Introduction to Electromagnetism
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Lecture - 40
Vector potential from Current Densities – 1

Recall that in the past few lectures, we have been talking about the vector potential for a magnetic field, which I denote by A r.

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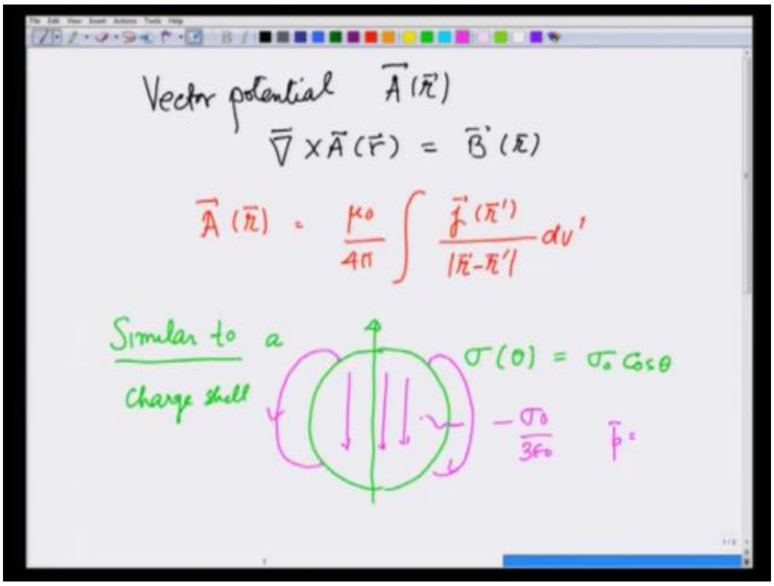

Such that curl of A is B; in particular in the previous lecture, what we found is that, A can be expressed as mu 0 over 4 pi integration over the volume. The current density j at r prime over r minus r prime over the volume and we solved the couple of examples for different current distributions giving rise to A. In this lecture, I want to take one particular example; that is similar to the example of a charge sphere or charge shell.

Similar to a charge shell and I will take this vertical axis to be the z axis with sigma, the surface charge is sigma theta being equal to sigma naught cosine of theta. Recall that, this give us the electric field inside the shell as minus sigma 0 over 3 Epsilon 0, a constant electric field and outside, it was like a dipolar field, with a dipole movement p, whatever that comes out to be.

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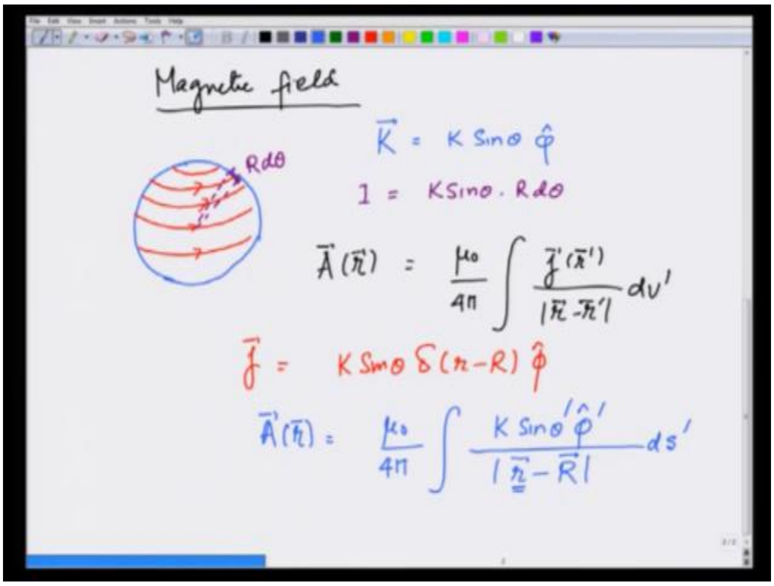

Now, a similar situation occurs in magnetic field, if I take a sphere, so for a magnetic field a similar situation occurs, if I take a sphere, rather a shell and I have a surface current density K on it, which is K sine theta in the phi direction. Let me show this current here with red color, it is a current at everywhere, it is only on the surface it is flowing in the phi direction, if I use this spherical polar coordinates.

The current could arise out of charged shell rotating with certain angular speed or as we will see later also, this is equivalent to a magnetized sphere. So, this is only on the surface. Therefore, if I were to ask how much is the current at a certain point passing through this line, then you recall that this length is going to be R d theta. So, the current through this small r d theta is going to be K sine theta times R d theta in the phi direction. In any case, I want to now calculate the vector potential for this.

By definition the vector potential A r will be equal to mu naught over 4 pi integration j r prime over r minus r prime d v prime. I could write j for this as j, since this is confined to the surface it will be K sine theta delta of r minus r, where R is the radius of this sphere as you already seen in phi direction. And therefore, I will quickly change this integral to a surface integral and write this as A r equals mu naught over 4 pi integral K sine theta.

In phi direction, it should be sine theta prime, in phi prime direction divided by r minus vector R d s prime, where vector R denotes the vector on the surface. Vector R denotes the vector on the surface, r is the point, where I am calculating the vector potential. And

K sine theta prime gives me the magnitude of the surface current and phi prime is the phi unit vector prime is the direction of it and we wish to calculate this.

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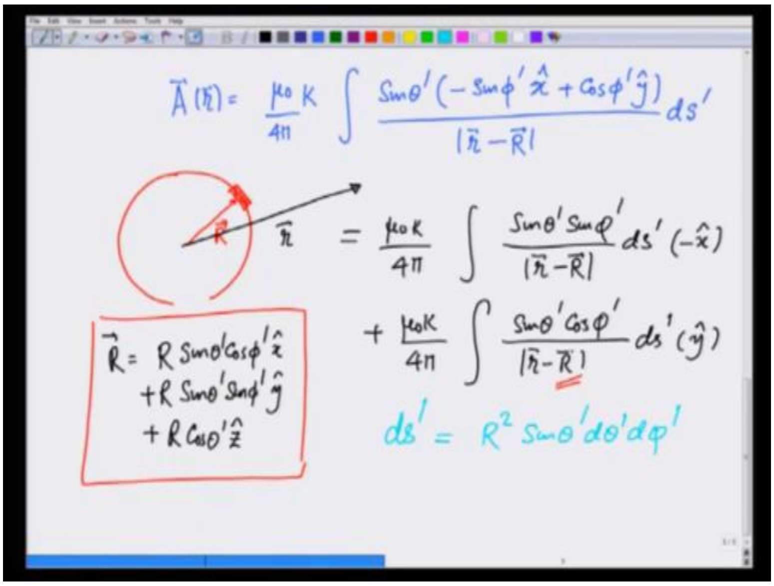

If I expand this, I am going to get A r equals mu naught over 4 pi, K comes out inside, I am left with sine of theta prime. Phi prime, I can write as minus sine phi prime in x direction plus cosine of phi prime in y direction divided by r minus vector R d s prime, where d s prime is now, let me show it again over this sphere, d s prime is a surface area over the sphere. This vector is R and I am calculating the vector potential at sum vector r.

So, this becomes equal to mu naught K over 4 pi integration sine theta prime, sine phi prime over r minus R d s prime in minus x direction plus mu naught K over 4 pi integration sine theta prime, cosine phi prime over r minus R d s prime in y direction. Keep in mind that this vector capital R that I have written, vector R is nothing but R is fixed sine theta prime cosine phi prime in x direction plus r sine theta prime sine phi prime in y direction plus r cosine theta prime in z direction.

Why I am writing this is because, I want to emphasize that this R out here includes these theta prime and phi prime the integration variables. Let us also write d s prime. What is d s prime? D s prime is R square sine theta prime d theta prime d phi prime. So, this is because this is the integration over this spherical surface. Now, you must be wondering how to calculate this integral.

So, let me write this again and give you the answer first, because right now I want to focus on the vector potential. And in the next lecture, I will tell you how to calculate these potentials, using analogy with the electrostatics.

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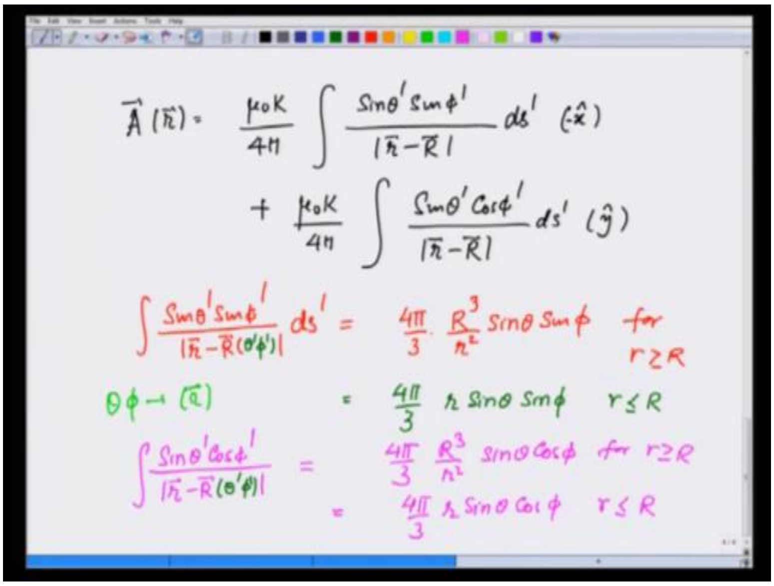

So, right now what I have is, I have A r equals mu naught K over 4 pi integration sine theta prime sine phi prime over r minus R, where this r capital R vector includes d s prime minus x. This R capital includes theta prime and phi prime mu naught K over 4 pi integration sine theta prime cosine phi prime over r minus R d s prime in y direction. Let me quickly tell you, what these integrals will be and I will explain them in the next lecture.

This integral sine theta prime sine phi prime over r minus capital R, let me write to show explicitly. That this capital R is a function of theta prime and phi prime, d s prime is equal to 4 pi by 3 R cubed over r square sine theta sine phi for r greater than r or greater than or equal to R, where theta and phi correspond to r vector. And this is equal to 4 pi by 3, r sine theta sine phi for r less than R, you can see for small r equal to R, both of them match.

Similarly, if I look at integration of sine theta prime cosine phi prime over r minus R. Again, I want to emphasize that capital R is the function of theta prime and phi prime is equal to 4 pi by 3 R cubed over r square sine theta cosine of phi for r greater than or equal to R and is equal to 4 pi by 3 r sine theta cosine of phi for r less than or equal to R.

So, you have got in these integrals for both, sine theta prime, sine phi prime and sine theta prime to cosine phi prime.

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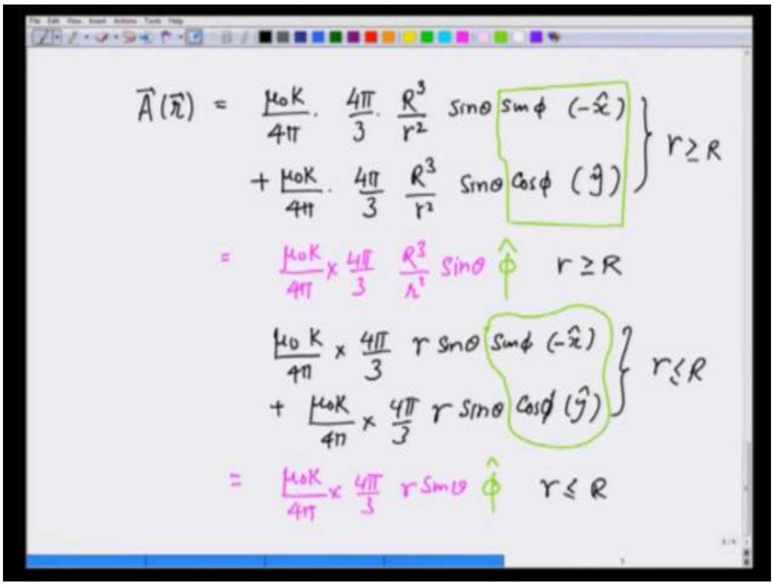

And therefore, my answer for A r is equal to mu naught K over 4 pi, then I have a 4 pi by 3 R cubed over r square sine theta, sine phi in minus x direction plus mu naught K over 4 pi 4 pi by 3 R cubed over r square sine theta cosine of phi in the y direction. This is for r greater than or equal to R. Let us simplify this, it will be mu naught K over 4 pi times 4 pi by 3 R cubed over r square sine theta, I can take common.

And this quantity sine phi minus x unit vector plus cosine phi y unit vector is nothing but phi vector unit vector. Because, this is for r greater than or equal to R and for the other limit, I have mu naught K over 4 pi times 4 pi by 3 r sine theta, sine phi in the minus x direction plus mu naught K over 4 pi times 4 pi by 3 r sine theta cosine phi in the y direction for r less than equal to R, which again comes out to be equal to mu naught K upon 4 pi times 4 pi by 3 r sine theta. And again I can write this quantity in the green enclosure as phi unit vector for r less than equal to R.

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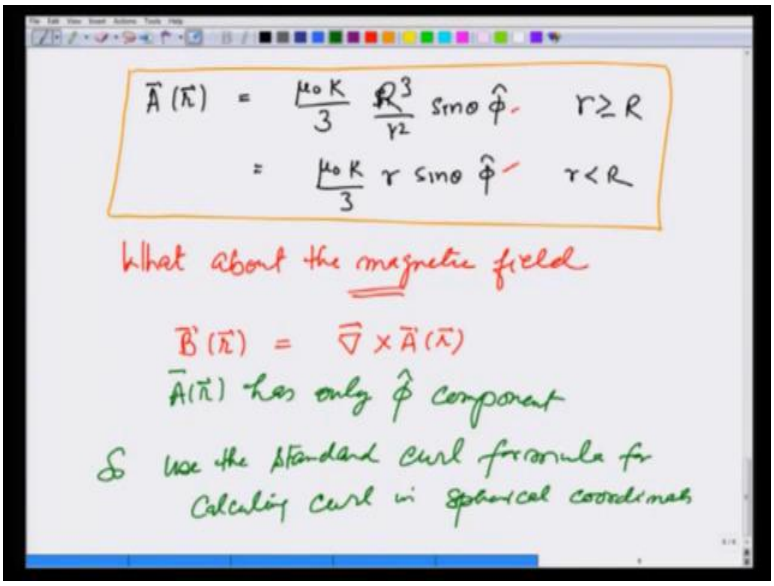

So, in the final answer I have A r equals mu naught K over 3 R cubed over r square sine theta phi unit vector for r greater than equal to R and is equal to mu naught K over 3 r sine theta phi for r lesser than equal to R. This is my final answer for the vector potential. What about the magnetic field? The magnetic field B r is given as curl of A r.

Notice that, curl and notice that A r has only phi component and we are writing this whole thing is spherical polar coordinates. So, use the standard curl formula for calculating curl in spherical coordinates.

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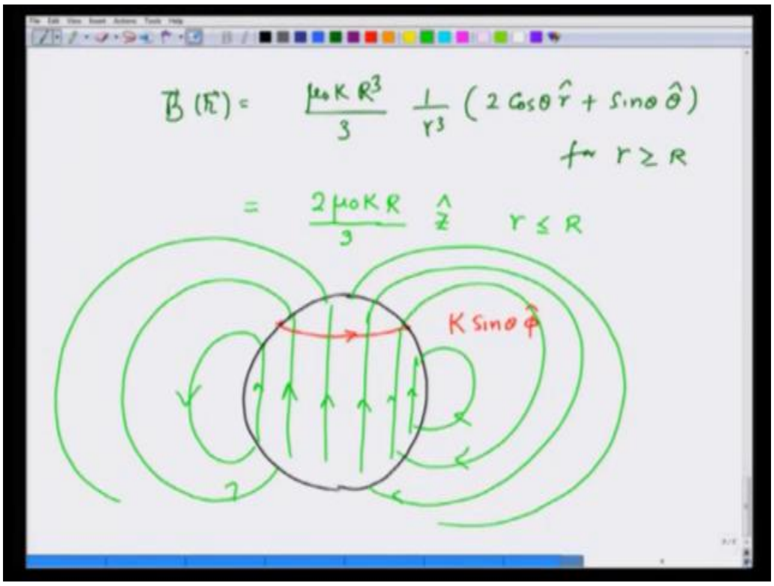

And that gives you the answer B of r is equal to mu naught K, R cubed over 3, 1 over r cubed 2 cosine theta r plus sine theta in theta direction for r greater than or equal to R. And this is equal to 2 mu naught K over 3 R in the z direction, for r less than or equal to R. So, you see, when I look at this sphere a spherical shell that has surface current K, going like K, sine theta in phi direction.

The field inside is constant, it is similar to electric field being constant, if a shell had a surface charge density, which was sigma naught cosine theta. What about outside, I show you that the outside field is like the dipolar field like this. So, this is a kind of problem that we solved in electric field and this is similar problem in the magnetic field, except that now, it has a surface current density with sine theta dependence. Let me before I finish this lecture; show you that outside field is really a magnetic dipole field.

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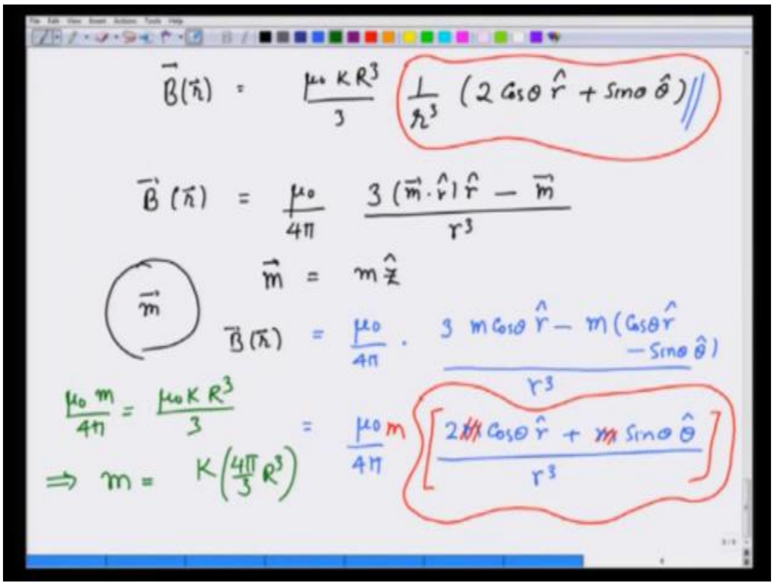

For that, let us write this B again, B r is mu naught K r cubed over 3, 1 over R cubed 2 cosine theta r unit vector plus sine theta unit vector. Recall that magnetic field due to a magnet; magnetic moment m is equal to mu naught over 4 pi, 3 m dot r unit vector r unit vector minus m itself, divided by R cubed. Let us take m in the z direction with magnitude m z, then I have B r, I use a different color is equal to mu naught over 4 pi times 3 m dot r.

Now, would give me m cosine of theta is r unit vector minus m z unit vector, I will decompose into theta and phi, which then gives me cosine theta in r direction minus sine theta in theta direction divided by R cubed. So, this gives me mu naught over 4 pi and I have 2 m cosine of theta r plus m sine of theta in theta direction divided by R cubed. You see this has the same dependence on cosine theta and sine theta as the answer here.

Let us compare terms, let me take this m out and write it here. So, that this factor enclosed in red is a same in both. So, what I am left with is, I will write on the left hand side. Mu naught m over 4 pi is equal to mu naught K R cubed over 3 or this implies that this sphere which carries a surface current of K has a magnetic moment of the magnitude, m equals K times 4 pi by 3 R cubed. So, outside the magnetic field, outside this sphere is like that produced by a point magnetic dipole with magnitude K 4 pi by 3 R cubed and inside, it is a constant field.