Engineering Mechanics

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Module 04

Lecture No 34

Properties of plane surfaces – V: Second moment and product of an area and radius of gyration

In the previous lecture, we have good docking about the first moment of a plane area and its centroid. Continuing on that, in this lecture we define some more mathematical quantities and just work out some examples with them. Utility of defining such quantities would be clear later when you do rotational dynamics and so on. But in this lecture we are just going to restrict to their definition and working them out.

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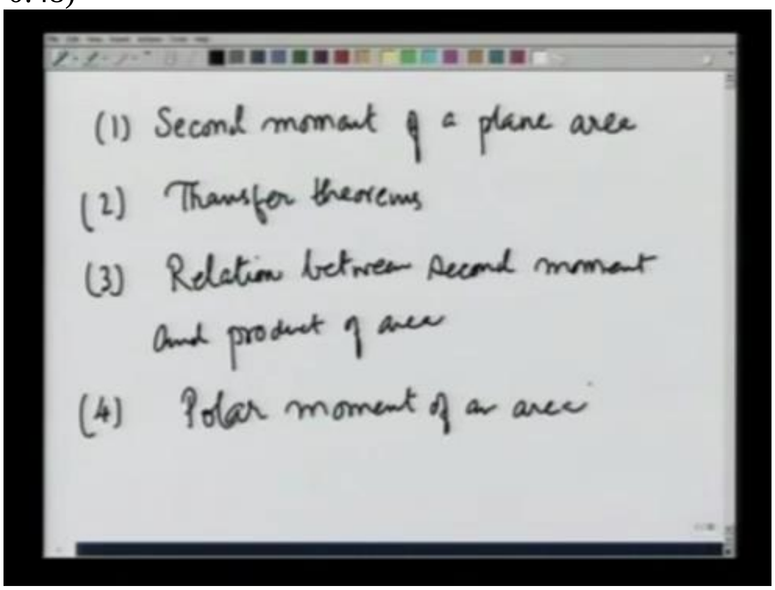

So quantities I will be talking about would be one, second moment of a plane area. Two, transfer theorems. Transfer theorems we mean how if we know the second moment of a plane area in one particular set of axis, how do we transfer them to another set of axis? Three, we will be talking about relation between second moment and product of area, in particular we will be focusing on how second moments and products of area which we will define later in the lecture change when we go from one set of axis to another set which are rotated with respect to the other. And then we will be talking about polar moment of an area.

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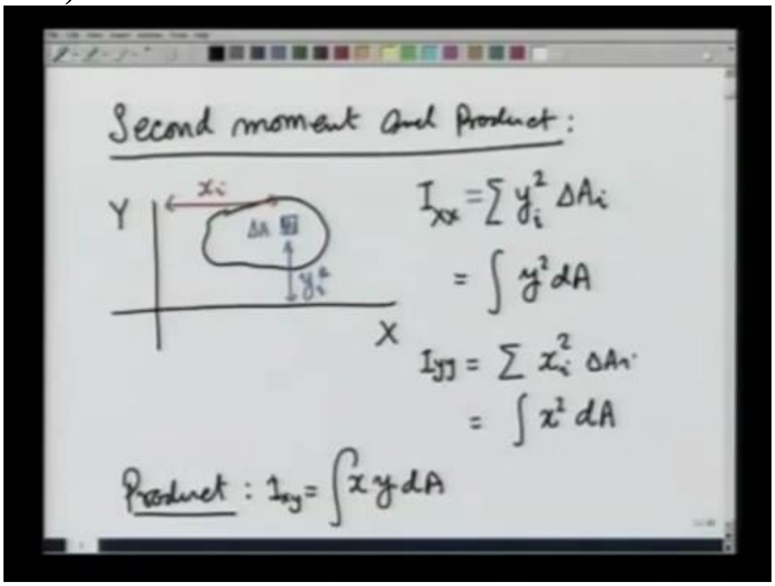

So to begin with, let us define what is second moment and product of an area. And product. Suppose I am given an area, a plane area in XY plane like this. This is the X axis, this is the Y axis. Then the second moment IXX about the Y axis is defined as we take a small area let me make it in blue. Delta A, multiply this by the perpendicular distance from the x-axis square. So IXX is defined as, take area Delta AI multiply by its perpendicular distance from the x-axis and add it up.

This is the second moment of this area with respect to the x-axis. And this obviously goes to the integration Y Square DA. Similarly, IYY, that is the second moment about the Y axis is defined as the distance of the area from the Y axis is chosen and then we write this as summation XI square Delta AI or limit of integration this becomes X square DA. These are just mathematical definitions.

And then the product of this area is defined as integral XYDA. And I am going to call this IXY equals this. So we have defined the second moment of a plane area and product of a plane area.

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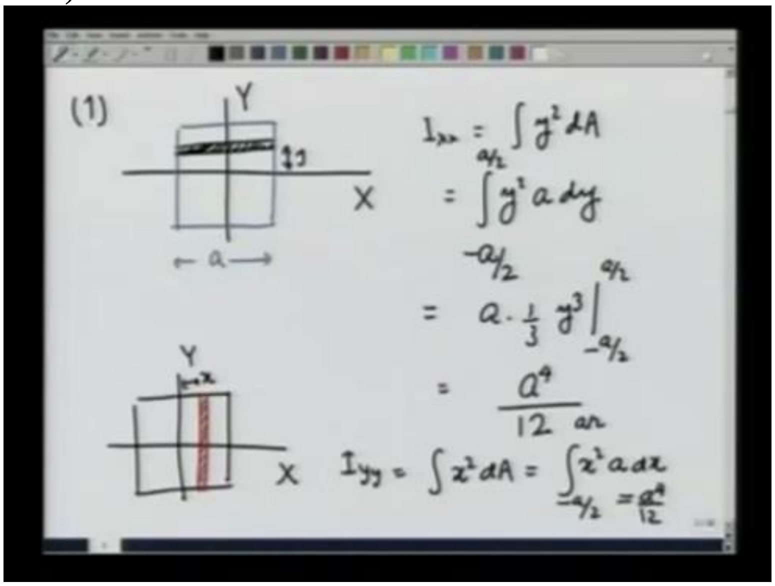

Let us now work out some examples of how to calculate this. As the first example, I take a square of side A and calculate for it the moment of area about the x-axis and about the y-axis and its product of area. IXX is equal to by definition Y Square DA. To calculate DA, I choose a strip at height Y because for this entire strip, the moment of the area is going to be the same. So this becomes integral Y Square ADY and Y changes from - A by 2 to A by 2 and therefore IXX is going to be A times one third Y cubed - A by 2 to A by 2.

Or this comes out to be A raised to 4 over 12. Similarly to calculate IYY, this is the square, I choose a strip like this and calculate IYY as integral X Square DA which in this case would become integral, this is at distance X. So integral X square A DX from - A by 2 to A by 2. And this also in this case by symmetry would come out to be A raised to 4 over 12.

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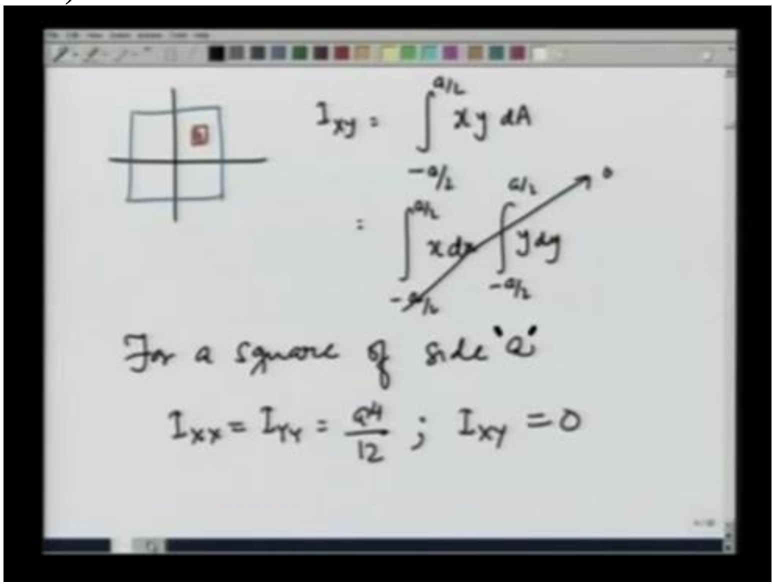

third, for the product of area, you see that IXY is equal to integral XYDA - A by 2 to A by 2. And if I choose this small area DA here, this will be equal to integral - A by 2 to A by 2 XDX integral - A by 2 to A by 2 YDY. And the symmetry of the function X and Y, this goes to 0. So for a square of side A, IXX equals IYY equals A raised to 4 over 12 and IXY equals 0.

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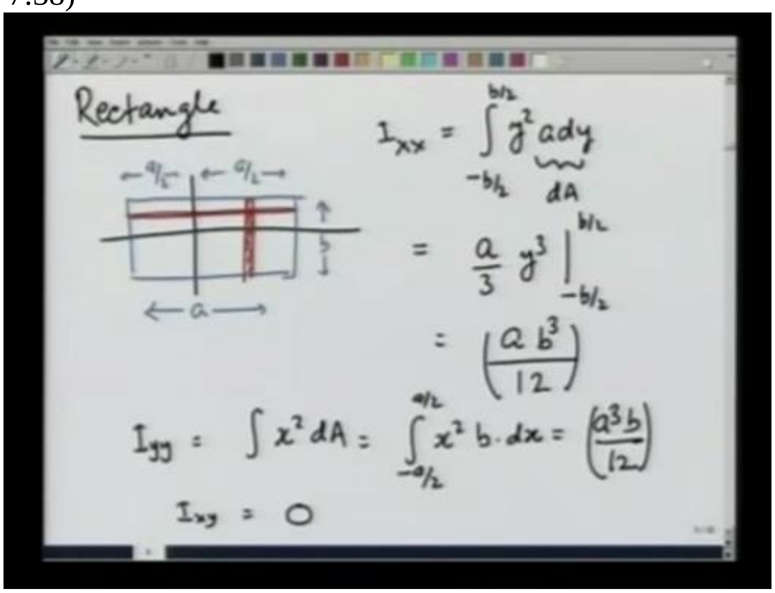

As a second example let me take a rectangle of length A and width B placed symmetrically about the origin. So this side is precisely A by 2 and so this is not made to scale. But please understand this is A by 2, A by 2 divided on both sides. And I want to calculate IXX. For that again I choose a strip here width DY because for this entire strip, Y Square is the same and calculate integral Y Square A DY is that small area DA and Y changes from - B by 2 to B by 2.

And this comes out to be A over 3 Y cubed - B by 2 to B by 2 or AB cubed over 12. Similarly when I calculate IYY, for that I choose a strip parallel to Y axis and calculate integral X square DA which now becomes X square BDX X varying from - A by 2 to A by 2 and this comes out to be AQ cubed over 12. How about IXY? IXY again you will see by symmetry because the area is equally distributed on the negative side and the positive side of the Y and X axis comes out to be 0. So you have also calculated the second moment and product of area for a rectangle.

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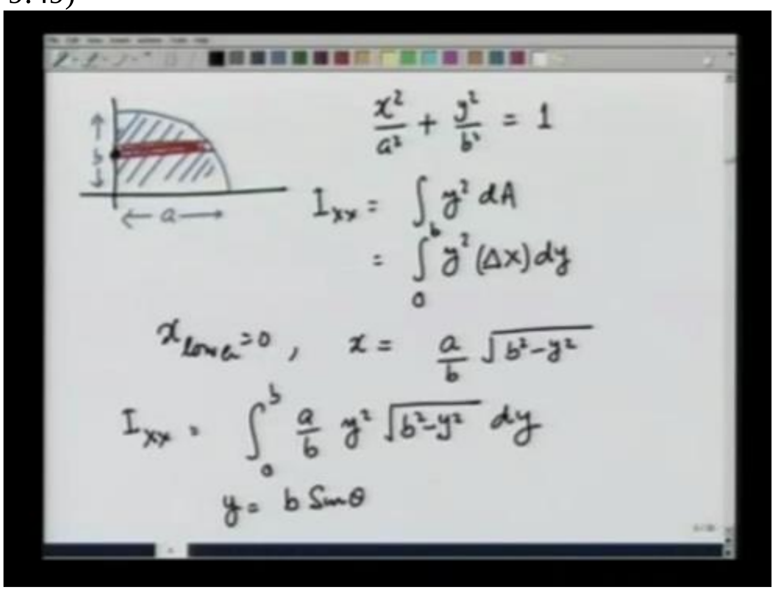

As a third example we make it slightly more complicated and I wish to calculate second moment and product of area for an ellipse for quarter of an ellipse here with semi-major axis A and semi-minor axis B. The equation for the ellipse is X square over A square + Y Square over B square is equal to 1. To calculate IXX which is integral Y square DA I choose a strip parallel to the x-axis of width DY and write this quantity as Y Square.

The length of the strip, let us call it Delta X DY. My job is to calculate Delta X and Y varies from 0 to B. From the equation X lower is 0, that does this point. And I also know from the equation of the ellipse that X is equal to A over B square root of B square - Y Square. Therefore, IXX is going to be 0 to B A over BY Square square root of B square - Y Square DY. This is what we got to integrate.

To do this, we substitute Y equals B sine of theta.

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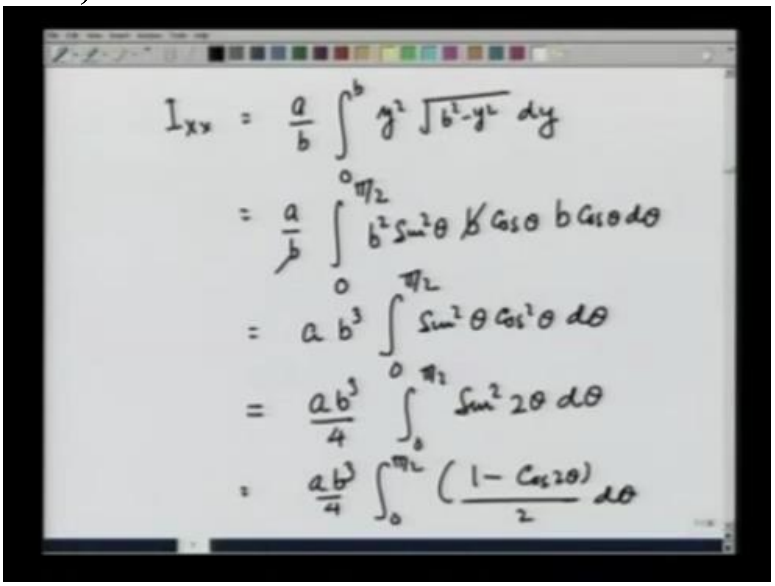

By doing so, I get IXX which is A over B integral Y Square square root of B square - Y Square DY 0 to B as A or B integral 0 to pi by 2 Y Square is B square sine square theta. B square - Y Square square root is going to be B cosine theta and DY is again B cosine theta D theta. And this gives me, this B cancels and I get AB cubed integral sine square theta cosine square theta D theta 0 to pi by 2.

This can be written as A B cubed divided by 4 0 to pi by 2 sine square 2theta D theta which is AB cubed over for integral0 to pi by 2 1 - cosine 2 theta over 2 D theta. Doing this integral, when we do cosine 2 theta integral, that gives me 0 and the first endeavours going to give me pi by 4 .

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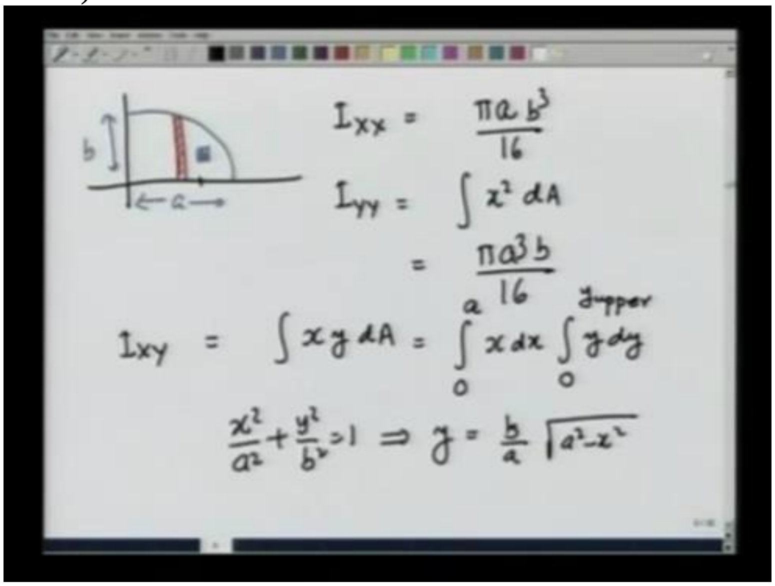

And therefore I get for this shape where this is A, this is B, IXX equals piA B cubed over 16. Similarly I can calculate IYY which is integral X square DA where now I am going to choose a strip parallel to the y-axis. The integral is very similar to what we just now did and this will come out to be pi A cubed B over 16. How about IXY? To calculate IXY, I have to calculate the integral XYDA where DA I take to be a small area at point X and Y.

This can therefore be written as XDX YDY, X varies from 0 to A and for any given X, a given X here, Y varies from 0 to Y upper where Y upper is by the equation X square by A square + Y Square over B square equals 1. And that gives for any given X that Y is going to be B over A square root of A square - X square.

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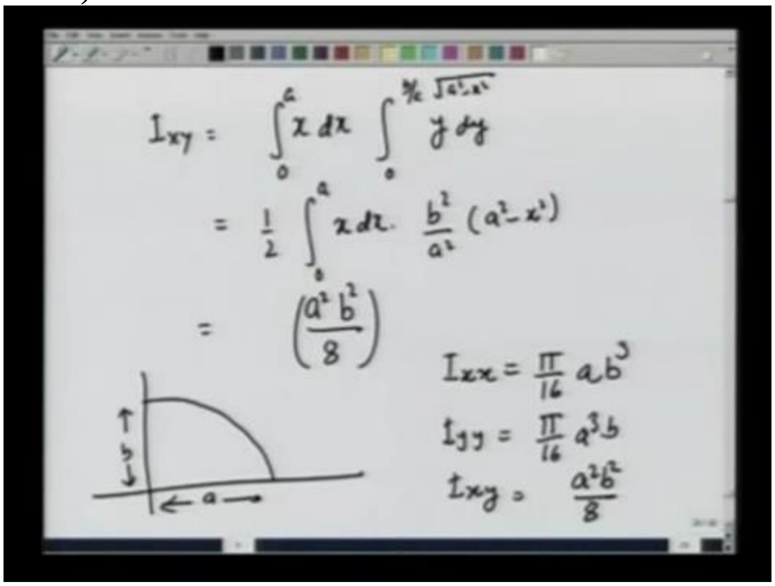

And therefore IXY for this is going to be integral XDX 0 to A integral 0 to B over A square root of A square - X square YDY which is nothing but one half 0 to A XDX B square over A square times A square - X square. This is a standard integral. So this gives me the answer A square B square divided by 8.

Therefore what we have determined that for quarter of an ellipse semimajor axis A and semi-minor axis BIXX is pi over 16 A B cubed and IYY is pi over 16 A cubed B and IXY is A square B square over 8. I will now leave it for you as an easy exercise to calculate IXX and IYY that is the second moment of an area with respect to X and Y axis for the entire ellipse. And also show that IXY for the entire ellipse is going to be 0 because of the symmetry between the X and Y axis.

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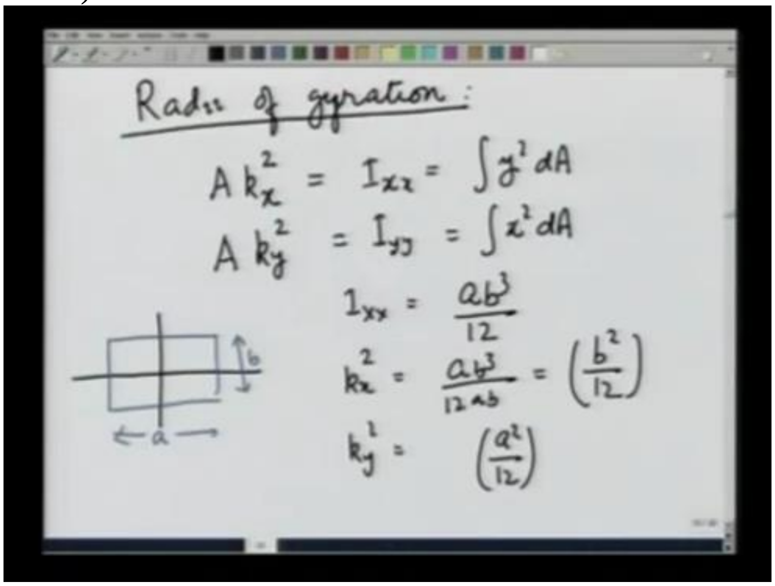

Using a second moment of Asia, we can also define, let me call the radii of the gyration. By gyration you can already see that these quantities are going to be useful when we describe rotation. The radius of gyration KX off and area A about the x-axis is defined through the relationship A KX square equals IXX equals integral Y square DA and radius of gyration for about the Y axis is defined as AKY square equals IYY equals integral X square DA.

Thus, for example when we look at a rectangle of length A and width B and we have already calculated that IXX is equal to A B cubed over 12 and therefore K X square is going to be AB cubed over 12 AB or B square over 12.. Similarly KY Square is going to be A square over 12. And from these radii of gyration about the x-axis and the y-axis can be calculated.