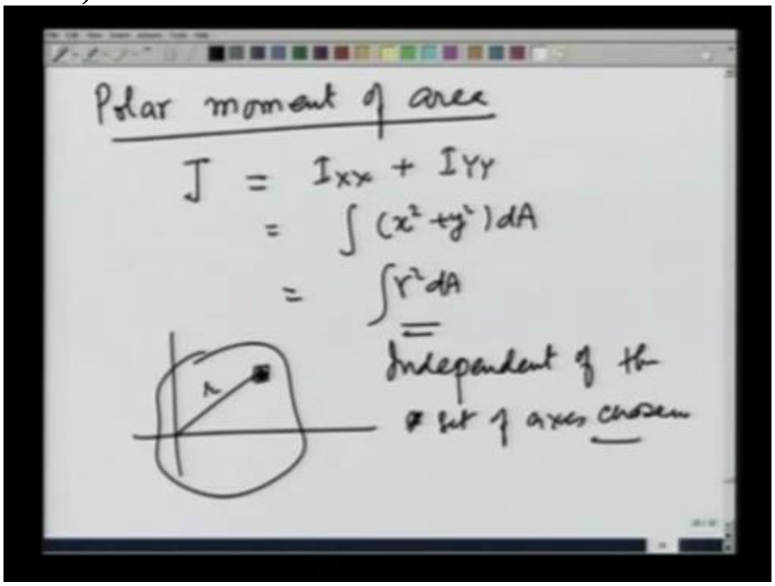

Having made this point, let me nowdefine something for you but is known as the polar moment of area. This is usually written as J. This is nothing but IXX + IYY. And therefore is equal to integral X square + Y square DA or R square DA. Given any area, R square for a small area chosen is independent of which set of axis we are talking about.

So this is independent the set of axis chosen. And through this discussion, we also see that for a square, the any set of axis is the principle set of axis because as we have seen earlier, the principle set of axis gives product of area 0 and second moment of area maximum or minimum. For a square any set of axis gives you product of area 0. So therefore any set of axis chosen for a square or a circle is the principal axis.

What we have covered in this lecture so far is the second moment and product of any area are related quantity which we will talk about in later lectures when we discuss dynamics of rigid bodies would be the moment of inertia and the product of inertia and we will be using it then in describing the rotational motion of a rigid body.