Introduction to Electromagnetism
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Lecture - 56
Linear Momentum and Angular Momentum Carried By Electromagnetic Fields

We have learnt about Electromagnetic Field Carrying Energy.

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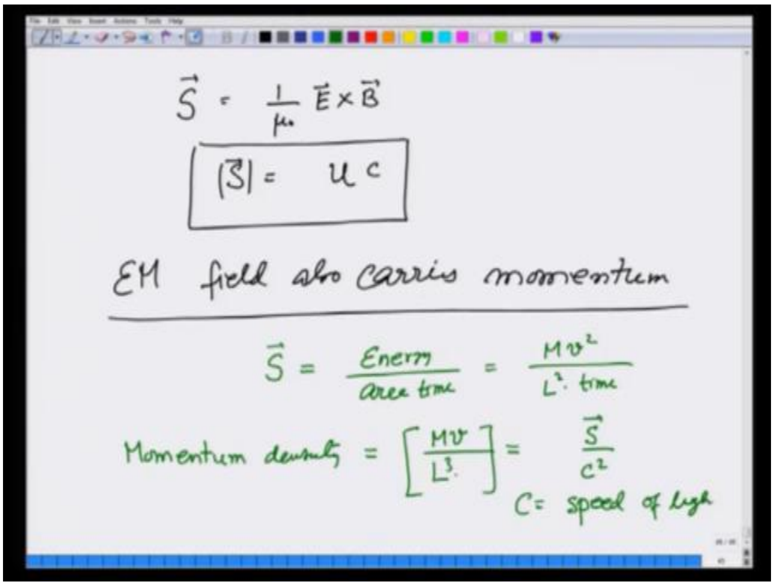

I just want to tell you that S which we wrote as 1 over mu 0, it is easy to show that the magnitude of S is also equal to the energy content, energy density times c, I leave this as an exercise for you. So, it is like you know if there is water in some container or some liquid of density rho and if it is following with a speed v. Then, the flow of the water moving per unit time is the density times the velocity, in the same manner energy flow per unit time per unit area is u times c.

I am saying this now, because I will use it in the momentum equation that I am going to use now, introduce now. Electromagnetic field also carries momentum E M filed also carries momentum and this can also be derived exactly in the same manner as we derived the poynting vector, except that the derivation is a little more complicated and slightly beyond the level of this course. So, I will just state in dimensional way that this S which is nothing but, energy flow per unit area per unit time, which has the dimensions of M v square over L square time is related to the content of momentum or momentum density as follows.

Momentum density which is going to be M v over L cubed that is the dimension. So, this is related to this is the dimension of this, the S vector as S over c square, where c is the speed of light.

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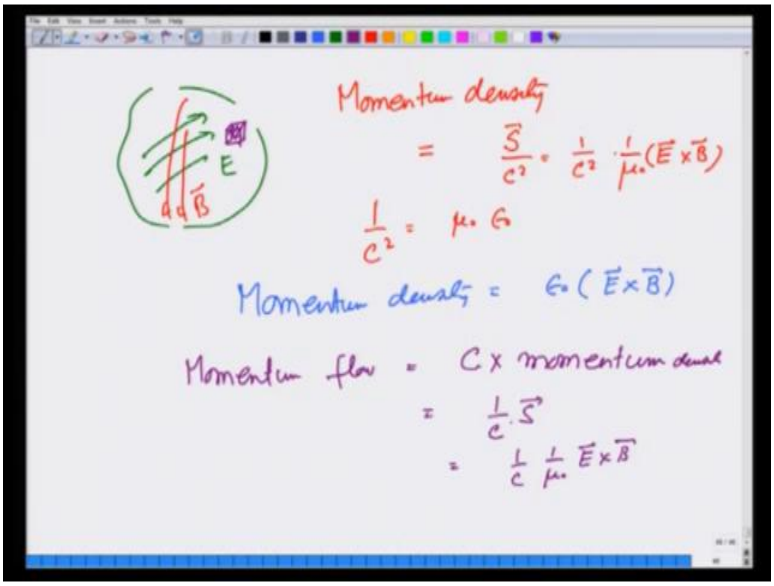

What it means is, if there is in some region of space, there is E field and there is B field, then in this region, there is momentum content in this field. So, momentum density per unit volume, momentum is going to be S over c square equal to 1 over c square times 1 over mu 0 E cross B. Since, 1 over c square is mu 0 Epsilon 0, this can also be written as momentum density can also be written as Epsilon 0 E cross B.

So, what it means is if I take a small volume here in this region, where the E and B field exists. The momentum contained with this volume is going to be per unit volume Epsilon naught E cross B. This is momentum density; obviously, there is going to be momentum flow also which is nothing but, c times the momentum density as I argued for the energy flow only. And this is going to be nothing but, 1 over c times S which is 1 over c times 1 over mu 0 E cross B.

What I am concerned about right now is a momentum density that electromagnetic field carries momentum and I am going to now demonstrated this formula through an example.

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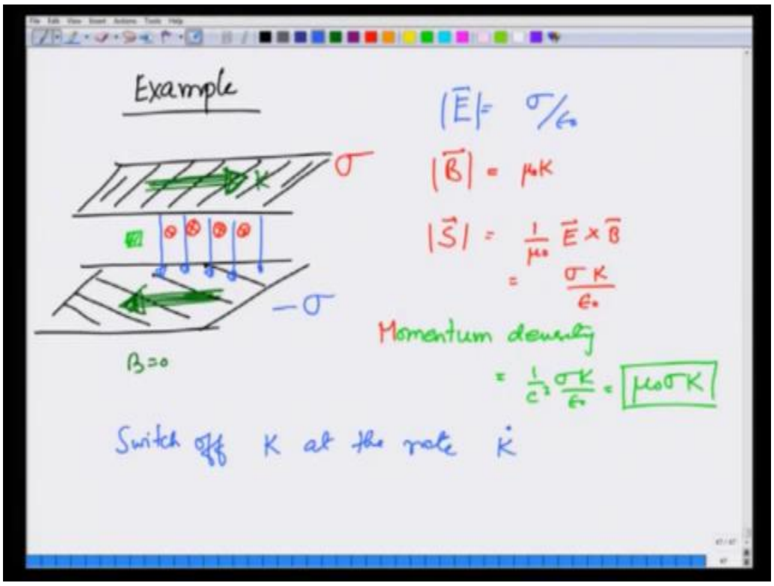

So, let us solve that example. Example I take is, suppose I have two sheets of charge, the upper one carrying sigma per unit area and the lower one carrying minus sigma per unit area. So, that there is an electric field that exist between these two. The electric field is going to be sigma over Epsilon 0 the magnitude. Suppose, now they also carry a surface current K going to the right side in the upper plate and to the left in the lower plate.

Then, by applying Ampere's law you can easily figure out that B outside will be 0 and B inside is going to be this is the direction. So, it is going to be, going into this screen and it is magnitude is going to be K mu 0. Therefore, S vector magnitude which is 1 over mu 0 E cross B is going to be equal to sigma K over Epsilon 0 and momentum density is going to be 1 over c square sigma K upon Epsilon 0 which is mu 0 sigma K. That is a momentum density or momentum contain per unit volume in this region.

How we are going to show that this is, really the momentum density contain here is, I am going to switch off K at rate K dot. When we switch off this current K, this is going to reduce the magnetic field, if you reduce the field which is going into the board let us go to the next slide.

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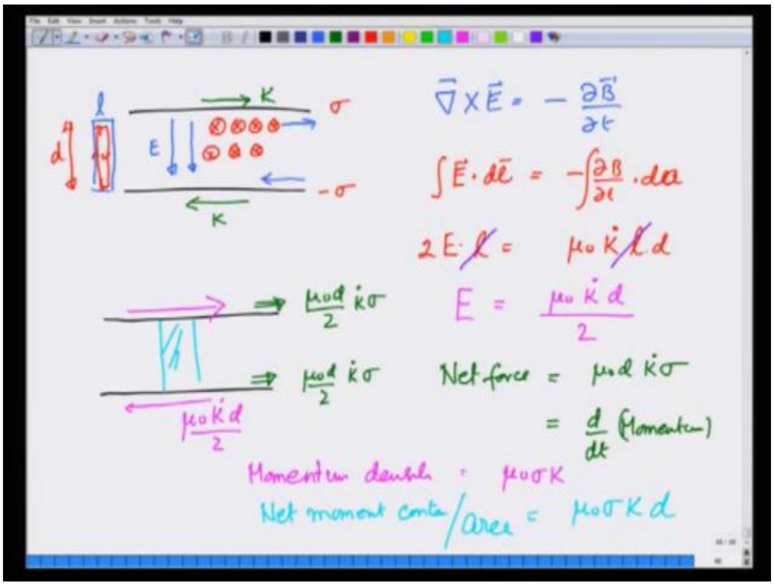

So, what we are doing is we have these two sheets which I am writing just showing by line now minus sigma. We have the electric field coming down, we have the surface current K going this way, K going this way and we have the magnetic field going in. Although as K is brought down from Faraday’s law, because B is now reducing we are going to have electric field induced inside and you can easily work it out.

Now, B is going in I would like to have an electric field that produces B in the same direction, because it wants to keep the same. So, the electric field produce is going to be induced, electric field is going to be like parallel to the plate going to the right on the upper plate and going to the left on the lower plate. This comes from curl of E induced is equal to minus d B by d t. Using Stokes theorem I can calculate the magnetic of the electric field. How do I do that?

I will take a small loop of length l on the upper and the lower side and the distance between the plates I will take to be d. Then, integral E dot d l equal to minus d B d t dot d a gives me that E times l times 2, because this contribution come from the upper one and the lower one and I am traversing at like this. This I am going little fast, because you use it by now 2 E l is equal to change in the flux, which is mu 0 K dot times the area, which is l times d.

Let us again cancel l and I get this induced electric field E to be equal to mu 0 K dot d over 2. This is the induced electric field, the direction of the electric field on the upper plate is like this, let me make this clear again and on the lower plate it is in the opposite direction and it is magnitude is mu 0 K dot d over 2. So, it will apply the force on the sigma and minus sigma, the force on the upper plate is going to be E times sigma per unit area.

So, this is going to be mu 0 d over 2 K dot sigma per unit area, on the lower plate again the force is going to be to the right, again mu 0 d over 2 K dot sigma. So, the net force on the whole system is going to be mu 0 d K dot sigma and this should be equal to d by d t of the momentum of the plates of the system, where does this momentum come from. I am going to show that momentum comes from some electromagnetic field which is reducing.

And therefore, the momentum of the electromagnetic field is reducing and that goes into this mechanical momentum. We saw in the previous slide, that the momentum density was mu 0 sigma K. So, momentum density is mu 0 sigma K, net momentum content per unit area, this volume per unit area which is going to be mu 0 sigma K, area is unity and the height is d mu 0 sigma K t.

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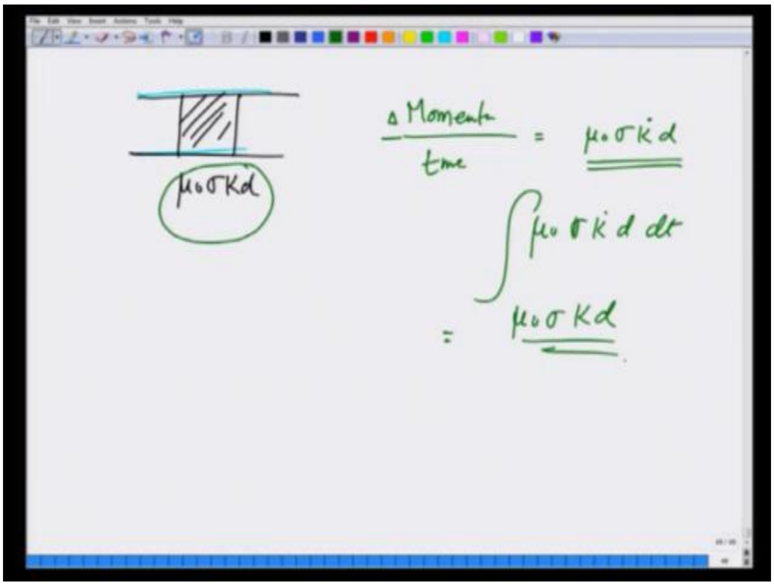

So, what we have in these plates is that the momentum contain per unit area is mu 0 sigma K d and as we switch off the K, then the momentum change per unit time delta momentum per unit time comes out to be mu 0 sigma K dot d that is the momentum that the plates came. So, we can easily see as K goes down the momentum of the field between the plates goes down and that is transferred to the plates.

So, what we have shown through that formula that field carries momentum and it is transferred to plates. Finally, when by integrate over the entire time, entire momentum mu 0 K sigma d will go into the plates momentum. Because, if I integrate mu 0 sigma K dot d over time, this comes out to be mu 0 sigma K d which is indeed the momentum of the fields that has been transferred to the plates.

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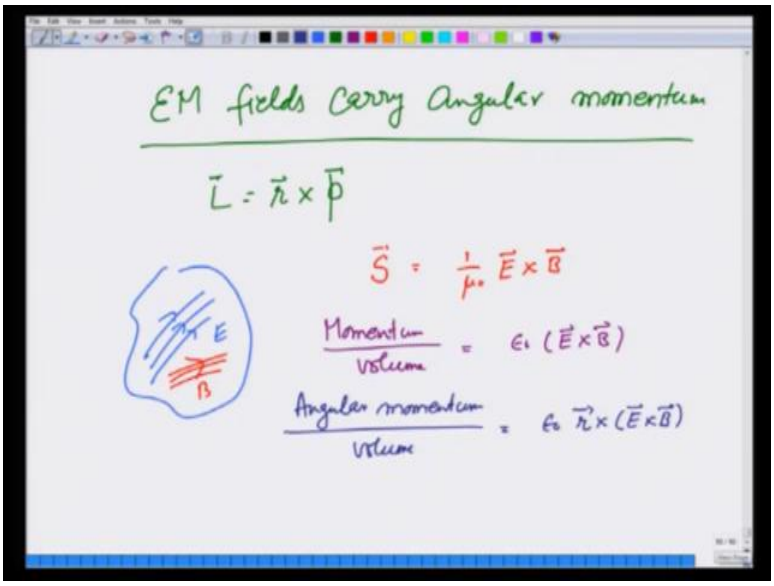

Finally, when we are talking about the energy content, linear momentum content of the electromagnetic field I will conclude this set of three lectures by showing that E M fields carry angular momentum also, that should not be surprising. Because, if there is momentum p than angular momentum is defined as r cross p and so with linear momentum, angular momentum is always associated, I will also demonstrate this through an example.

So, if there are these electromagnetic fields E and B in certain region, we have seen that S is 1 over mu 0 E cross B. The momentum content per unit volume is equal to Epsilon 0 E cross B and therefore, angular momentum contained per unit volume will be Epsilon 0 r cross E cross B. Can this be true? Is this true? Again, we will show it through an example.

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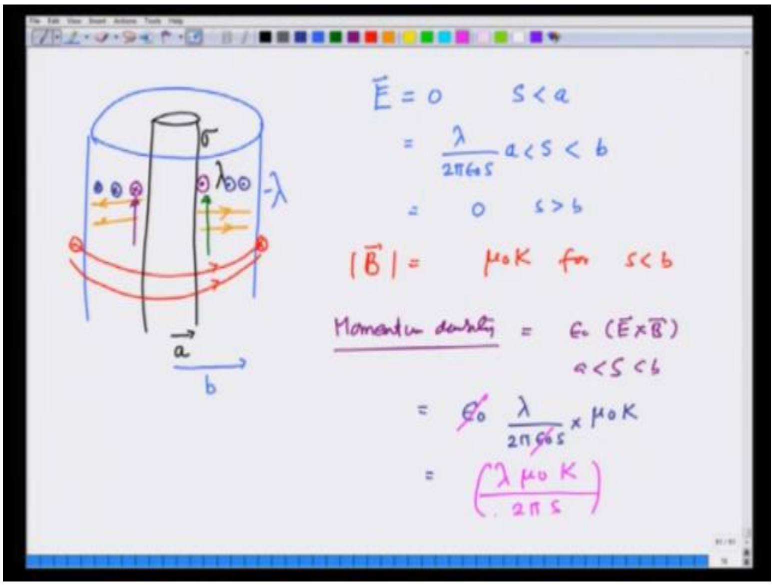

The example I take is we take a solenoid which is carrying surface charge sigma or charge lambda per unit length. This is surrounded by another solenoid which is also carrying charge lambda with the minus lambda and also carries a surface current say going in on the right side coming out in the left side. So, surface current like this it is only the outer one that carries the surface current.

Let radius of the inner cylindrical shell be a let the radius of the outer 1 B B, then electric field E is 0 for S I am using cylindrical coordinates less than a, this is equal to lambda over 2 pi Epsilon 0 S or S between a and b and this is 0 again for s greater than b since there is a current in the outer shell. So, the magnetic field B is equal to it is magnitude is equal to mu 0 K for S less than B. And therefore, there is going to be momentum density which is going to be Epsilon 0 E cross B between a and B it will be non 0 only between a and B and 0 otherwise.

Let us see the directions e fields is pointing away from the inertia p field pointing up. So, E cross B is coming out here and going in here, because B is like this. So, E cross B is going in, so what I have in this combination is that there is electromagnetic momentum coming out on the rights side of the shell and going in the left side of a shell. And since this is going around it gives rise to an angular momentum.

So, let us calculate that now, so momentum density E cross B is going to be Epsilon 0 E is lambda over 2 pi Epsilon 0 S times B which is mu 0 K. So, this comes out to be Epsilon 0 cancels lambda mu 0 K over 2 pi S that is the momentum density.

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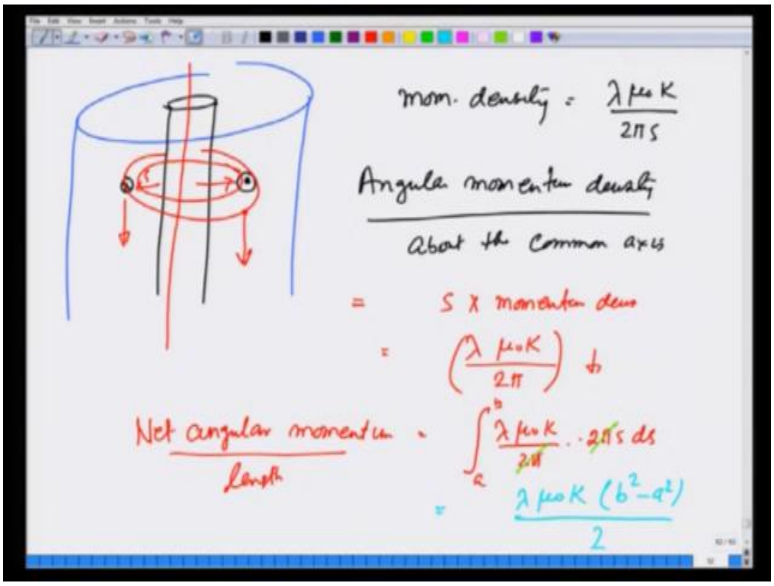

So, what I have is these the solenoid and an outer solenoid and in this region there is this momentum density which is coming out here and going in here and it is magnitude is we have just calculated is nothing but, lambda mu 0 K over 2 pi S. How about the angular momentum density about the common axis, let me make this common axis which is right in the middle is going to be s times the momentum density. So, this is going to be lambda mu 0 K over 2 pi, how about the net angular momentum contains.

So, net at this point the direction of the angular momentum density is pointing down, because this is s cross k is pointing down on this side also this is S cross K is pointing down. So, this is pointing down and net angular momentum therefore, is going to be lambda mu 0 K over 2 pi. Then, out here in this shell and let say net angular momentum per unit length is going to be 2 pi S d s times unit length which is 1 and this is going to be from a to b, we can cancel 2 pi and this comes out to be lambda mu 0 K b square minus a square over 2 that is the net angular momentum content between these two shells.

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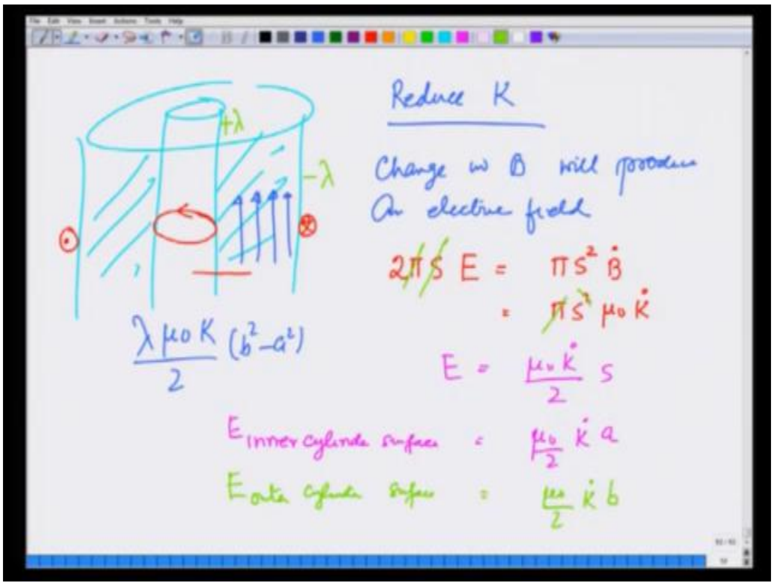

So, if I make this again per unit length in this region, per unit length the angular momentum content is lambda mu 0 K over 2 b square minus a square. Again we will use the same reduce now K, which use K as you reduce K the field B which is like this is going to start going down, as the field goes down as the field becomes smaller and smaller. So, change in B will produce and electric field and how the electric field be electric field will be such that it generates field in the same direction.

And therefore, the direction of the electric field is going to be like this going in here and coming out on this side by applying stoke theorem I can calculate electric field at distance S from the center 2 pi S E is going to be pi S square times B dot which is going to be pi S square mu 0 K dot that is the electric field formula. So, again if I cancel as one has goes away, pi goes away I get electric field is equal to mu 0 K dot over 2 S that is electric field produce.

So, electric field at the E at the inner cylinder surface is going to be mu 0 over 2 K dot a and E at the outer cylindrical surface is going to be mu 0 over 2 K dot b. However, the inner surface has charge plus lambda, the outer surface has charge minus lambda and therefore, the forces are going to be in different directions. The electric field is going in on the outer surface and coming out on this side. So, because of the minus sign will give a force in the opposite direction, on the inner circle you will be in the same direction as the E.

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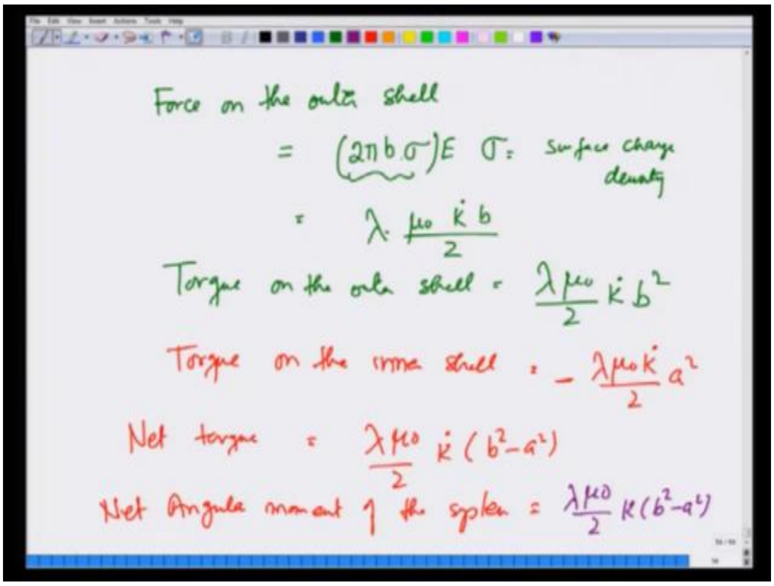

Because of this electric field the force on the outer shell is the lambda per unit length is going to be equal to 2 pi b per unit length is, per unit length time sigma with sigma is a surface charge density times electric field and this is nothing but, lambda. So, this is lambda times E which we have already seen is equal to mu 0 K dot b over 2 and therefore, torque on the outer shell is going to be lambda mu 0 over 2 K dot b square. In the similar manner you can calculate torque on the inner shell and it will be lambda mu 0 K dot over to a square with the minus sign.

And therefore, the net torque on the system is going to be lambda mu 0 over 2 K dot B square minus a square, net angular momentum of the system is going to be when I integrate this by the time K comes to 0 for starting form some initial value of K is going to be lambda mu 0 over to K b square minus a square, where did this angular momentum from. We have already shown that the electromagnetic field carries this angular momentum which we calculated earlier.

So, therefore, that angular momentum is now transfer to the cylinders and so what we have shown is that electromagnetic field carries angular momentum which by some mechanism can be transferred to the shells or the systems around it and that is equal to the mechanical angular momentum again by this.