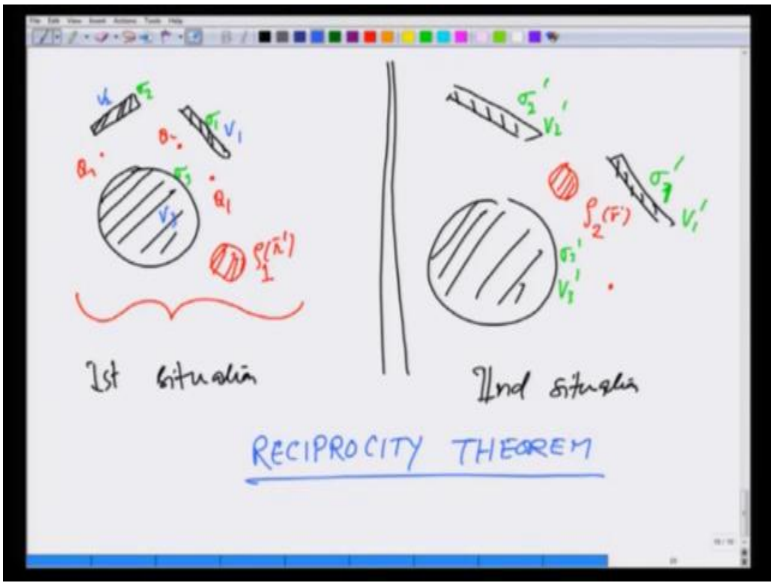
Introduction to Electromagnetism

Prof. Manoj K. Harbola
Department of Physics
Indian Institute of Technology, Kanpur

Lecture - 27
Reciprocity theorem for conductors-I

(Refer Slide Time: 00:34)

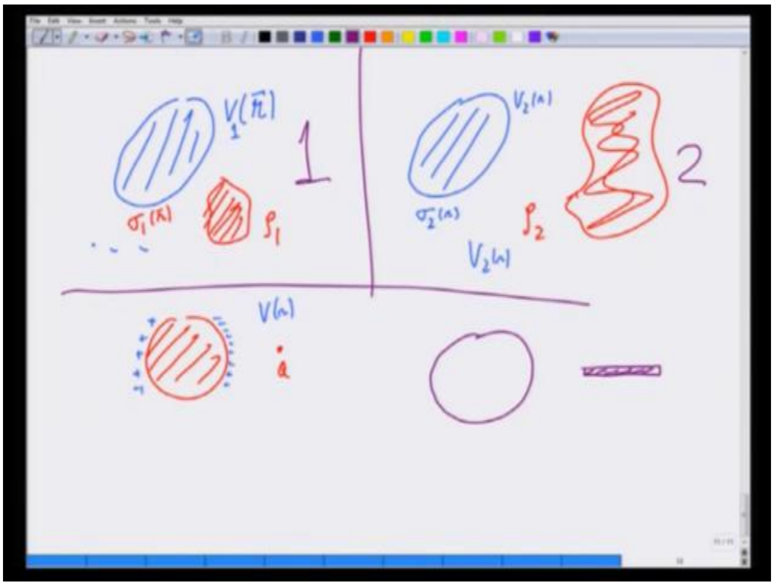


We have discussed how potential and electric field and charges behave near a conductor. In this lecture, I am going to now ask a question, suppose I am given some fixed conductors and space, they could be any shape and in sizes, but these are fixed. And I bring some charges, let us say Q 1, Q 2, Q 3 or a charges distribution along with it, some rho r prime and find that this situation I get, because this charge will produce electric field and they will produce sigma. The surface charges on the surface on all these and some potential V could be V 1 here, V 2 here, V 3 here, sigma 1, sigma 2, sigma 3 here and we know this answer.

From this, can I calculate for the same situation, same fixed charges, fixed conductors, if I now change the density and bring in some other rho 2, r prime, let us call this rho 1 and some other charges. So, this is second situation, so this is first situation, this is a second situation and now, I get sigma 2. Let us say prime here, sigma 1 prime, sigma 2 prime, V 1 prime, V 2 prime, V 3 prime, sigma 3 prime.

Can I relate the two? Because, after all it is a same situation, can I relate same situation as far as the conductors are concerned, can I relate the two and we will show that this can be done beautifully through energy considerations. And what we get as a result is, something called the reciprocity theorem. Although, this can also be proved mathematically using some theorem called Greens Theorem, we are not going to do that. We are going to look at it very physically through energy considerations and see, how the two situations can be related, it is a nice, beautiful exercise.

(Refer Slide Time: 03:13)

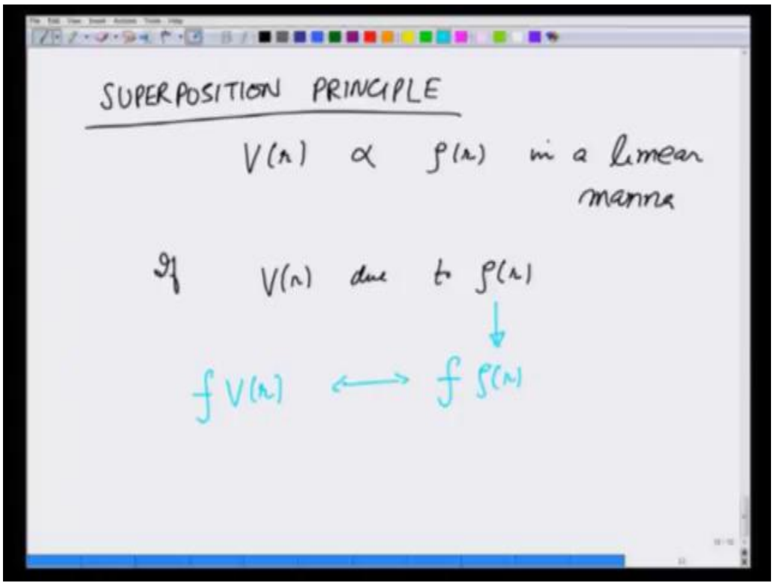


So, now I am going to change notation a bit and let me again say there is some conducting surface. If there are more I am just calling everything same conducting surface, which has potential on it V 1 r not only on it, all over the place, there is V 1 r potential. Because of some charge distribution rho 1 and some charge distributions sigma 1 r, for example, I could have a spherical surface and put a charge in front of it.

In fact, I am going to solve that example later; this will induced some negative charges here and positive charge on the backside and some potential all over the place. So, this is situation 1, I take the same conductor and change my charge distribution to rho 2. So, that the potential on the surface becomes V 2 r on the surface and in general V 2 r and charge distribution on this becomes sigma 2 r.

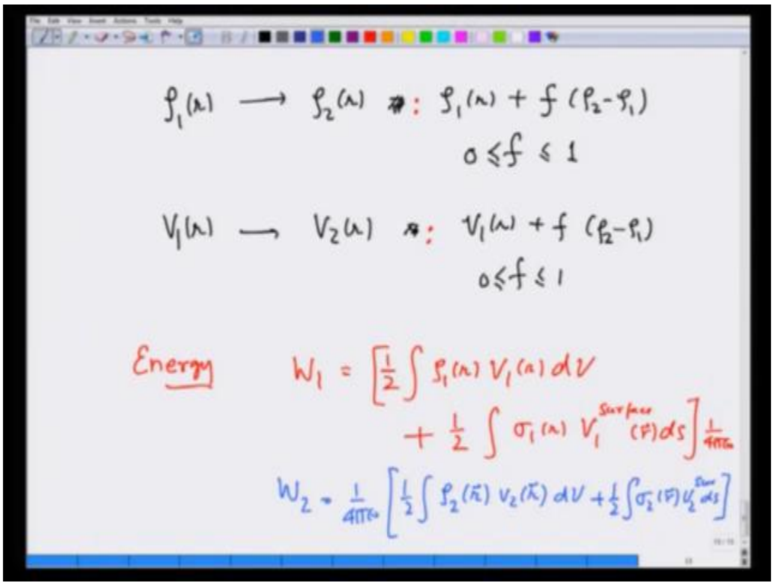
For example, analogy out here would be again take this sphere and instead of this Q, suppose I put a line charge here, can I relate the two situations. So, from situation 1 to situation 2, I have rho 1 in this, which gives me potential V 1 and surface charge sigma 1 on the surface of the conductor. I have situation 2 here in which I have charge distribution rho 2 which gives me potential V 2 and surface charge sigma 2 on the conductor. To derive Reciprocity theorem, we will start with an observation and we have talked about it in earlier lectures.

(Refer Slide Time: 05:02)



This observation is based on superposition principle, which says that, if there are two charges net field is the addition of the two fields, what it means basically is that the potential V r depends on rho r in a linear manner, that is why I can just E add it up. And what it means is, if there is some potential due to rho r, if I change the density to some factor f, rho r, the potential correspondingly will change the potential V r. That is a linear dependence; f could be greater than 1, smaller than or whatever.

(Refer Slide Time: 06:07)

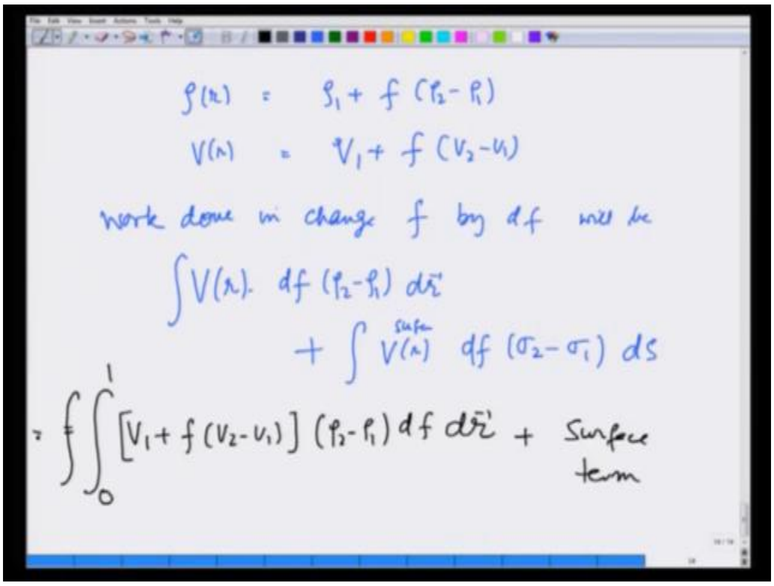


Now, let us take these 2 situations and change density rho 1 r to rho 2 r by writing this as rho 1 r plus a factor f, rho 2 minus rho 1, f varies between 0 and 1. If f is 0, I have the charge density rho 1, if f is 1 the charge density becomes rho 2. Consequently, the potential V 1 r goes to V 2 r, which is V 1 r plus f, rho 2 minus rho 1, through this f, which is between 0 and 1. So, this is not exactly equal, this is how we are doing it by changing f.

So, the energy of the initial distribution, let us call it W 1 is nothing but 1 half integration rho 1 r V 1 r d v plus, because they are surface charges, there is going to be energy due to surface contribution which is going to be 1 half integration sigma 1 r V 1 on the surface r d s. Of course, there is a factor 1 over 4 pi Epsilon 0, which I am just keeping outside, I do not want to write so much, I will bring it in later.

Let us see what happens when I change the charge distribution to the next charge distribution W 2. W 2 happens to be that same 1 over 4 pi Epsilon 0 is there, 1 half integration rho 2 r V 2 r d v plus 1 half sigma 2 r V 2 on the surface d s. So, I have changed energy from W 1 to W 2, I could also change it by changing rho slightly.

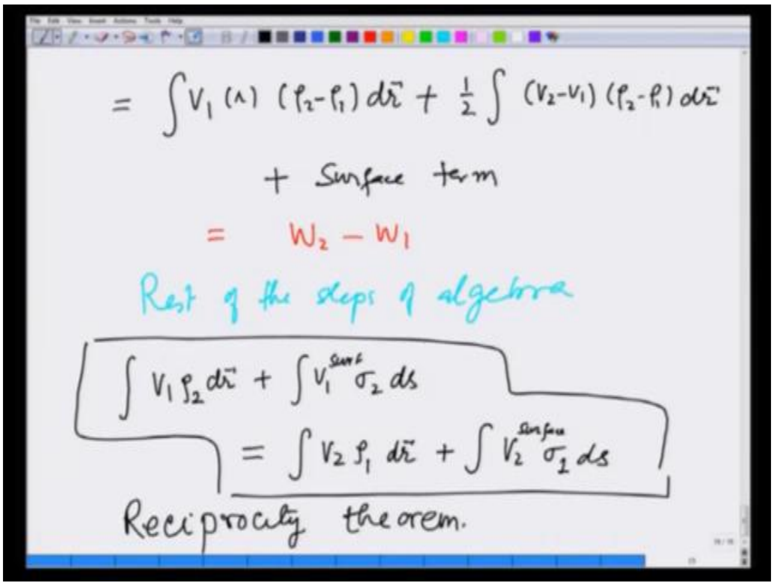
(Refer Slide Time: 08:25)



So, what I will do is, if I have take some intermediate density rho r, which is rho 1 plus f, rho 2 minus rho 1. The intermediate potential will be equal to V 1 plus f, V 2 minus V 1, then the work done in changing f by d f will be the potential V r, extra density I am bringing in which will be d f, rho 2 minus rho 1, integrated d r plus V r on the surface. And how much extra charge is coming on the surface, it is coming out to be d f sigma 2 minus sigma 1, I have put in some extra charge d s, this is a change.

Let us write these explicitly, when I do that V r is nothing but V 1 plus f V 2 minus V 1, I am multiplying this by rho 2 minus rho 1 and I am integrating over d f. Of course, there is a volume integral f varies from 0 to 1 and there is a volume integral d r. So, let me put that here plus a surface term, similar surface term that structure is the same as this, except rho 2 and rho 1 get replaced by sigma 1 and sigma 2 and V remains v on the surface, let us calculate this.

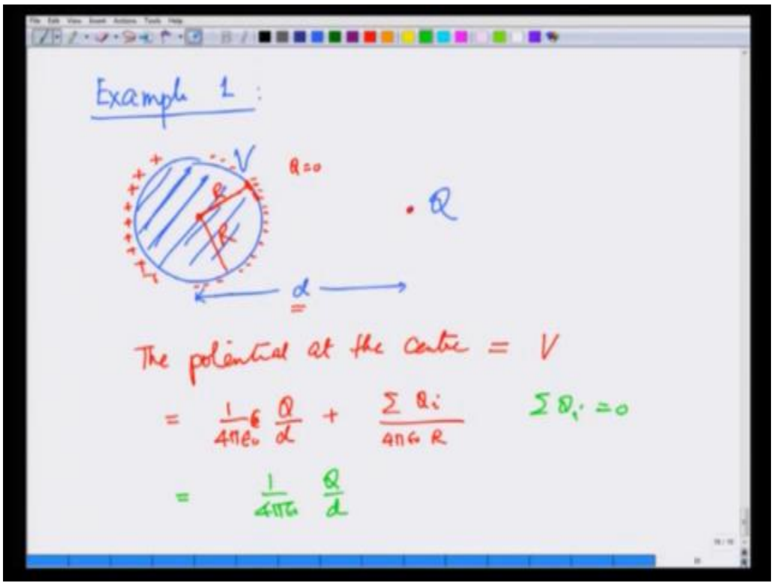
(Refer Slide Time: 10:15)



So, this comes out to be V 1 r rho 2 minus rho 1, I am doing the integration over f d r plus now I have an integration over f d f. So, I will get a 1 half integration V 2 minus V 1 times rho 2 minus rho 1 d r, f integral has been completed plus a surface term and this must equal W 2 minus W 1 that we have already calculated. I will now leave the rest of the steps for you, rest of the steps of algebra and write the final result.

The final result will come out to be that I have integration V 1 rho 2 d r plus integration V 1 surface sigma 2 d s over the surface of the conductors is going to be equal to integration, V 2 sigma 1 d r plus integration V 2 surface sigma 1 d s. So, we have related potential and density from one situation, potential and density to the other situation and this is the statement of Reciprocity Theorem. And this can be used very effectively, if I know the answer in one situation to know the answer in the other situation, if for a given geometry, if the geometry is given.

(Refer Slide Time: 12:14)



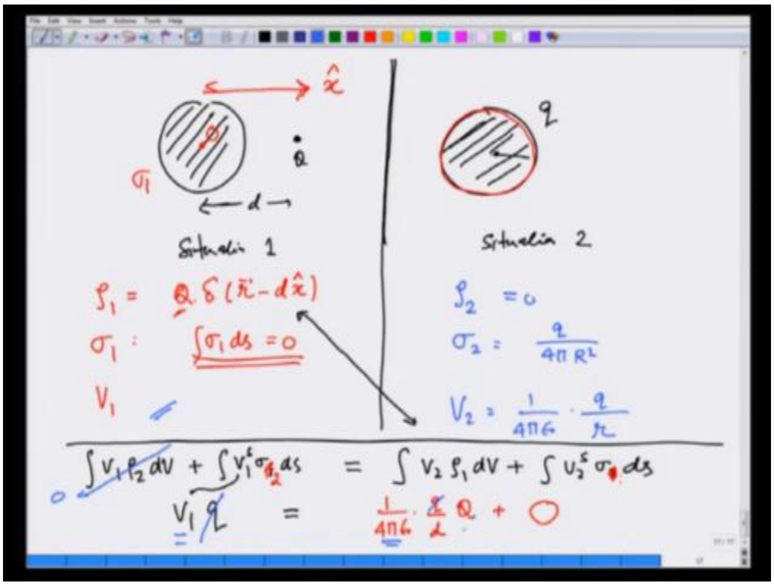
Let us take an example, as a first example; it is a very simple example, if I want to calculate the potential on a conducting sphere due to a charge at a distance d from it. This charge is Q and this is at uncharged sphere, I want to know the potential on the uncharged sphere. And as we discussed in the previous lecture, this entire sphere would have the same potential.

The radius of this sphere is given to be R and the distance from the sphere centre to the charge is d as already shown here. Let us just first use a different trick rather than we using Reciprocity's Theorem and then we will show that the two lead to the same answer. This charge induces charge on the surface of the conductor. Now, on this side there will be negative charge, more dense here, less dense in the back side and on back side will be positive charge, so that net charge Q is 0.

Now, since the entire sphere, it is at a same potential, the potential at the centre is easy to calculate is also the same as V. Now, potential at a center is very easy to calculate. What will it be, it will be 1 over 4 pi Epsilon 0, Q over d; that is potential due to this plus these small, small charges here that will be summation. Let us make it divide into small, small surface elements, Q I on each surface element divided by 4 pi Epsilon 0 r, because the distance from the each element to the center is same as r.

However, notice that summation Q i is 0, because this sphere is uncharged and therefore, this answer comes out to be 1 over 4 pi Epsilon 0, Q over d only. Let us now do this problem using Reciprocity Theorem.

(Refer Slide Time: 14:37)



In a Reciprocity Theorem, I will take two situations, one situation will be where this charge is, Q is given at a distance d and the second situation will be, when this charge there is no charge. But I give charge, let us say Q to the sphere, this is my situation 2, this is my situation, and let us see what can I write in the two situations. In this case, the charge density rho 1 is equal to let us take the center to be, center of this sphere to be the origin.

So, center is the origin, then rho 1 can be written as let this be the x direction, then I can write rho 1 as Q delta of r minus d x. There is some sigma 1 on the surface; however, sigma 1 d s is equal to 0 and there is some potential V 1, which I do not know there will be some V 1 on the surface and there will be V 1 all over the place. Let us take situation 2, in situation 2, rho 2 is 0, there is no charge anywhere outside.

However, there is sigma 2, which is Q over 4 pi r square over the surface and I also know V 2 outside, V 2 is 1 over 4 pi Epsilon 0, this is due to a spherical charge on the surface, so therefore, this is going to be q over r. So, I know V 2 sigma 2 rho 2 known situation, I want to know, what the answer in this case would be. Apply Reciprocity Theorem, so let me erase this, I will apply it right here, what does it say it says integration V 1 rho 2 d V plus integration V 1 surface sigma 1 d s is equal to integration V 2 rho 1 d v plus integration V 2 surface sigma 2 d s, rho 2 is 0.

And therefore, this term goes to 0, this should be sigma 2 here, V 1 surface is what, I want to calculate. So, and that is the same all over the surface, because it is a conducting surface times sigma 2 d s sigma 2. So, V 1 I can take out, this fellow comes out, it is constant, sigma 2 d s is nothing but this entire charge and the second situation. So, V 1 Q gives me V 2 rho 1, V 2 is 1 over 4 pi Epsilon Q over r and this, when I multiply by rho 1 here and integrate gives me r equals d.

And therefore, I get in the other case, I get 1 over 4 pi Epsilon 0 Q over d; that is V 2 and rho 1 gives me a Q, because of this q plus V 2 is, this should be sigma 1, I wrote it wrongly, V 2 is again is a constant, V 2 is a constant over this. And sigma 1 integrated over is 0 and therefore, this gives me 0. Now, you can see that this q cancels and I get V 1 equals 1 over 4 pi Epsilon 0 Q over d and that is the reciprocity theorem. In the next lecture, I will solve one more example using Reciprocity Theorem.