Introduction to Electromagnetism
Prof. Manoj K. Harbola
Department of Physics
Indian Institute of Technology, Kanpur

Lecture - 50
Energy stored in a magnetic field – I

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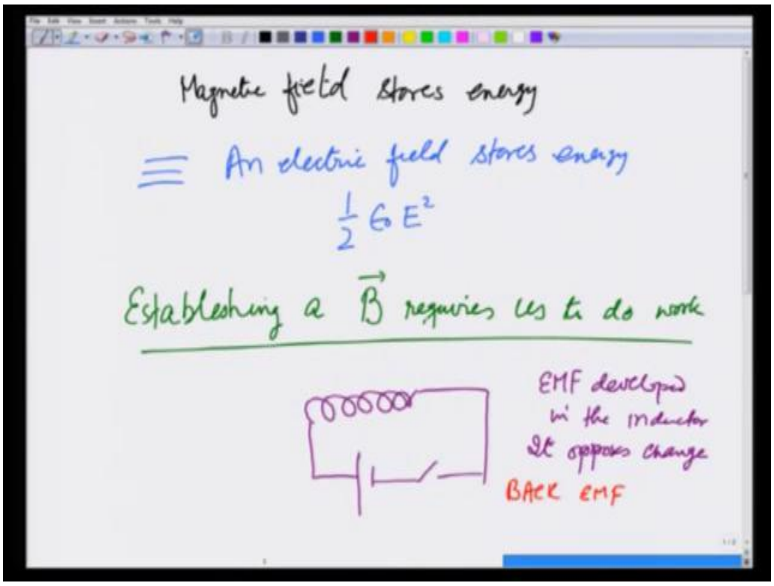

We have learnt that magnetic field stores energy exactly in the same way similar to electric field stores energy, and that is given as 1 half epsilon 0 E square which we had already seen. And today's lecture, we are going to use Faradays and Lenz’s law to establish, what should be the energy. There should be some energy stored in the magnetic field and what should it be. So, let us start why should there be energy in magnetic field. The reason is that establishing a magnetic field, I will just use B for it requires us to do work.

For example, if I take an inductor and we have seen if I now try to establish the current through it, maybe through a switch, then what happens is, there is a back EMF or EMF developed in the inductor and this EMF is such that it opposes, change that is establishing it and this is also therefore known as BACK EMF.

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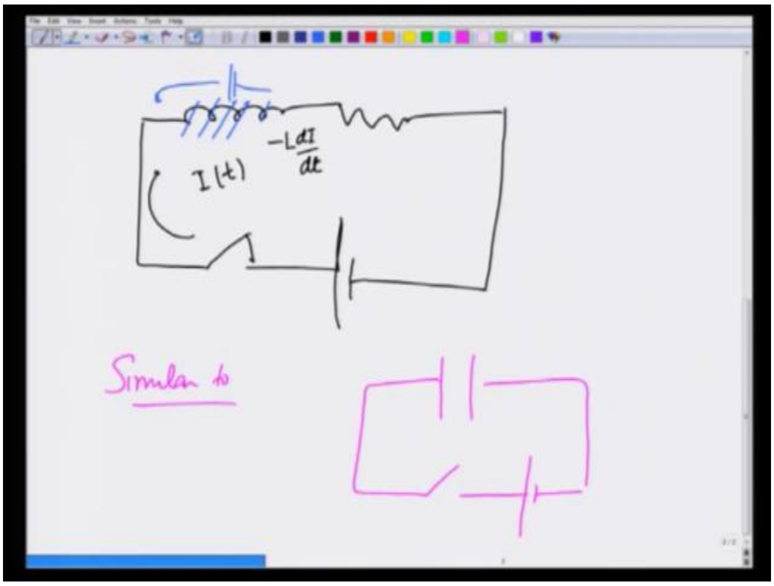

So, what is happening is that I am taking this circuit. Let me also put a resistor here, put a key, put a battery, as soon as I turned this key on, the current tries to increase, I t tries to increase. And as it increases, there is a BACK EMF which is developed here, which we write to the minus sign L d I by d t, I will explain this again and that opposes this change.

We have to work, then we have to supply energy to overcome this EMF, so that if you make an analogy, this can be replaced by a battery which is opposing the current and therefore, I have to supply energy, so that the current passes through the this battery. And that work that I am doing goes into establishing the magnetic field which arises out of this current. So, finally, what we interpret this as, as if this energy is stored in the magnetic field.

Very similar to, recall if I have a capacitor and I turned a current on through a switch, the current builds up slowly or the charge in the capacitor builds up slowly and the process I end up supplying energy to it which we finally, interpret as if, it is stored either in the capacitor or in the electric field that is there. So, we are going to do a similar calculation now, how to find out how much energy do we supply that is stored, so let us do that.

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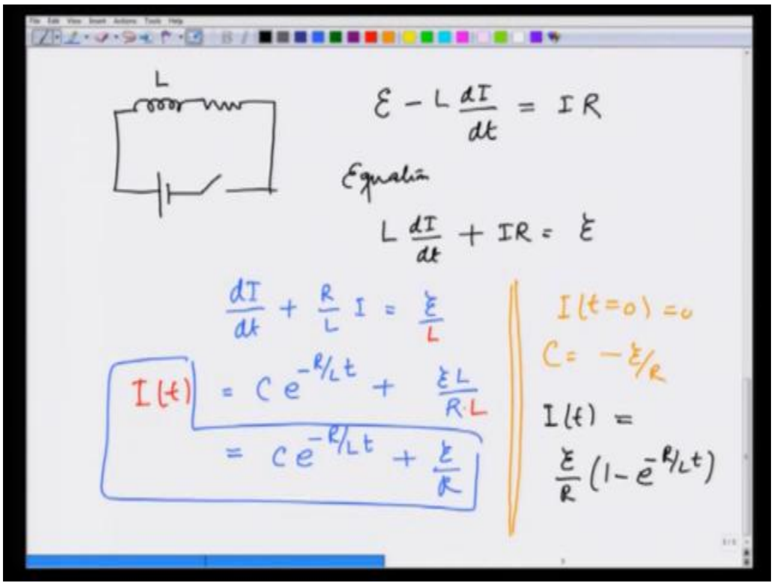

We had this inductor with inductance L, a resistance r and a battery and let us turn this switch on at t equal to 0. So, by Kirchhoff's rule I have this EMF E and a BACK EMF L d I d t and this should equal I R, this is Kirchhoff's rule. So, the equation that we get for I is L d I d t plus I R is equal to the EMF or the voltage applied or d I d t plus R over L I is equal to E.

The solution for this is the solution for the homogeneous part which is very straight forward, which is a constant times e raise to minus R over L t plus the other part which is for the inhomogeneous term E, which comes out to be E L over R and this should be equal to this is E over L top. So, E L over R and there is this L, this cancels and so we get I t equal to this is equal to C e raise to minus R over L t plus E over R, this is the current as a function of time.

Now, let us apply the initial conditions. The initial conditions are I t equal to 0 is equal to 0. When I substitute this n, I get C equals minus E by R and therefore, my final answer for the current I t is equal to E over R 1 minus e raise to minus R over L t, this is a result that you know very well.

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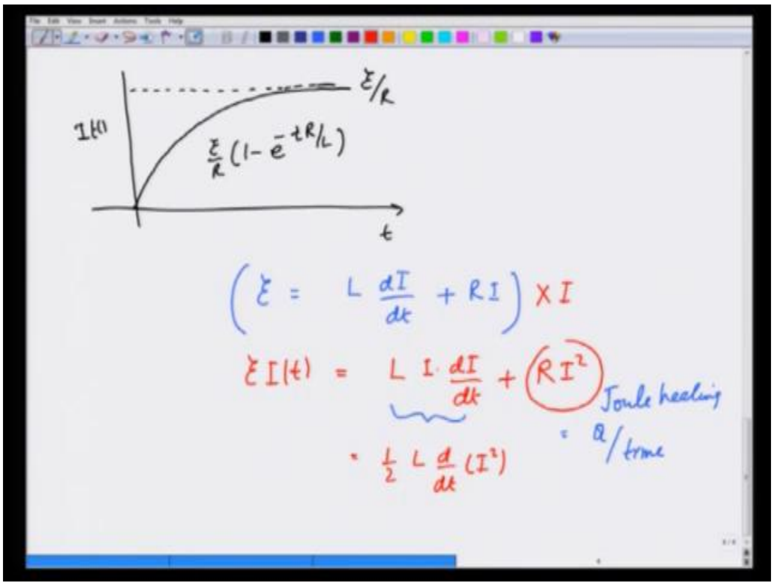

If I plot this I t versus time, it goes up and finally, as t goes to infinity goes to the value of E over R and this function is E over R 1 minus e raise to minus t R over L. So, initially as soon as the switch is turned on, there is no current and finally, as if there is no inductor there, the current becomes E over R, let us now see the energy supply. So, if I take this equation again, EMF is equal to L d I d t plus R I and multiply both sides by I, remember I is a function of time, I get e I t is equal to L I d I over d t plus R I square.

This term I already know is Joule heating, which is heat produced per unit time or power consumed in the resistance. The other term is this one, which I can write as 1 half L d by d t of I square.

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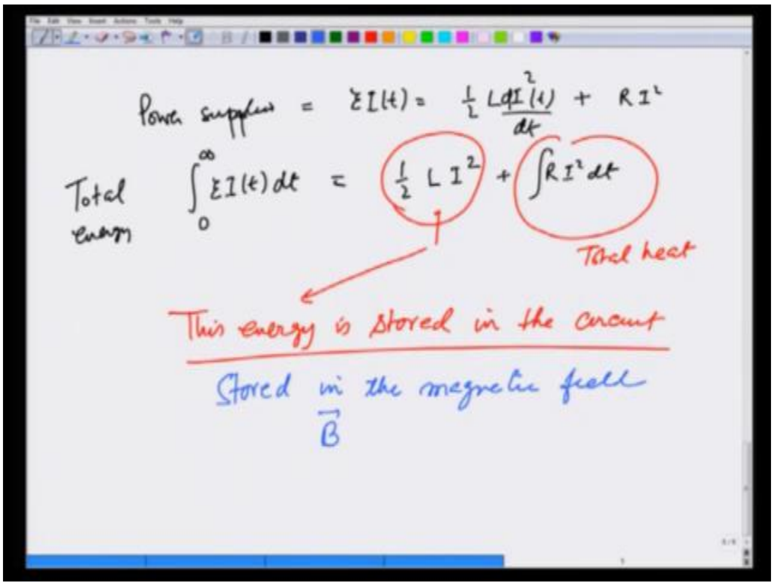

So, what we get then is power supplied is equal to E times I t is equal to 1 half L I t square plus R I square. As time goes to infinity, I can find the total power supplied E I t, 0 to infinity d t, this is the total energy not total power, total energy supplied by the time the current becomes I, it is equal to 1 half, this was d by d t, 1 half L I final square plus R I square d t integrated, so this is a total joule heating, total heat produced.

What about this term? This is the energy which is stored somewhere. How do I know, it is stored? I know, if I turned the battery off, somehow the current does not go down immediately, it still remains that energy is supplied by this stored energy. So, we say this energy is stored in the circuit and we take it to be stored in the magnetic field B that is produced by this current finite. So, now what we want to do is, convert this formula into energy in terms of magnetic field. In particular, I want to get the energy density energy per unit volume in terms of magnetic field.

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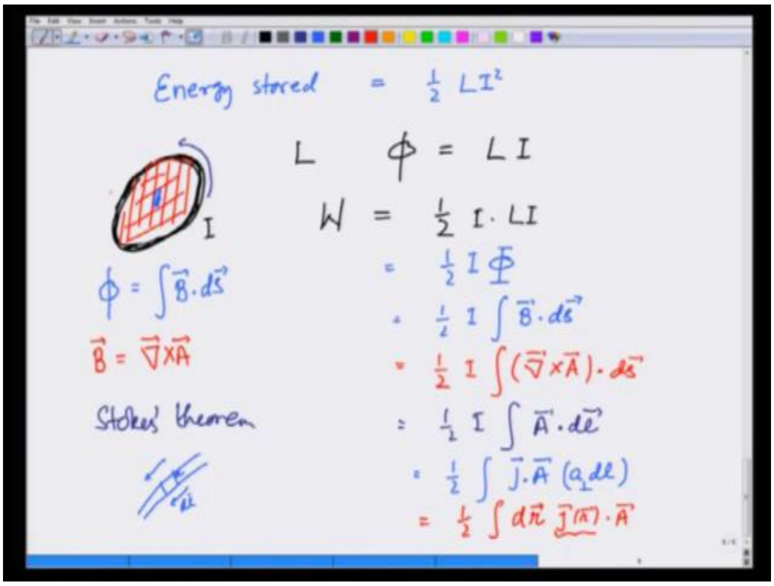

To convert the formula of energy stored which we just saw is equal to 1 half and L I square, we go about it in the following way. Let me take this wire as an inductor, let me give it some thickness, very small thickness and let it be carrying current I. By definition of inductance L, the flux phi passing through the surface of the, this surface which I will show by red color, the surface area covered by the wire is going to be equal to L I. Therefore, I can write this energy stored, let me write this as W is equal to 1 half I times L I, which we can write as 1 half I times the flux phi.

Now, I know that flux phi through this surface is equal to the magnetic field through the surface dot d s, where d s is the element small area here, elemental area and then we integrate all over. Again, remember d s and I R related through the conventional directional sense. So, I am going to write this as 1 half I integral of B dot d s, which I am going to use another relationship B equals curl of A and doing this, so that I can use the Stokes theorem and then the next step I write this as 1 half I integral curl of A, where A is the vector potential dot d s and this I can rewrite using Stokes theorem as equal to 1 half I integral A dot d l.

Remember, again d l and d s are related such that the directions are properly related through the conventional sense of direction. And since we have been talking about this direction I is in the same direction is d l and therefore, we again use a trick that we used earlier. If I look at this wire, here is I going through this, here is small area A, here is direction d l.

So, this can be written as 1 half I is nothing but, j A dot the direction of current which now I am going to put over j and therefore, I am going to remove this dot from here. Put the dot near j a d l, a perpendicular and this is nothing but, the volume and therefore, the final expression I get is going to be 1 half integral d volume which I write as this j r dot A. This is volume over the entire universe or entire space here, but it is non zero only where j is non zero and therefore, in the end it becomes non zero only over the wire.

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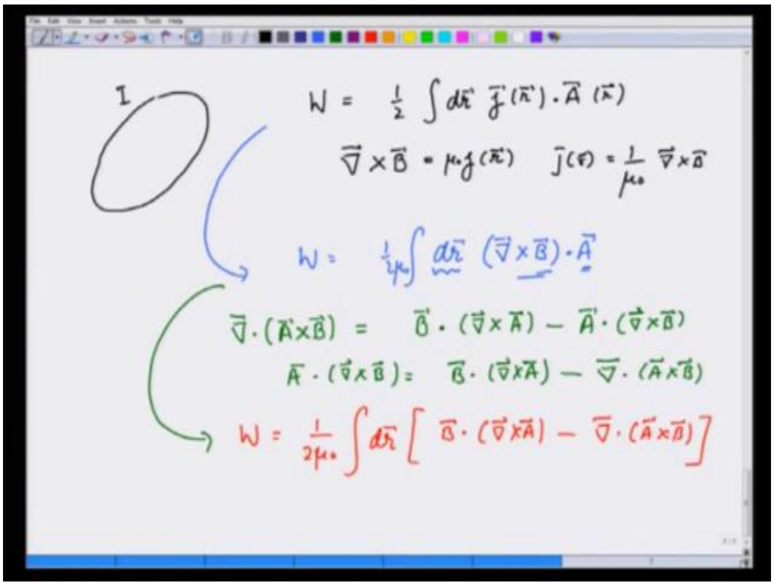

So, we get for this wire that is carrying the current I that the energy is stored by the time, I finish establishing this current is 1 half a volume integral j r dot A r and that point. We not yet done, because we want to convert this whole thing into the magnetic field. So, I am going to use another identity which says that curl of B is equal to j of r times mu 0 and therefore, j of r can be written as 1 over mu 0 curl of B, and therefore this work I can write as 1 half the volume integral curl of B dot A 1 over mu 0 here.

Remember, we are taking these steps as we did in deriving the electrostatic energy, where we kept converting the integrals or the integrals over charges into integrals over the field. Now, you see it make sense when I talk about d r which is the volume integral over the entire space that it will be non zero wherever curl of B is non zero and where are wherever A is non zero.

Now, we again convert this entirely into B, we are going to use an identity which says that divergence of A cross B two vectors is equal to B dot curl of A minus A dot curl of B, which immediately tells me that A dot curl of B is equal to B dot curl of A minus divergence of A cross B. And therefore, this integral again can be written as the work done is equal to 1 over 2 mu 0 integral of the volume integral inside I get B dot curl of A minus divergence of A cross B.

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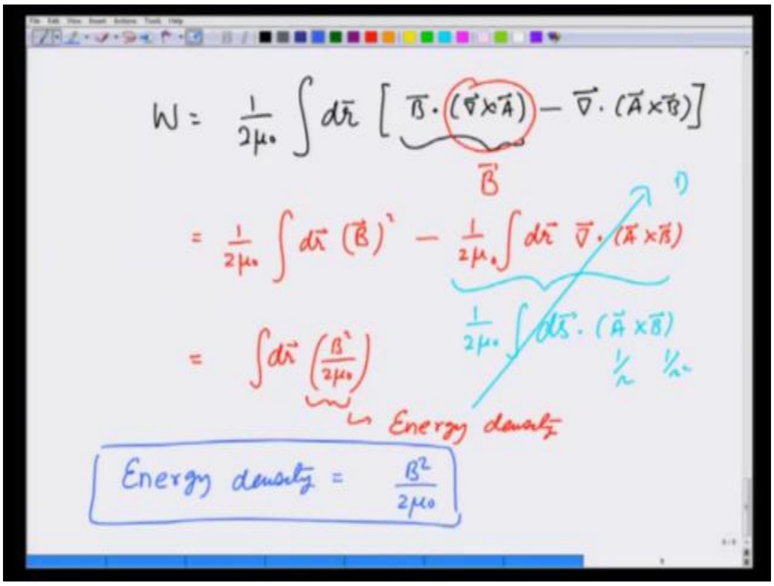

So, let me go to the next slide now and write this work done as 1 over 2 mu 0, integral the volume integral B dot curl of A minus divergence of A cross B. Let us look at this term, curl of A is nothing but, B again, so this becomes 1 over 2 mu 0 integral, the volume integral B square minus 1 over 2 mu 0 volume integral divergence of A cross B. I can use Divergence theorem now to convert this into a surface integral, which becomes 1 over 2 mu 0 integral d s.

Remember, I said this integration is over the entire space, so d s is really far, far away dot A cross B. B roughly goes as 1 over r square, A goes as 1 over r, so this integrand goes as 1 over r cubed, d s far away goes as r square, so this whole integral goes as 1 over r and therefore, we can write this to be 0. And therefore, finally, the energy comes out to be equal to the volume integral of B square over 2 mu 0 and this we are going to call the energy density.

Because, integrated this over the entire volume gives me the total energy. So, in the magnetic field the energy density or energy stored in a magnetic field is such that the energy density is given as B square over 2 mu 0. This is similar to the electrostatic energy which was given as 1 half epsilon 0 e square. In the next lecture, we are going to solve some examples using this.