Introduction to Electromagnetism
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Lecture - 39
Equation for the Vector Potential in terms of current density

We have been discussing vector field A r that is related to magnetic field B r to Curl equation.

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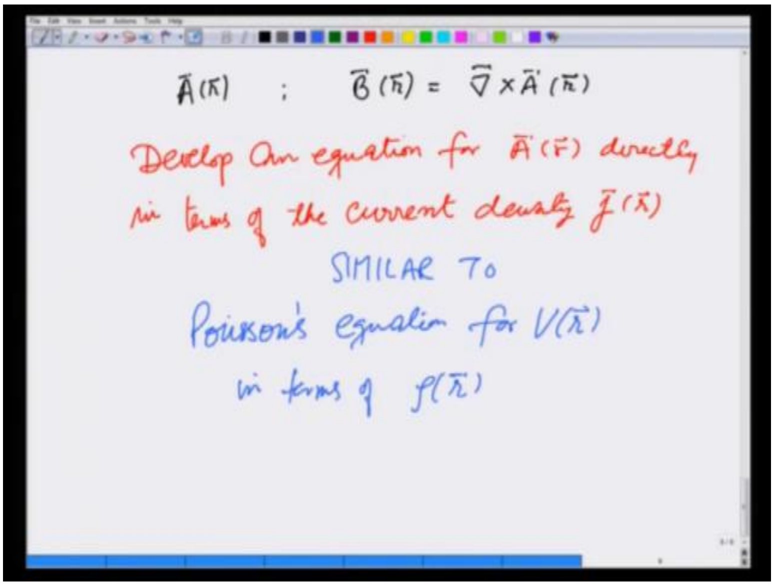

What we want to do in this lecture is, develop an equation for the vector field A r, directly in terms of the current density j of r. Something similar to Poisson's equation for V r, the electro static potential in terms of rho r. And then if I can calculate A directly in an easier way then I calculate B and then take it is curl to get B that will be a good way of calculating B for a given current distributions. So, that is the motivation.

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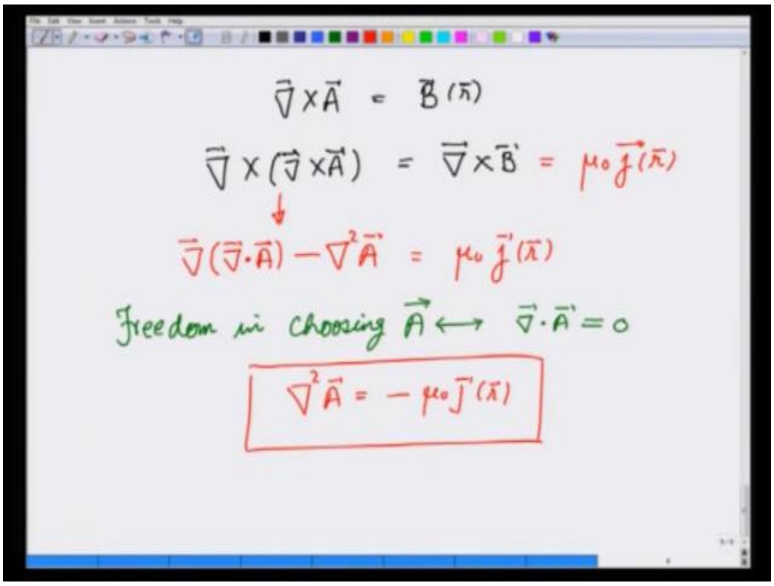

To develop this equation, we start with the equation, curl of A is B of r and take curl again on both sides. The curl of A is equal to curl of B, for curl of B, I know in terms of the current density, which is nothing but mu naught j of r. And the left hand side, I know from vector identity it is nothing but gradient of divergence of A minus Laplacian of vector A.

By taking Laplacian of vector A, one means that you are taking Laplacian of each component of A and this is equal to mu naught j of r. Now, recall from my earlier lectures that there is a freedom in choosing A in that, we can add a gradient to it and therefore, I can choose A in such a way that divergence of A is 0. If I do that, then I am left with the equation Laplacian of A is equal to minus mu naught j of r; that is the equation, Poisson’s equation for vector field, let us understand what it means.

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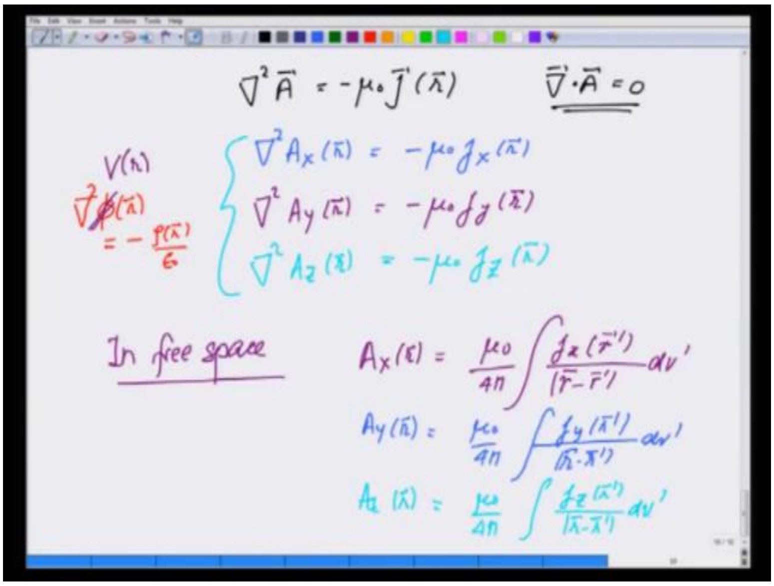

The equation we have is del square of A is equal to mu naught minus mu naught j of r, under the condition that divergence of A is 0 and keep in mind, that this is all for a steady state current. What it means is that each component del square A x is equal to minus mu naught j x r, del square A y at r is equal to minus mu naught j y at r. And finally, del square A z of r is equal to minus mu 0 j z of r.

All these equations are absolutely similar to this Poisson's equation, phi of r is equal to minus rho of r over Epsilon 0, all we are doing is 1 over Epsilon 0 is being replaced by mu 0 and rho is being replaced by j x, phi is being replaced by A x and y and z component. Therefore, the equation has the same solution or similar solution as for the Poisson's equation.

So, in free space, we already know the solution for the electrostatic potential, I have been using V r, so I should be using V r. In free space therefore, A x at r is going to be equal to mu 0 over 4 pi integral of j x r prime over r minus r prime d v prime. Similarly, A y at r is going to be mu naught over 4 pi integral j y at r prime over r minus r prime d v prime. And similarly equation for A z at r is equal to mu naught over 4 pi integral j z r prime over r minus r prime d v prime.

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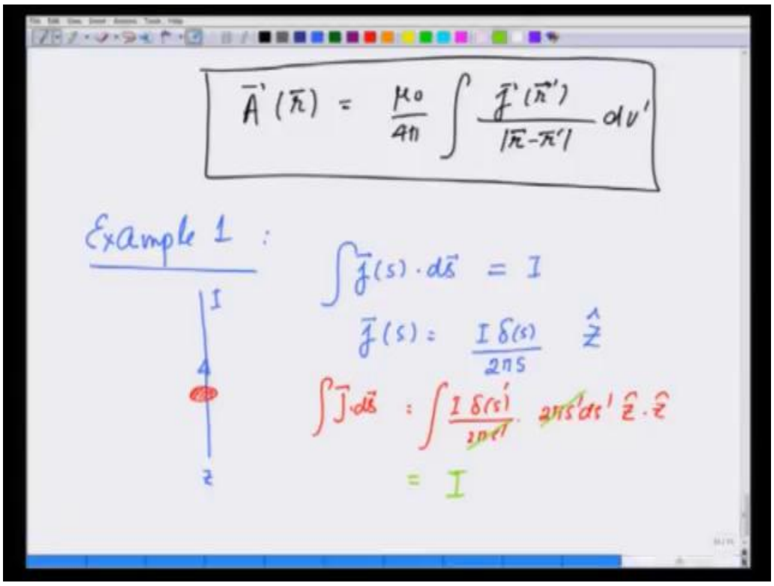

To write it altogether, what I have is then A vector at r is given as mu naught over 4 pi integral j vector r at r prime over r minus r prime d v prime; that is the equation for vector potential. Very similar to the equation for electrostatic potential except that now, I have three components, because this is the vector potential. Once, I know the current density I can calculate from this the vector potential and taking it is curl, I can always get my magnetic field.

Let us now solve a few examples, as example 1; I am going to take current I, going in a wire along the Z axis. So, this is the well known example, we have been solving so far and I want to calculate for the vector potential directly. Let us write the current density for this, you see current is going only at S equals 0 and I know by definition, current density rho d s should be equal to I.

Current density is going in the Z direction, it is magnitude is I, it is only at S equals 0, where the density becomes infinite and to normalize, I am going to do it divide by 2 pi S and that is it; that is the current density, let us check, if this is correct. If I take this area across the wire, j dot d s gives me I delta s prime over 2 pi S prime, area is going to be 2 pi S prime d s prime Z dot Z, because area is in the Z direction integral.

2 pi S prime cancels with 2 pi s prime and integral over S prime gives me 1 and therefore, this is equal to I. So, I have got the correct current density; let us then directly use this to calculate the vector potential.

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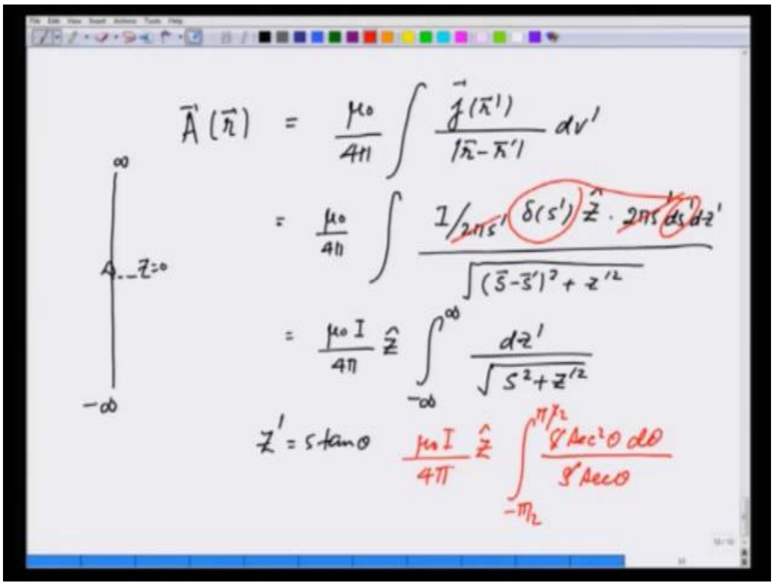

A vector potential A r is equal to mu naught over 4 pi integral j r prime over r minus r prime d v prime. Remember this is a current carrying wire going from minus infinity to plus infinity. This is equal to mu naught over 4 pi integral I over 2 pi S prime, because this is at S prime, delta S prime, this is a vector in Z direction, d v is going to be 2 pi S prime d s prime d Z prime over.

Since, this is at infinite wire at any Z, the answer for A is going to be the same, therefore I may as well calculate it at Z equal to 0. So, this is going to be square root of S vector minus S prime vector square plus Z prime square. Since, S prime goes to 0, this becomes mu naught I over 4 pi in Z direction and you can see this 2 pi S prime and 2 pi S prime cancel, this delta S prime with delta S prime here gives me S prime equal to 0. So, I am left with integral minus infinity to infinity d Z prime over square root of S square plus Z prime square.

Let us do this integral, I do this integral by taking Z prime equals S tangent theta and this gives me mu naught I over 4 pi Z integral S secant square theta, d theta over S sec theta minus pi by 2 to pi by 2. This S cancels and you can see this is an integral of secant theta, d theta will gives me log tangent plus secant theta and that blows up in this case. So, I have to actually do this calculation by taking a long wire going from minus L to L and then taking the limit L going to infinity and that will give me the S dependence. So, let us do that properly.

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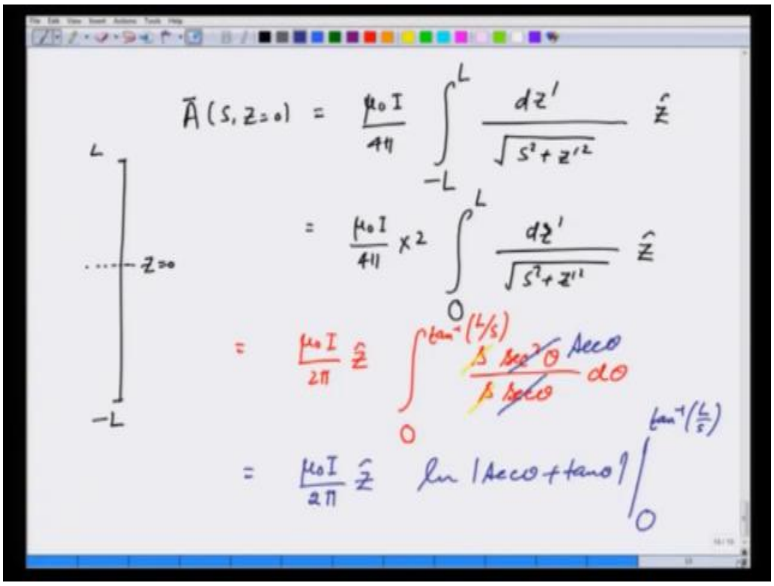

Therefore I am going to write A at S, Z equal to 0, because now I am taking a long wire extending from minus L to L and I am calculating A at Z equal to 0. This is going to be equal to mu naught I over 4 pi integral minus L to L d Z prime over square root of S square plus Z prime square. This is in the Z direction, which I can write as mu naught I over 4 pi times 2, because the integral is even in Z prime, I can make it from 0 to L, d Z prime over square root of S square plus Z prime square in the Z direction.

So, I get this to be equal to mu naught I over 2 pi, in the Z direction integral again making substitution Z prime equals S, tangent theta I will get S, secant square theta over S, secant theta, d theta, tangent theta is going to vary from 0 to tan inverse L over S. So, now let us do some cancellations again S cancels the secant theta square cancels here and I get secant theta.

And therefore, my answer comes out to be mu naught I over 2 pi in the Z direction log of secant theta plus tangent theta 0 and tan inverse L over S, which I can simplify to at 0 tangent theta is 0, secant theta is 1, so that gives me 0.

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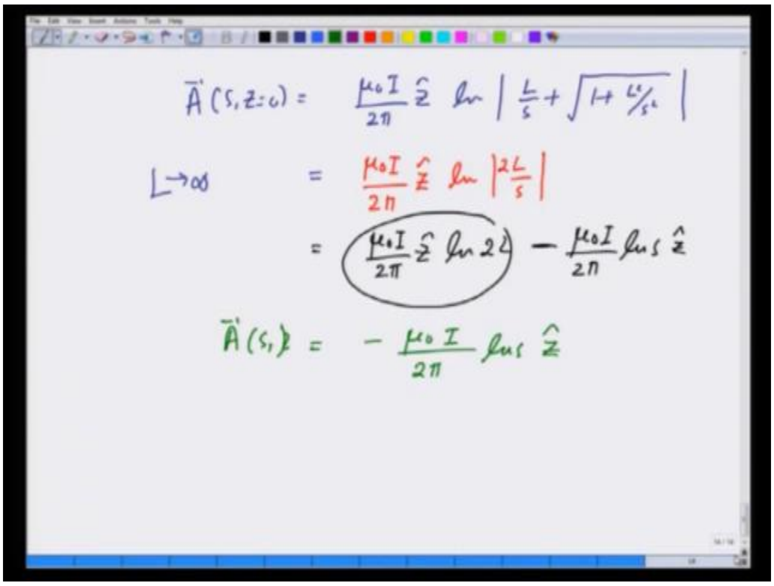

And so this comes out to be A at s z equal to 0, comes out to be mu naught I over 2 pi in Z direction log of tan inverse L over S tangent of that gives me L over S plus square root of 1 plus L square over S square is the secant theta. For L going to infinity, this can be written as mu 0, I over 2 pi, Z log of L over S plus L over S again, because I can ignore that 1.

So, I will erase this and write this as 2 L over S, which is equal to mu naught I over 2 pi Z log of 2 l minus mu naught I over 2 pi log of S Z. This part goes to infinity as L goes to infinity. So, I can ignore this, I mean this is just addition of infinity. So, the S dependence actually comes out to be A S, Z it does not depend on. So, I may write, may not write it is equal to mu naught I over 2 pi log S in the Z direction indeed the answer we had obtained earlier.

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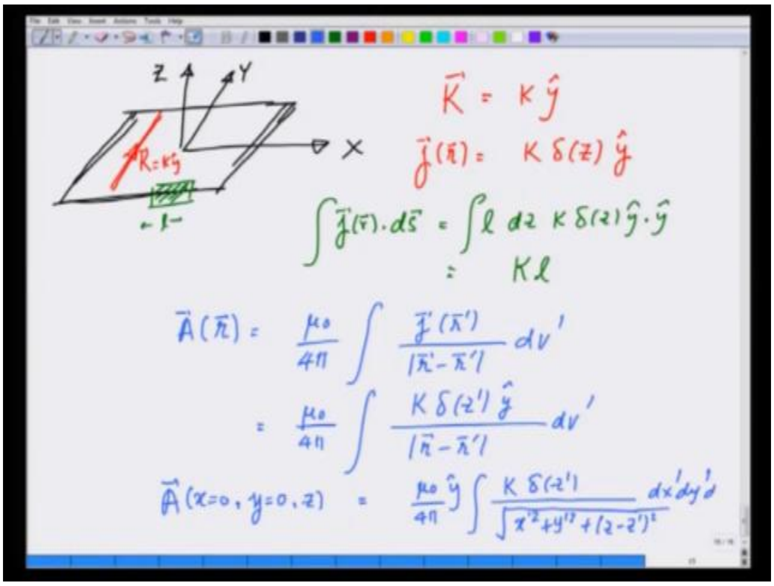

Let us do one more example and that is going to be that of a sheet of current for which we have already calculated magnetic field and we also calculated A, but now we want to do it directly. This is X direction, this is Y direction and to remind yourself, you may want to look up the lectures we have done earlier. This is Z direction and this is carrying a surface current K, which is K in the Y direction.

So, this carry is a surface current K is equal to K in the Y direction. The current density can therefore, be written as K, the area across which it is going is the Y Z plane. So, in the Z plane, it is confined to Z equal to 0. So, I can write K delta Z in the Y direction. How do we check this, let me take a cross section across which the current is flowing. So, if I do j r dot d s across this cross section, I am going to get if this length is L, L d z K delta Z y which gives me y is the direction of area also.

So, this gives me K L, which is the current indeed passing through length L of the sheet. So, our j is correct, therefore, A the vector potential A at any point r is going to be mu naught over 4 pi integral j r prime over r minus r prime d v prime, which is equal to mu naught over 4 pi integral k delta Z prime in the y direction over r minus r prime d v prime.

Now, let us see this is a plane sheet. So, the answer should not really depend on with which x, which y, am I calculating it at, because it is invariant, it does not matter, which x which y, because in x y plane, this is infinite extent. So, I may as well calculate A at x equal to 0, which maybe the same as any at any other x, it just makes my life little easy, y equal to 0 and z, which will be equal to mu naught over 4 pi integral k delta Z prime, it is in Y direction divided by r minus r prime.

Since, I am calculating it at x and y equal to 0 is going to be only x prime square plus y prime square plus z minus Z prime square d x prime, d y prime, d Z prime; that is the expression, let us write this again.

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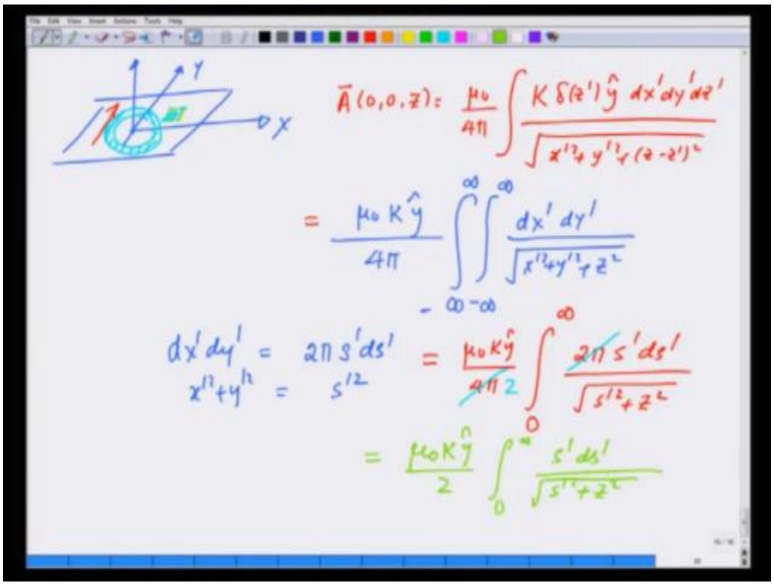

So, I am calculating due to this sheet of charge, which is carrying the surface current in direction Y. We have calculated A 0, 0 it does not matter any x, y with the same answer at Z is equal to mu naught over 4 pi integral K delta Z prime, it is in y direction d x prime, d y prime, d Z prime over square root of x prime square plus y prime square plus z minus Z prime square.

Now, the integral over Z prime because of delta Z prime gives me Z prime equals 0 and therefore, this becomes equal to mu naught K comes out Y direction comes out. It is a fixed direction over 4 pi integral of d x prime d y prime over square root of x prime square plus y prime square plus z square, both x and y going from 0 or minus infinity to plus infinity minus infinity to plus infinity.

I can make this integral very easy, if I realize that this actually is an area integral. So, I can convert this area integral into in terms of the cylindrical coordinates or planar polar coordinates, whereby d x prime, d y prime, I can write as 2 pi S prime d s prime, x prime square plus y prime square is s prime square. So, instead of taking this d x prime d y prime, let me make in a different color.

Like this, since this is an integral which is symmetric, I am actually taking this area and integrating over it is the same thing as long as I covered the entire area. And therefore, I can write this expression as equal to do mu naught K y over 4 pi integral 0 to infinity 2 pi S prime d s prime over square root of S prime square plus z square this 2 pi cancels with this I get 2.

And therefore, the expression becomes mu naught K y over 2 integral 0 to infinity s prime d s prime over s prime square plus z square root and this is very easy to calculate, let us do that.

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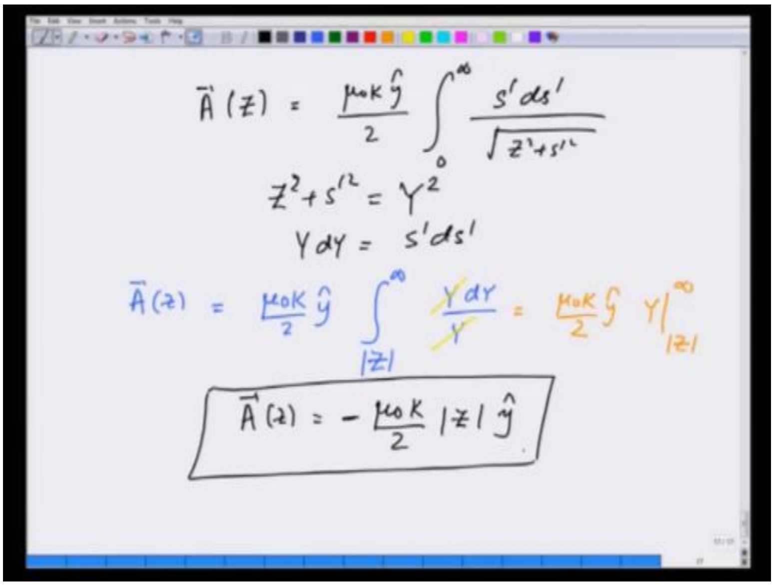

So, you have got in A at any point, but depends only on Z, x y does not matter is equal to mu naught K in y direction over 2 integral 0 to infinity S prime d s prime over square root of Z square plus s prime square. Let us take Z square plus S prime square to be sum y square, so that y d y is equal to S prime d s prime. So, I get A z is equal to mu naught K over 2 y integral modulus of z to infinity y d y over y. This y cancels and I end up getting mu naught k over 2 y at mod Z an infinity.

Again, at infinity it is a constraint which I will ignore and therefore, the Z dependence of A finally, comes out to be A z is equal to mu naught K over 2 modulus Z with the minus sign in the Y direction, indeed the answer we had obtained earlier looking at b. So, you see I have given you two examples where we can calculate A directly from A current distribution, and now I can take it is curl and get my answer for B.