Engineering Mechanics
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Module 03
Lecture No 30
Properties of plane surfaces-I- First moment and centroid of an area

In the previous lectures, we have dealt with forces which act on a point.

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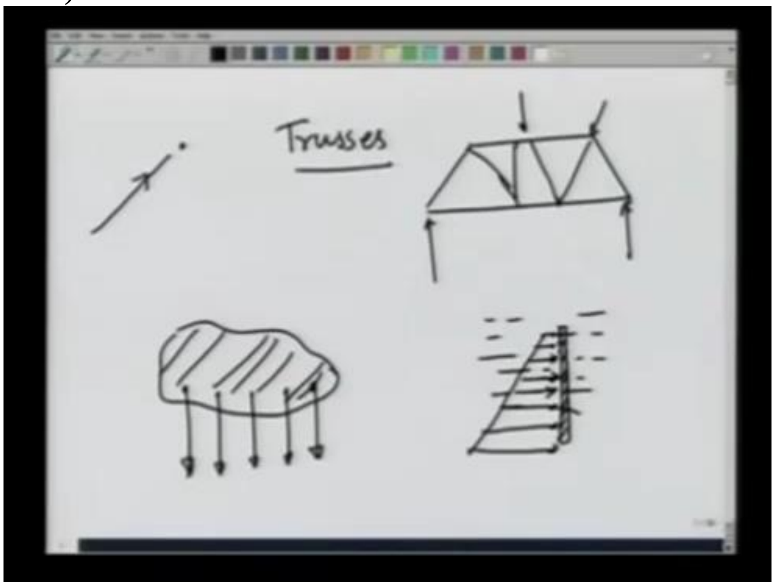

So for example, I took a mass on which a force is acting. Then we took trusses and dealt with forces acting on the joints of the trusses. For example one force would act here, one force would act at this point, one force would act at this point and so on. Similarly when we considered frictional forces, we considered the forces as if they were acting at one particular point.

However, in nature, the forces are distributed. For example if I take an extended body and consider the gravitational force on it, on each point, there is a force acting. It is net effect is felt at a certain point which we know as the centre of gravitation. Similarly if I take a plate submerged say in water, the force acting on it due to the pressure of water is also distributed.

Not only it is distributed, its magnitude at different points. So deeper you go, larger it is. Near the top, the force is not that large. So these are the arrows that are showing the force due to the pressure. This is also a distributed force which varies with the depth of water. And now we want to develop mathematical techniques to deal with such forces.

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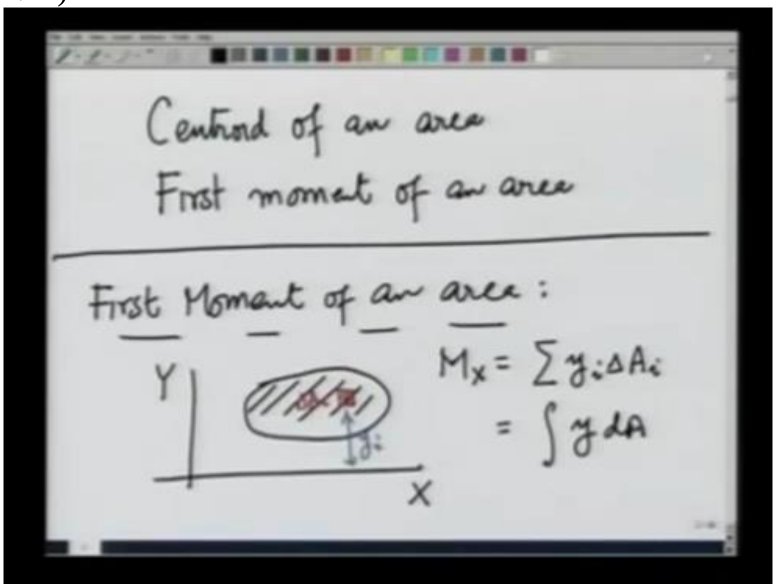
e 

In this lecture, I am going to develop concepts such as centroid of an area which is related to the First moment of area and then see how these concepts can be applied to mechanics or statics in particular. So let us start with the concept of the First moment of area. I am first going to give you the definition and then later we will exploit in certain static situations. Suppose I am given a plane area like this in a coordinate axis aet X and Y.

I define the first moment of this area MX about the X axis as summation YI delta AI where delta AI is a small area here, delta AI at a distance of YI from the x-axis. So what am I doing? I am taking small elemental area delta AI and multiplying it by the distance from the x-axis and summing it up. In a continuous distribution, you know from your course in calculus, this is nothing but YDA. That is the first moment about the x-axis.

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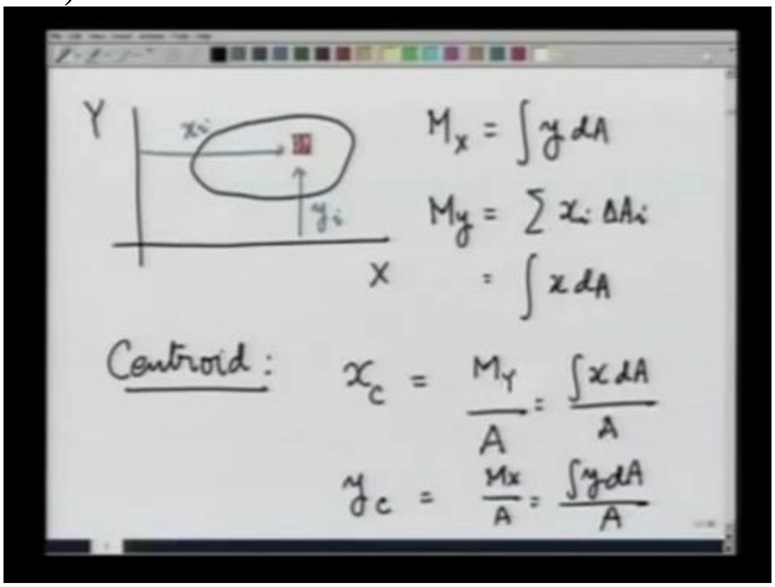

In a similar manner, we define the first moment about the y-axis. This is the area and I took a small elemental area here at a distance YI from the x-axis and a distance XI from the y-axis. This is x-axis, this is y-axis. Define why first moment about the x-axis as YDA and similarly I can define MY that is the first moment about the y-axis as summation XI Delta AI which in the limit of continuous area becomes XDA. These are the definitions.

So we have first moment about the X axis and first moment about the y-axis. Using the concept of first moment, I can define the centroid of this area. Centroid coordinate, X coordinate XC is defined as first moment about the y-axis divided by the total area which is summation XDA divided by A. And similarly the Y coordinate is defined as an MX divided by A which is summation of YDA divided by A. Now let us do certain examples using this definitions to calculate these quantities.

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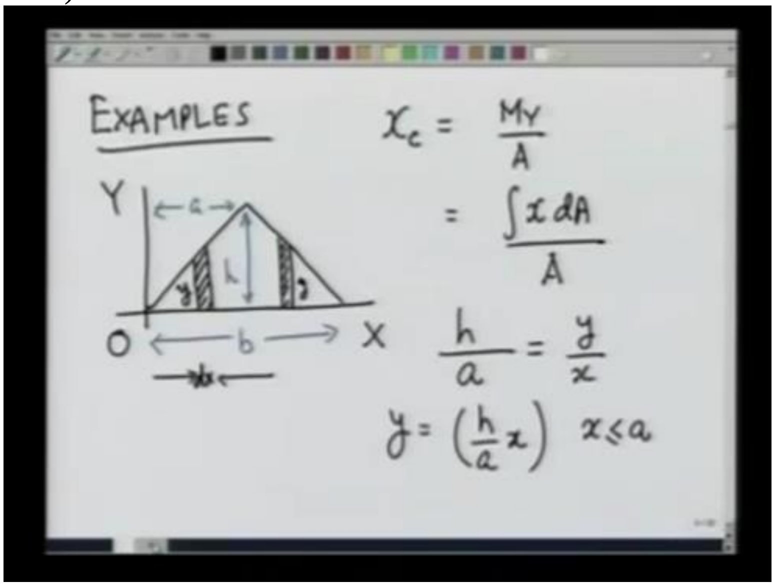

As the first example I take a triangle. One corner of this triangle at is at the original. This is the X axis, this is the Y axis. The height of the triangle is H. This distance of the vertex from the Y axis is A and the other corner is at distance B. I wish to calculate its first moment about the X and Y axis and it is centroid XC and YC. So let us do that.

Let us was calculate XC which is related to MY divided by the total area which is equal to integration XDA over the total area. To calculate XDA of the moment about the y-axis, let me take a small area BA like this, parallel to the y-axis. This distance I will take to be the X. The height of this is Y. From similarity of triangles, I know that H divided by A is going to be equal to Y divided by X. And therefore Y equals H over A X.

This is true as long as this X is below A. For X greater than A, I take an small area here whose Y is going to be different and that also we can calculate. Let me now go on to the next page so that calculations become more clear.

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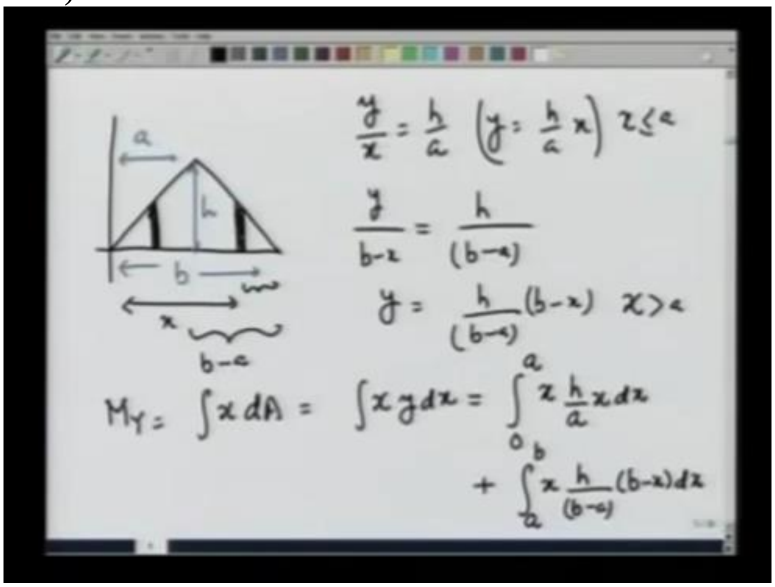

This is the y-axis, x-axis. This is the triangle. This is B, this is A, this is H. One step I took here, other step I am taking here. The first step I know that Y divided by X was equal to H over AY equals H over A X. X less than or equal to A. For the second step, I am going to have this distance is X. So I am going to have Y divided by B - X, that is this distance Y divided by B - X is going to be H divided by B - A.

B - A is this distance. And therefore Y equals H over B - A times B - X. Now I am ready to calculate moment about the Y axis. Moment about the Y axis MY is equal to integration XDA which I am going to write as integration XYDX where YDX represents the area of this strip or this small strip. This I am going to divide into two parts because behaviour of Y with respect to X is different for X up to A and for X greater than A.

This by the way is for X greater than A. So I am going to write this as integration 0 to A XY up to A Y is H over A XDX + A to B XH over B - A B - X DX. that is MY. And you calculate this. Integrations you can perform easily. You get an answer for MY.

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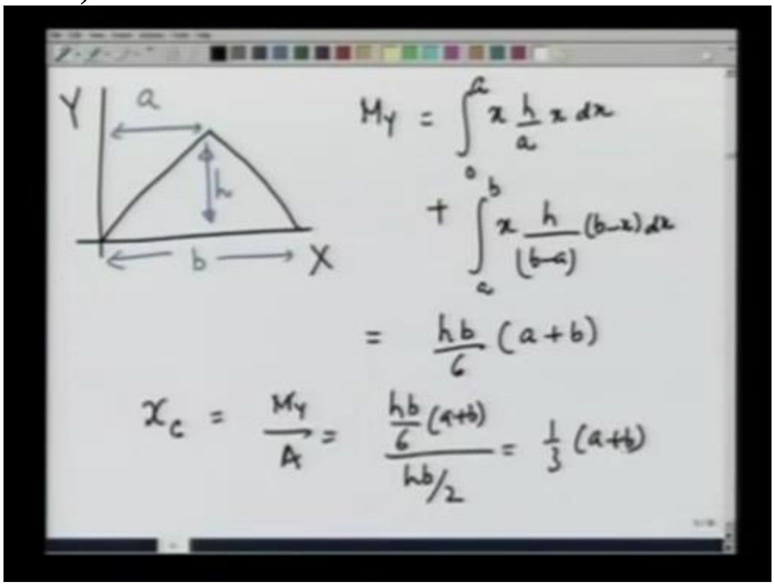

Let me make the figure again. This is B, this is A, this is the height H, this is the X axis, this is Y the axis. And for MY which I wrote as 0 to A X H over A times XDX. Let me see I indeed wrote H over A, yes. And + A to B X H over B - A B - XDX and **we** we do this integration it comes out to be HB divided by 6 times A + B. And therefore, X centroid is equal to MY divided by the area which is HB over 6 A + B divided by HB divided by 2.

And this comes out to be one third of A + B. That is, the X coordinate of the centroid of the triangle. Let us not calculate the Y coordinate.

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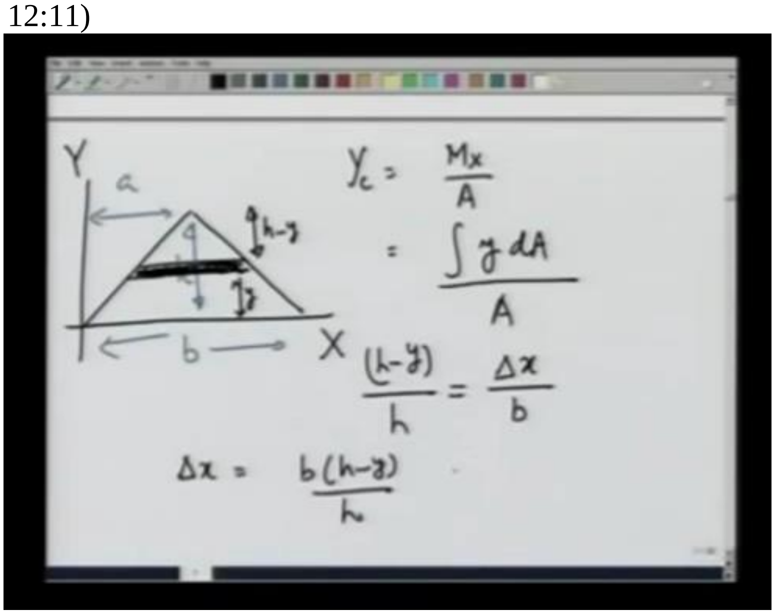

Again I make this triangle. X, Y, B, A, H. And to calculate the Y coordinate, I take the elemental area DA like this. Recall that YC is equal to MX divided by the area which is integration YDA divided by the area. So at a height Y, I take this is small step multiplied by Y and then calculate the area. We can see from similarity of triangles that I am going to have Y H - Y. This is H - Y divided by H is equal to the width of the strip which let me call Delta X divided by B.

And therefore Delta X is equal to B H - Y divided by H. This is the length of the strip.

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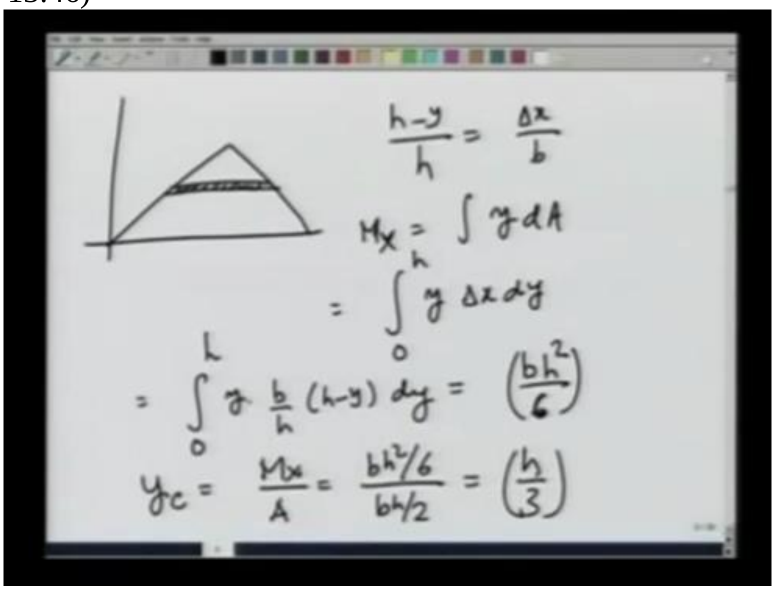

And therefore, we can write, let me make the picture again. I am taking this trip. I am not writing B and A this time. We got H - Y divided by H equals Delta X divided by B. And MY or MX is going to be equal to YDA which I can then write as Y Delta X DY. Y varying from 0 to total height H and this comes out to be Y Delta X is B over H H - Y DY.

And you do this, this comes out to the B Y varies from 0 to H, this comes out to be BH square over 6. And therefore YC is going to be MX divided by the total area which is BH square divided by 6 over BH divided by 2 which comes out to be H over 3. So we have calculated the centroid of a triangle and what is it?

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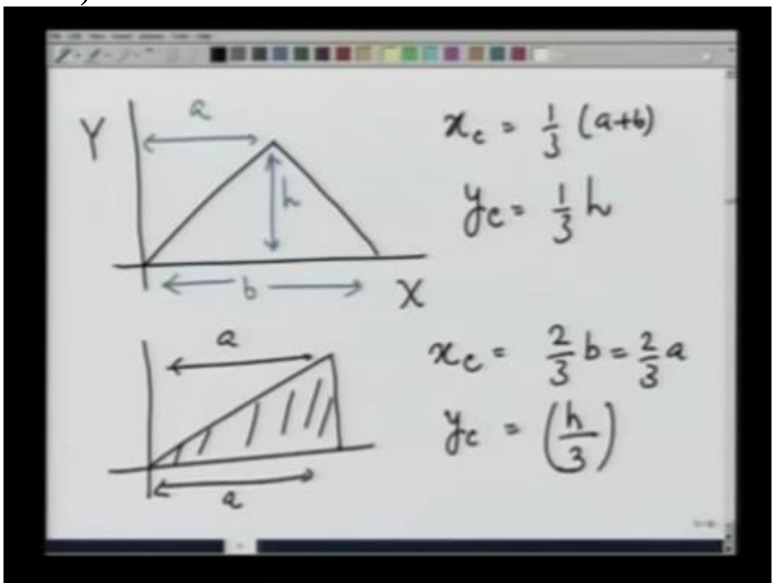

It is it is for our triangle it is given like this. With this distance being B, this being A, this height being H, the X , Y. The X coordinate is one third of A + B and Y coordinate is equal to one third of H. You can see that Y coordinate is one third of the height and X coordinate is one third of B + A. So suppose I had a triangle like this.

An area which is triangle of this shape. In that case A and B are going to be the same so that XC in this case would become to thirds of B or two thirds of A and YC is obviously equal to H over 3. So we have calculated the centroid for a triangle.

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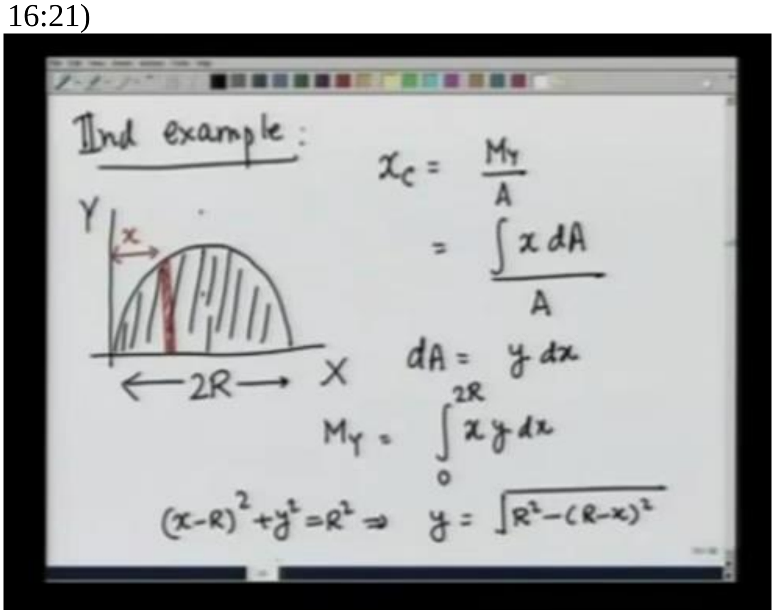

The second example, let me take a semicircular area like this of radius R. So obviously its diameter is 2R. This is the x-axis, this is the Y axis and I wish to calculate its centroid. From symmetry, you can make out that the centroid must be on this axis somewhere but I want to explicitly calculate it to show you how to do it. So again XC is going to be equal to MY over the total area.

To calculate MY, we do integration XDA over the area. To calculate the DA, I take a strip like this which is a vertical strip at a distance X from the Y axis. You can see that DA in this case therefore is going to be the area of this strip which is YDX. And therefore, MY is going to be integration XYDX X varying from 0 to 2R. What about the value of Y at a given X?

The equation of the circle whose centre is at R and Y0 is going to be X - R square + Y square equals R square. And therefore Y is equal to square root of R square - R - X square.

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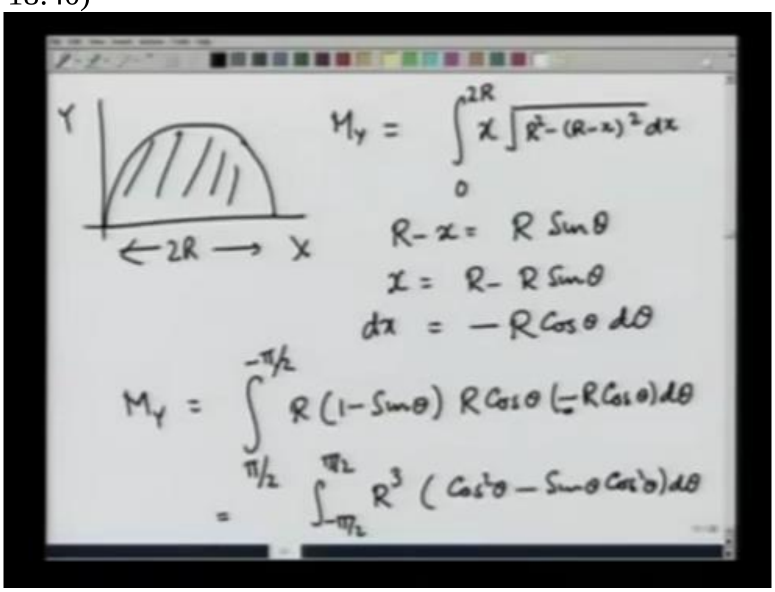

And therefore MY for the semicircle age is going to be equal to, this is 2R integration XY is the square root of R square - R - X square DX 0 to 2R. Let me take R - X to be equal to R sine theta so that X equals R - R sine theta. In this case, DX is going to be equal to - R cosine theta D theta. Substituting this, we get MY equals when X is zero sine theta is going to be one and therefore theta is equal to pi by 2.

When X equals 2R, sine theta is going to be - 1 and therefore this is going to be - pi by 2. X is R 1 - sine theta. X equals R 1 - sine theta times R square - R - X square is going to be R cosine of theta. And then for DX I have - R cosine of theta D theta. Because of this - sign here, I can write this as - pi by 2 to pi by 2 R cubed cosine square theta - sine theta cosine square theta D theta. This is a simple integration which we can perform.

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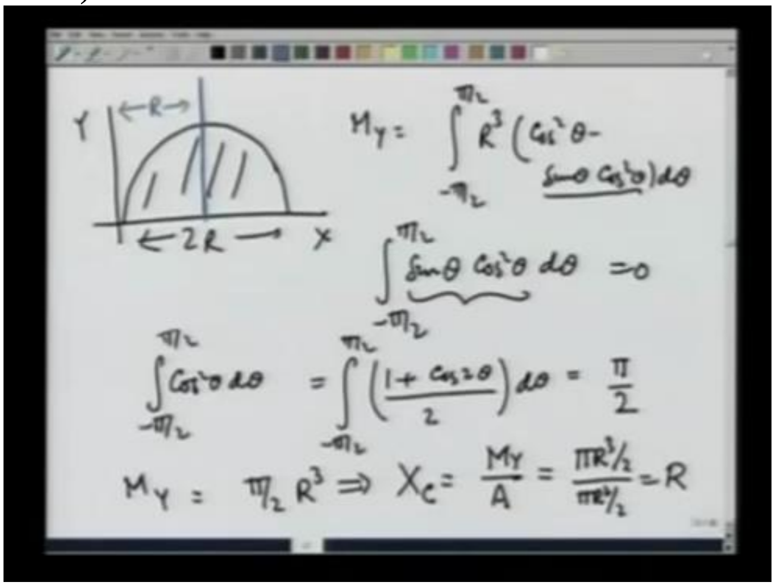

So I have MY for the semicircle of diameter 2R MY equals integration - pi by 2 to pi by 2 R cubed cosine square theta - sine theta cosine square theta D theta. Cosine square theta sine theta is an odd function with respect to theta. And therefore this integration sine theta Cosine square theta D theta - pi by 2 to pi by 2 is going to be zero because as you go from theta to - theta this integrand change the sign.

And integration cosine square theta D theta - pi by 2 to pi by 2 comes out to be one + cosine 2 theta over 2 D theta - pi by 2 to pi by 2 which is nothing but pi over 2. And therefore, we get MY is equal to pi over 2R cubed or XC witches MY divided by the area. Area is pi R square by 2. So pi R cubed divided by 2 divided by pi R square by 2 equals R. So as we noticed earlier, the X coordinate is on this axis. This is R. How about the Y coordinate of the centroid? Let us calculate that.

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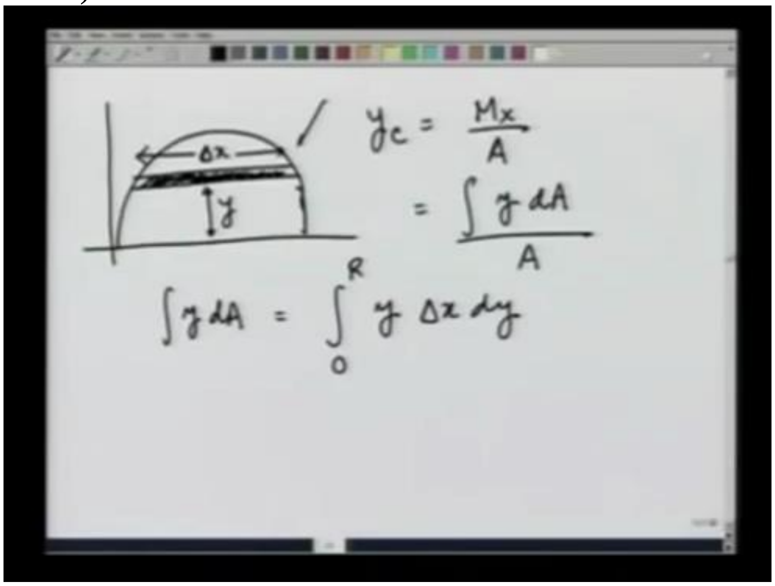

To calculate the Y coordinate I take a strip like this at a height Y. So YC is nothing but MX divided by the total area which is integration YDA divided by the total area. YDA you can see is going to be equal to the area of this strip here, this one times the Y this strip area is going to be this if this distance is delta X delta X DY and Y varies from 0 to R.

To calculate the value of delta X we again use the equation for a circle, X - R square + Y square equals R square and that gives me X - R is equal to + or - square root of R square - Y square. I get two values of X, namely X1 I have shown here equals R + sorry - square root of R square - Y square and X2 which I have shown here which is equal to R + square root of R square - Y square.

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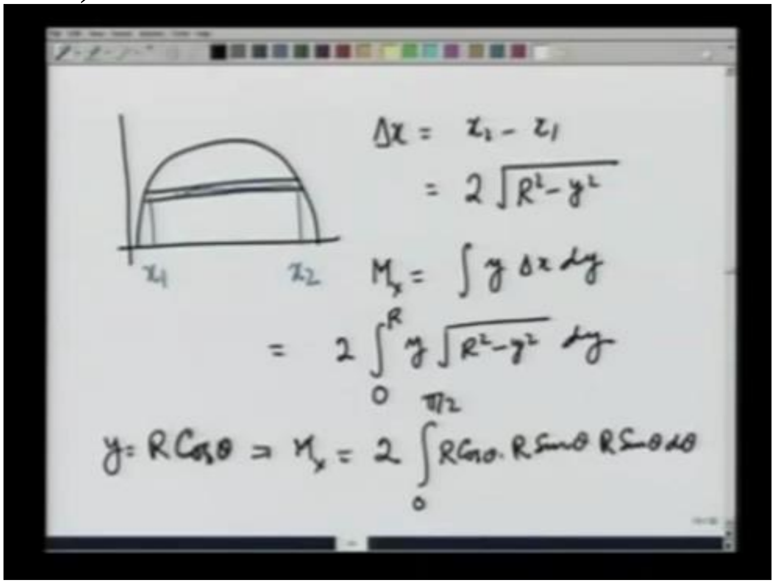

And therefore delta X again let me make this figure. This is X1, this is X2. Delta X is going to be X2 - X point which is equal to 2 square root of R square - Y square. Remember we are after MX which is equal to Y delta X DY which in this case is therefore going to be 2 integration Y square root of R square - Y square DY Y varying from 0 to R.

Let us take Y equals R cosine theta so that MX becomes equal to 2 integration 0 to pi by 2 R cosine theta times R sine theta times DY is R sine theta D theta.

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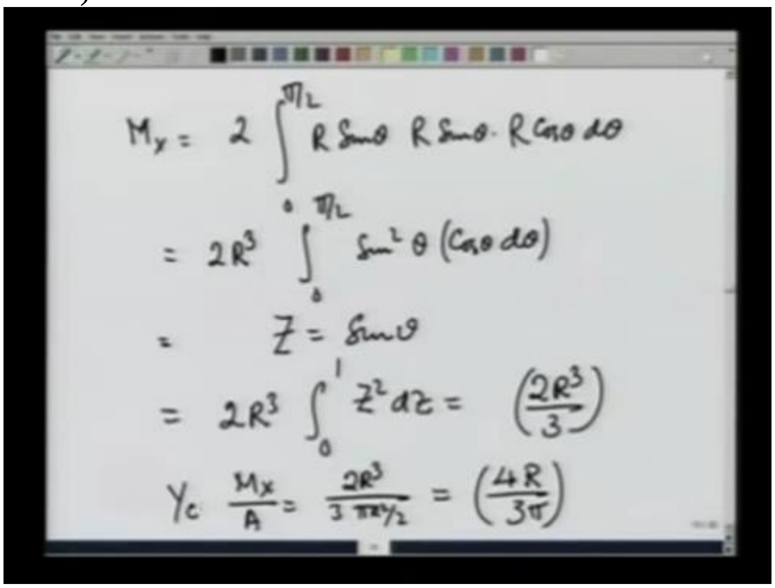

So we have M this is MX equal to integration 2 0 to pi by 2 R sine theta R sine theta R cosine theta D theta which is 2 R cubed integration 0 to pi by 2 sine square theta cosine theta D theta. Cosine theta D theta is D sine theta. So I can write this as by substituting Z equals sine theta I can write this as 2 R cubed 0 to 1 Z square DZ which is 2 R cubed divided by 3. That is MX and therefore XY centroid which is MX divided by the total area is going to be equal to 2R cubed over 3 divided by pi R square by 2 which is nothing but 4R over 3 pi.

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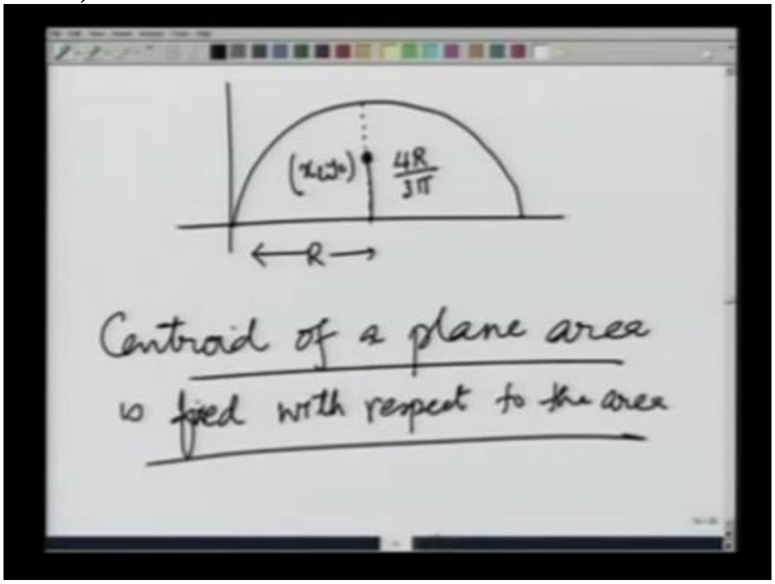

Therefore if I look at the semicircle, the semicircular area its centroid at a distance R and at a distance at a height of 4R over 3pi. Slightly 4 3pi is roughly 9. So 4R 3pi is slightly less than R by 2. So it is somewhere here. This is XC YC. Having defined the centroid of a plane area and solved two examples, let me make a point about the centroid of a plane area and that is that the centroid of a plane area is fixed with respect to the area.

That means no matter which coordinate system I calculate the centroid in, it will always come out to be the same point in the body or in that area. So centroid of a plane area is fixed with respect to the area. And let us see how we can improve this.

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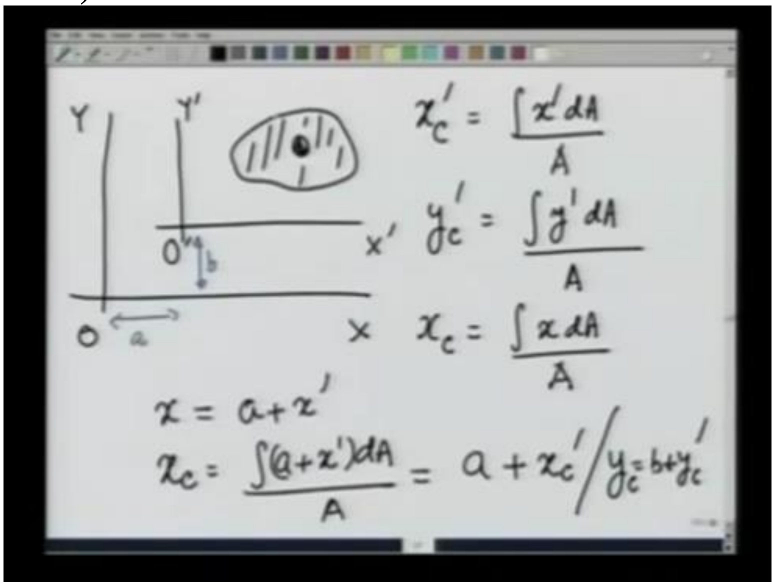

So let us take two set of axis, X, Y and origin O and another set of axis X prime, Y prime with origin O prime. The O prime is shifted with respect to O by X coordinate A and Y coordinate B. Let the centroid coordinate in O prime frame be XC prime which is equal to XDA over the area rather X prime. And similarly YC prime is equal to integration Y prime DA over the area where DA is some small area given in this plane area which I have drawn here.

Now let us calculate XC which is equal to integration XDA over the area but I know that X equals A + X prime and therefore XC is going to be equal to A + X prime DA over A. ADA integral, A comes out and integral DA is the total area. So this will come out to be A + X prime DA over A is nothing but C prime.

So you see if I calculate the centroid with respect to shifted coordinate system, the centroid shifts by appropriate amount so that to remain and the same point in the body. And similarly you can show that YC will be equal to be + YC prime which I leave as an exercise for you.