Introduction to Electromagnetism
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Lecture - 59
Electromagnetic Waves in Free space – II
Derivation of the Wave Equation from
Maxwell's Equations

We have been discussing electromagnetic waves.

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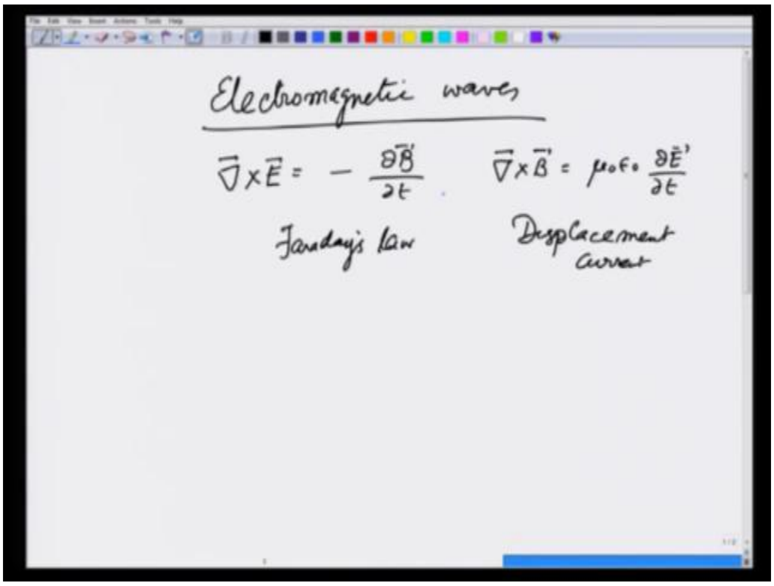

And, in the previous lecture, I argued… Electromagnetic waves; in the previous lecture, I argued on the basis of Faraday's law, which says del cross E is equal to minus partial derivative of B with respect to time and displacement current in free space. That it is possible to sustain electromagnetic field when they change and keep generating each other and it propagates. So, we use this source Faraday's law; and, this was based on displacement current. In this lecture, I want to use them the way Maxwell has written it. And, using those Maxwell's equations, we want to derive a wave equation and then discuss it further. So, we are going to be mathematically little more rigorous than the arguments that we gave in the previous lecture. But, I want you to appreciate what we did in the previous lecture; it gave you a physical feel of what electric and magnetic fields may look like and how they change with each other. In this lecture, we are going to see the same treatment in a more sophisticated mathematical method.

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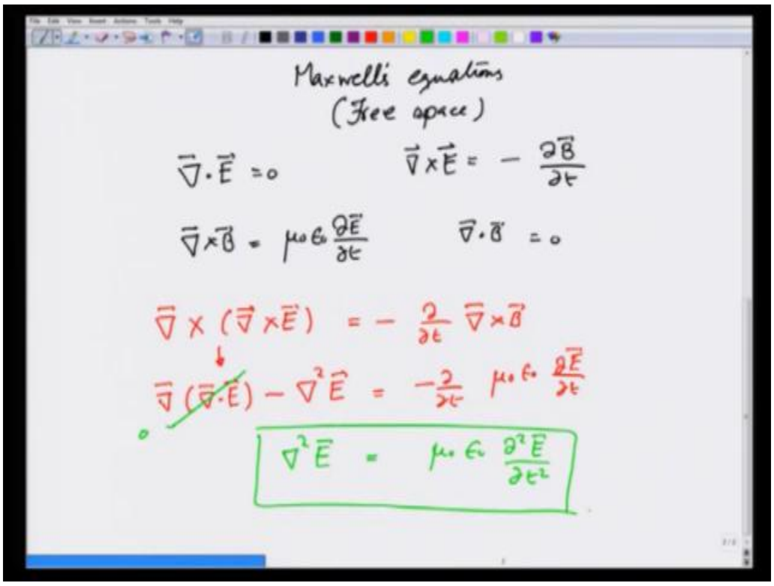

So to start with, let us write all the Maxwell's equations once more. And, I am writing them in free space; free space and there is no charge, no charge density. So, the equations I have is divergence of E is 0; curl of E is equal to partial derivative of B with respect to t. Partial derivative of B – since there is no real current, we have mu 0 epsilon 0 – the displacement current; and, divergence of B is 0. Let us look at the two curl equations and take the curl of this equation – the Faraday's law equation once more. This becomes equal to minus d by dt of curl of B. I have taken curl on both sides. This however is gradient of divergence of E minus Laplacian of E. This is equal to minus d by dt. And, curl of B from curl of B equation is equal to mu 0 epsilon 0 dE dt. From the first equation, Gauss's law, this term is 0. And therefore, I get del square E is equal to mu 0 epsilon 0 d 2 E over dt square.

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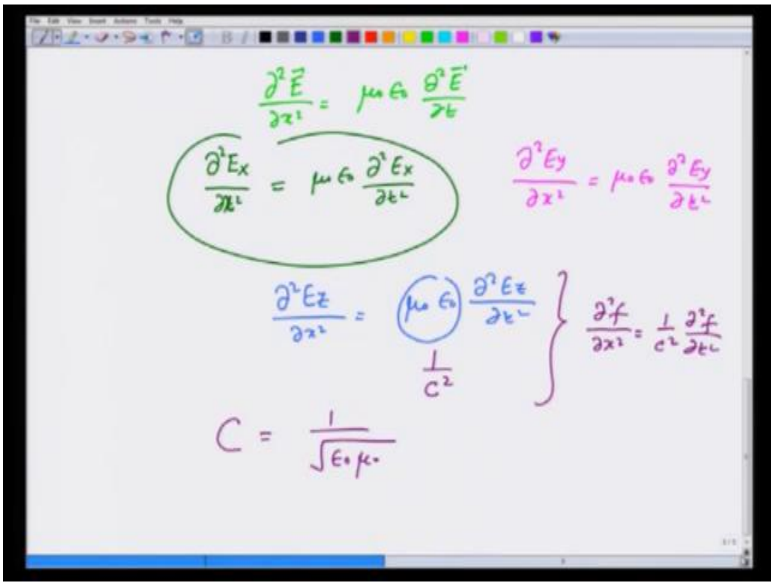

In particular, if the variation of E is only in one direction, let us say, x direction; then, I have partial E by partial x square is equal to mu 0 epsilon 0 d 2 E by d t square. When I am writing this E, that means each component satisfies this equation. Let us write this explicitly. I have d 2 E x d x square is equal to mu 0 epsilon 0 d 2 E x by d t square. We will see this equation actually is redundant, because there is going to be no x component that, d 2 of E y partial x square is equal to mu 0 epsilon 0 partial E y over partial t square. And finally, partial of E z partial x square is equal to mu 0 epsilon 0 partial E z over partial t square. Notice that, this is exactly like the wave equation with this quantity here being 1 over c square. So, with that, this becomes equals to d 2 f by d x square is equal to 1 over c square d 2 f over d t square. So, we immediately see that, the speed of this propagation or the electromagnetic wave is going to be 1 over square root of epsilon 0 mu 0 and, E propagates like a wave. We can do the same thing for B equation.

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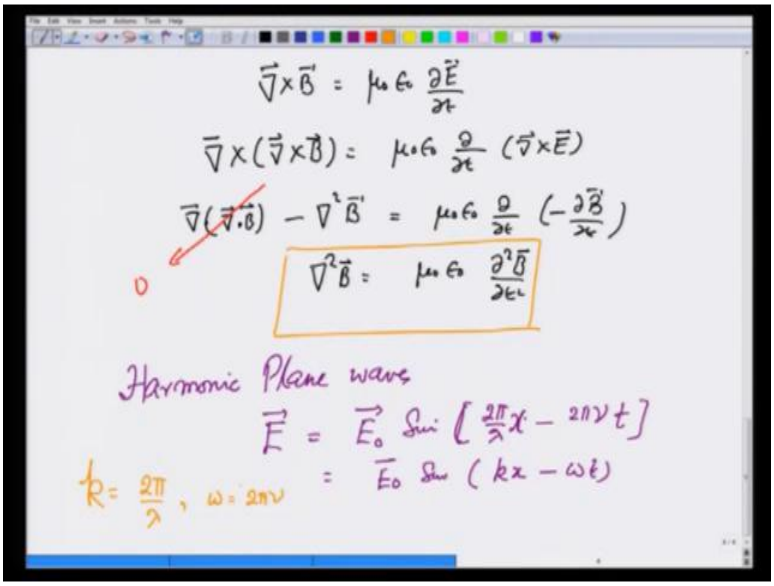

And, what we get in the b equation is – you start with curl of B is equal to mu 0 epsilon 0 dE dt and take curl on both sides. We get curl of curl of B is equal to mu 0 epsilon 0 d by dt of curl of E. This is gradient of divergence of B, which is 0 minus Laplacian of B is equal to mu 0 epsilon 0 d by dt of minus dB dt. That again gives me del square B is equal to mu 0 epsilon 0 d 2 B over dt square, which is the equation for B field. So, a and b propagate like a wave. What we are going to do now is focus on a particular solution of this harmonic plane waves in which E vector is going to be given as some amplitude E 0 sine of 2 pi over lambda x minus 2 pi nu t; which I can write as E 0 sine of k x minus omega t; where, k is 2 pi over lambda and omega is 2 pi nu.

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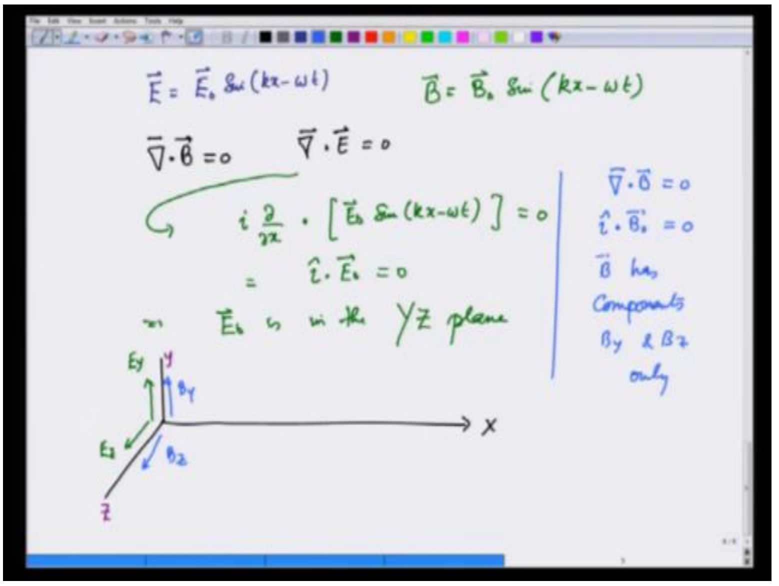

Similarly, I am going to write B as some B 0 sine of k x minus omega t. So, I have E is equal to E 0 sine of k x minus omega t; and, B equals B 0 sine of k x minus omega t. Let us see what equations tell us about these amplitudes and the fields if they exist like this. So, let us look at the equation – divergence of B is 0 and divergence of E is 0. If we do that… Let us look at this equation – divergence of E is equal to 0. I am going to have i d by dx dot E 0 sine k x minus omega t is equal to 0. Notice that, all the other components d by d y and d by d z – they vanish, because we have taken E naught to be same all over y and z. So, this gives you i – x unit vector dot E 0 is equal to 0. This means E 0 is in the yz plane. What it means is I have this wave propagating in the x direction; then, E can have components… Let me make… Since I am writing E with green, let me make – E can have components y and z. So, let us write this E y and E z and nothing else, because E dot i is 0. This is y; this is z. Similarly, if I look at divergence of B is equal to 0, I get i dot B 0 is equal to 0. This also means that, B has components B y and B z only. So, I have B y and B z. So, both E and B are perpendicular to direction of propagation as we had indeed argued earlier.

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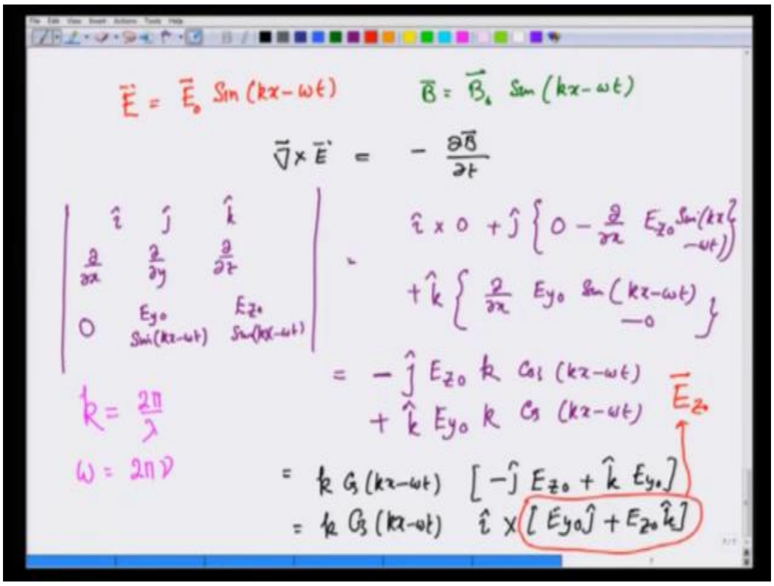

Let us now look at the relationship between E and B using other Maxwell's equations. So, we have now seen that, E is E in the yz plane 0 sine of k x minus omega t; and, B is B 0 sine of k x minus omega t. And, we want to see the relationship between B 0 and E 0. Just one point you may be wondering why I have kept sine k x minus omega t for both – same variation for both electric and magnetic field. If I did not do so, you will see later that, these terms will not cancel on two sides. And therefore, their dependence on k x minus omega t has to be exactly the same.

Now, let us look at the equation – curl of E is equal to minus the partial derivative of B with respect to time. If I calculate curl of E, this is equal to i j k d by dx d by dy d by dz; x component of electric field is 0; the y component is going to be E y 0 sine of k x minus omega t; and, the z component is going to be E z 0 sine of k x minus omega t. And, if I calculate the curl, it comes out to be i component times 0 plus j component times 0 minus d by dx of E z 0 sine of k x minus omega t plus k component d by dx of E y 0 sine of k x minus omega t minus 0. So, this comes out to be minus j E z 0 k cosine of k x minus omega t plus k unit vector E y 0 k cosine of k x minus omega t. I want to remind you that, this k is 2 pi over lambda and omega is 2 pi times a frequency nu. This can be further written as k cosine k x minus omega t minus j E z 0 plus k unit vector E y 0, which I can write as k cosine of k x minus omega t i cross E y 0 j plus E z 0 k. This last term here is nothing but E vector 0.

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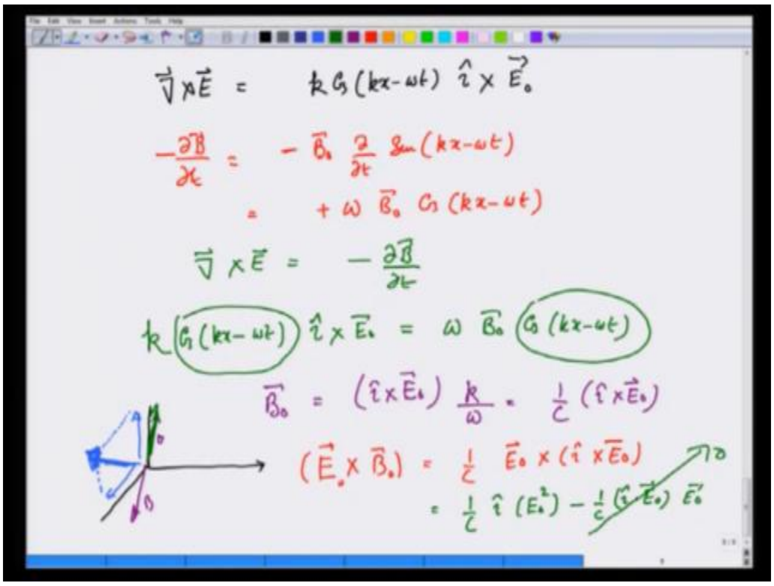

So, what we have obtained – go to the next slide that, curl of E is nothing but k cosine of k x minus omega t i cross E 0 vector. Similarly, minus dB by dt is going to be equal to B 0 vector d by dt of sine k x minus omega t with a minus sign in front; and, this is going to become then plus omega B 0 vector cosine of k x minus omega t. Notice that, earlier I had said – if the space-time dependence, that means, k x minus omega t was different; then, I could not have equated the space and time dependence of the two expression. More explicitly, if I write equate curl of E is equal to minus dB dt, I get k cosine of k x minus omega t i cross E 0 is equal to omega B 0 vector cosine of k x minus omega t. If this dependence, which is given by cosine was not the same on both sides although the equation could have been satisfied once in a while, but it will go out of phase further times. So, this dependence has to be the same.

The net result is that, we get B 0 vector is equal to i cross E 0 times k over omega. k over omega is nothing but 1 over c – i cross E 0. This you can easily see from definition of k being 2 pi by lambda and omega being 2 pi nu. So, what we have now is that, if the wave is propagating in the i direction; if E vector has components like this; the B vector has to be i cross E; it has to be perpendicular to it. So, if E vector is like this – let me make a little thick; then, B vector has to be perpendicular to this in the same plane. So, it has to be either this way or this way. i cross E gives me that. Now, which way would it go? For that, what we can do is we can multiply, take E cross B again; which will be equal to 1 over c E 0 cross i cross E zero. And, that gives me 1 over c i E 0 square minus 1 over c i dot E 0 E 0. So, this term is 0 because i – the propagation direction is perpendicular to E 0. So, what we see is that, E cross B should be I; E cross B should be in the direction of propagation. So, B has to be such that I would choose this particular B, so that B cross E gives me i.

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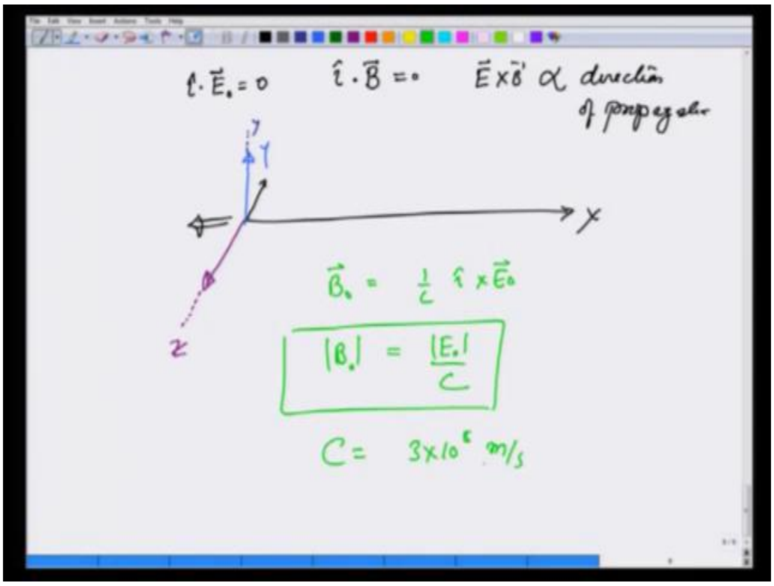

So, what we see is propagation direction and amplitude electric field are perpendicular. Propagation direction, magnetic field are perpendicular. Then, E cross B is – has the same direction as direction of propagation. So, if this is a direction of propagation x and if E happens to be in the y direction; then, B would be in the z direction. On the other hand, if the wave was propagating the other way – it was going to the negative x-axis; if the wave was propagating this way; then, if E y – E was in the y direction; B would be in the negative z direction. This is what we get from the Maxwell's equations. And finally, we saw that, B 0 is equal to 1 over c i cross E 0; which means that, magnitude of B is equal to magnitude of E over c smaller from E by the factor of speed of light, which is 3 times 10 raised to 8 meters per second. So, magnetic field in electromagnetic wave is much smaller than the electric field when they are measured in SI units.