Introduction to Electromagnetism
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Lecture - 25
Energy of a Charge Distribution-III
Energy Density in terms of Electric Field

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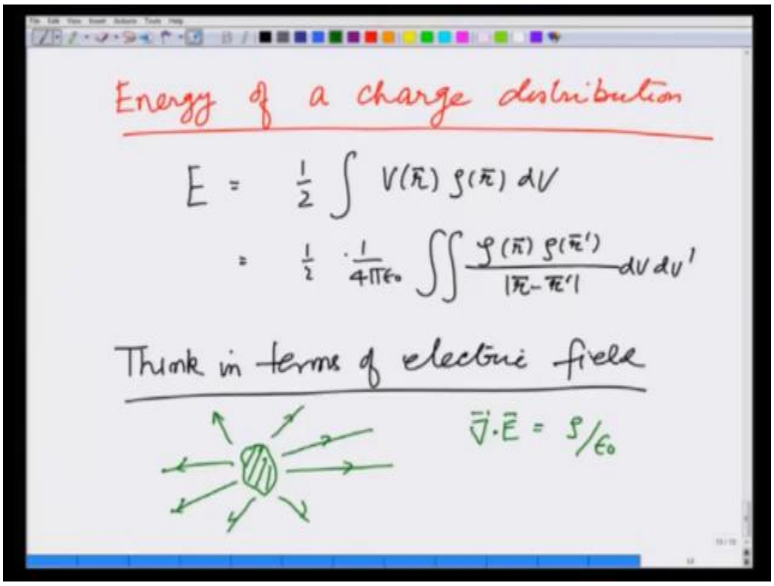

We have been talking about energy of a charge distribution. And what we have seen in the previous two lectures is that, this is given as one-half integration V r rho r dV or one half 1 over 4 pi epsilon 0 integration – double integration rho r rho r prime over r minus r prime dV dV prime. As we learnt earlier, we also like to think in electrostatics. And this becomes very important later in terms of electric field. So, think in terms of electric field. Conceptually, I will just recall what we had said earlier. What we had said is given a charge distribution, I can think of this creating a field around. And a charge distribution and field are uniquely related given boundary condition. So, if I know the charge distribution and the boundary condition, I know what the electric field is and vice versa. If I know the electric field, then divergence of electric field gives me the charge distribution.

So, can I think… If I am thinking of this charge distribution giving rise to this field all around it, can I think of this energy as if it has been stored in this electric field. There will be conceptually different way of thinking about it. And let us work on that. And in fact, what we will see is I can actually describe this energy as if it is stored in this electric field. And in turn, they will give me charge energy density per unit volume. This becomes conceptually very important later when we talk about electromagnetic waves of fields propagating. Then we will see that, these fields as they propagate and new fields are created, energy also gets transported by it. So, it is important to think in terms of the energy being carried by the fields. So, that is what we are going to establish in this lecture.

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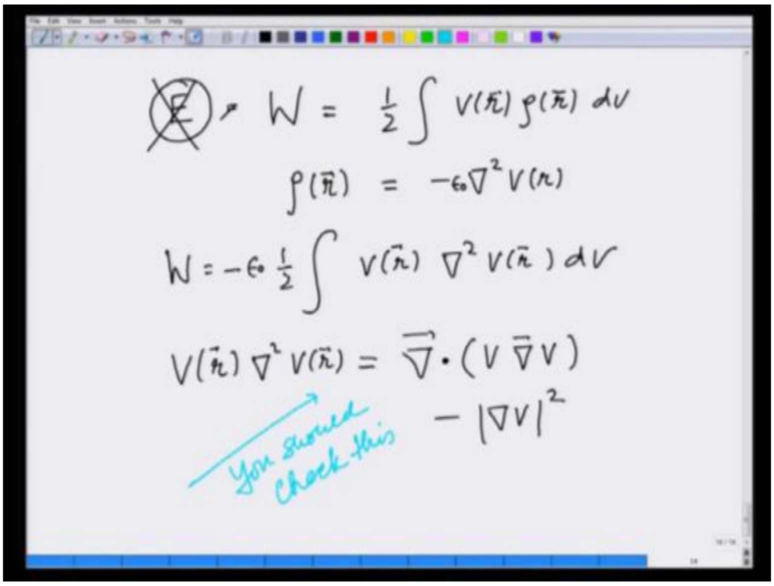

So, let us see what we had worked on. We said that, energy… I am using this E; do not confuse this with the electric field. And to remove that confusion, let us write this as W – the work, is given as one-half integration V r rho r integrated over the volume. Now, I am going to do some mathematical trick. I am going to write rho r as minus del square V r times epsilon 0. This is the Poisson’s equation. If we do that, then I am going to write W as one-half integration V r times del square V r – this is vector dV; and there is a minus epsilon 0 outside. All right? V r del square V r since del is a differential operator, I can write this as divergence of V gradient of V. And this will give me gradient V dot gradient V plus V del square V. So, I have to subtract that grad of V square from this. This I will leave for you should check this. All right? Remember – this del operator is actually a differential operator. So, I can I can write this and you please check this, because now I will be doing it in all three components.

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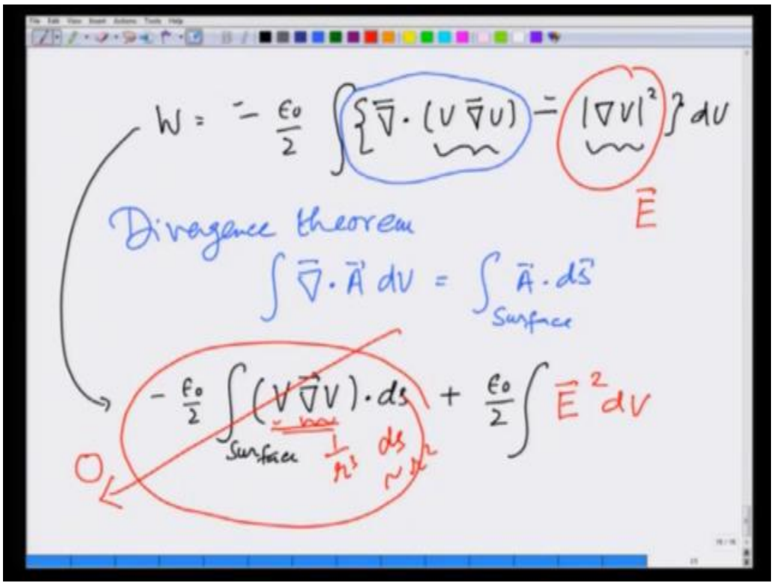

And therefore, using this, I can write the work as minus epsilon 0 over 2 integration of divergence of V del V minus grad of V square integrated over the volume. For the first term, I can now use divergence theorem. What does the divergence theorem tell me? It tells me that, divergence of a vector field integrated over the volume is nothing but A dot ds over the surface enclosing that volume. And therefore, I can write this work as minus epsilon zero 0 over 2 integration over the surface of the vector field, which is inside V gradient of V dotted with ds; minus-minus will give me plus epsilon 0 over 2. And what is grad of v? Let me show this here in the red, is nothing but the electric field. So, this is electric field square dV. If I am talking about a large space, then far away the surface is going to be at infinity; this V goes as 1 over r; grad of V is the electric field, which goes as 1 over r square. So, this term will be going as 1 over r cube; ds goes as r square. So, in the limit of r going to infinity, this vanishes as 1 over r. So, this goes to 0.

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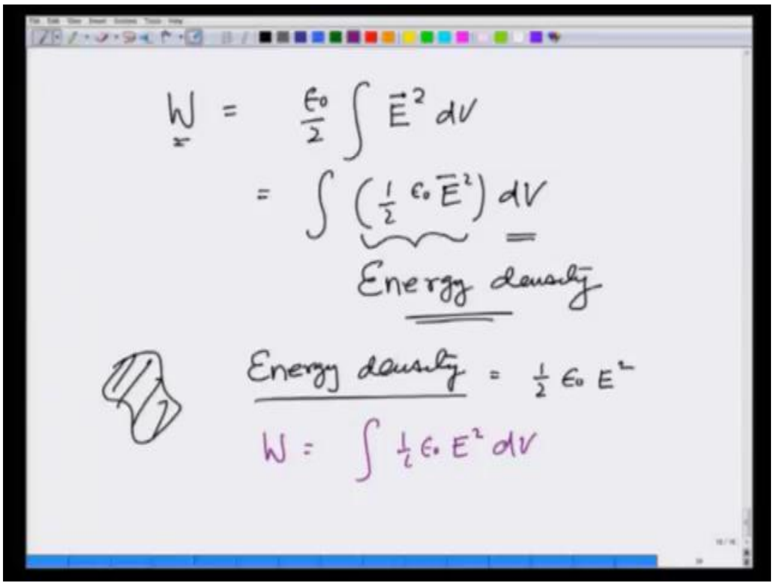

And therefore, I get the energy or the work done as epsilon 0 over to integration E square dV; which I can write as integration of one-half epsilon 0 E square dV. This is a quantity when integrated over the volume, gives me the total energy. And therefore, this quantity must be energy density. So, now I can think if there is a charge distribution; all right? then the energy density that it gives rise to all around including the space in which this charge distribution itself sits is equal to one-half epsilon 0 E square. If that is the energy density, the total energy comes out to be this energy density integrated over the entire volume. So, now I am thinking of this energy being sitting in this electric field or this electric field carrying this energy. Let us solve one example of this.

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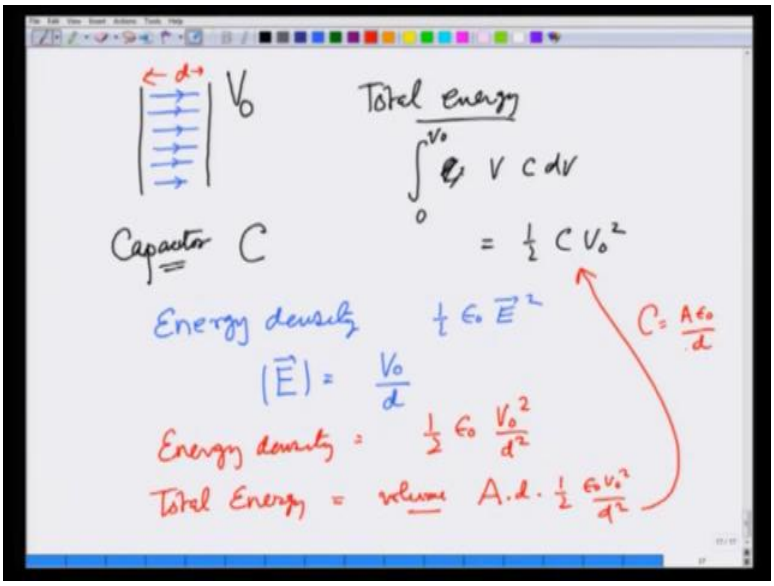

You have learnt in your twelfth grade – if I have a capacitor of capacitance c; and if I give voltage V to this; what is the total energy required to do so? It is… V is the potential somewhere… Let us say the potential given is up to V 0. So, V charge brought in is C dV 0 to V 0; which becomes half C V 0 square. That is the total energy in this. Now, according to what we just now said the energy density, it is one-half epsilon 0 E square. Now, in a parallel plate capacitor, the only place where the electric field is, is between the plates everywhere else is 0; and this electric field magnitude – it is nothing but V 0 over d; where d is the distance between the plates. And therefore, the energy density in this case is going to be one-half epsilon 0 V 0 square over d square. Total energy is going to be the volume, which is the area of the plates times d times the energy density. And this you can see immediately is same as this with C being equal to A epsilon 0 over d.

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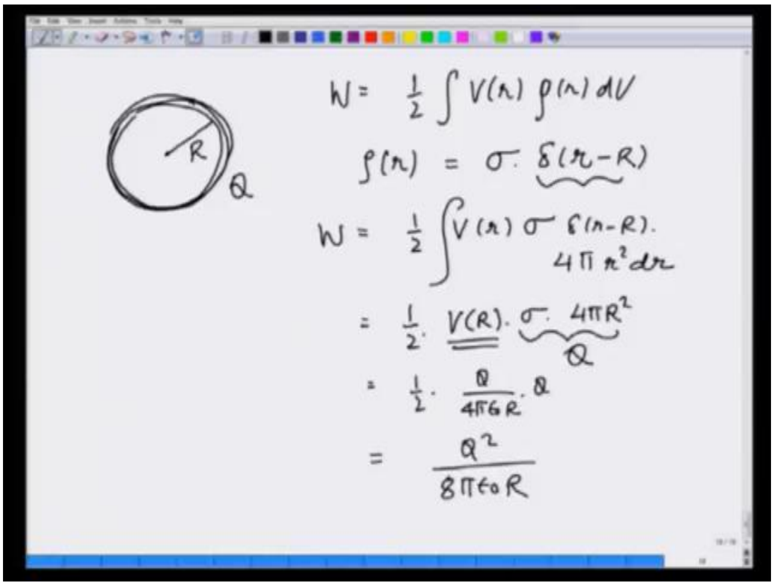

The next example I take is that of a charge shell on which there is a charge Q. Now, if you calculate the energy with using the formula one-half V r rho r dV; in this case, rho r happens to be the surface charge, which I can write as sigma times delta r – only r not vector r – minus R. This has dimensions of 1 over r. Therefore, this has dimensions of sigma times delta r minus R as dimensions of 1 over the volume – 1 over the volume. So, work is going to be therefore, one-half. If I put the integration V r sigma delta r minus R times 4 pi r square dr integration; which is one-half V at R times sigma times 4 pi R; square 4 pi R square is nothing but Q. So, this is one-half – V due to this Q is nothing but Q over 4 pi epsilon 0 R times Q. So, total energy is going to be Q square over 8 pi epsilon 0 R – a known result from your twelfth grade. So, this is… This is the energy of this system. Let us now calculate this using the electric field and the concept of energy density.

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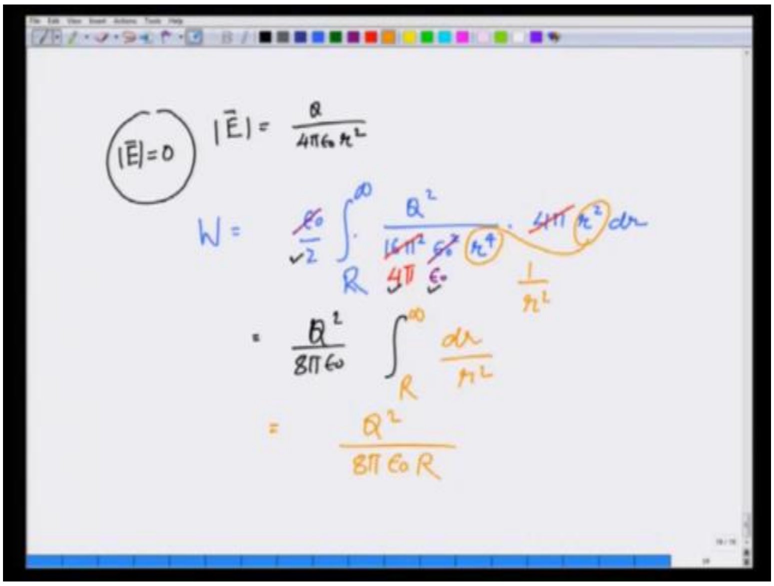

So, for the shell, the electric field magnitude outside is Q over 4 pi epsilon 0 r square; E inside is 0. And therefore, I get W equals integration epsilon 0 over 2; E square will give me Q square over 16 pi square epsilon 0 square r raised to 4 times 4 pi r square dr. And this integration is going to be from R to infinity, because inside the field is 0. Let us cancel a few terms; this 4 pi cancelling with this 16 pi square gives me 4 pi; epsilon 0 cancels with this gives me epsilon 0. So, I end up getting 1 over 8 pi epsilon 0; that is, this half this and this – Q square integration. This r square with this gives me 1 over r square dr over r square r to infinity, which is same as Q square over 8 pi epsilon 0 R – same answer as earlier.

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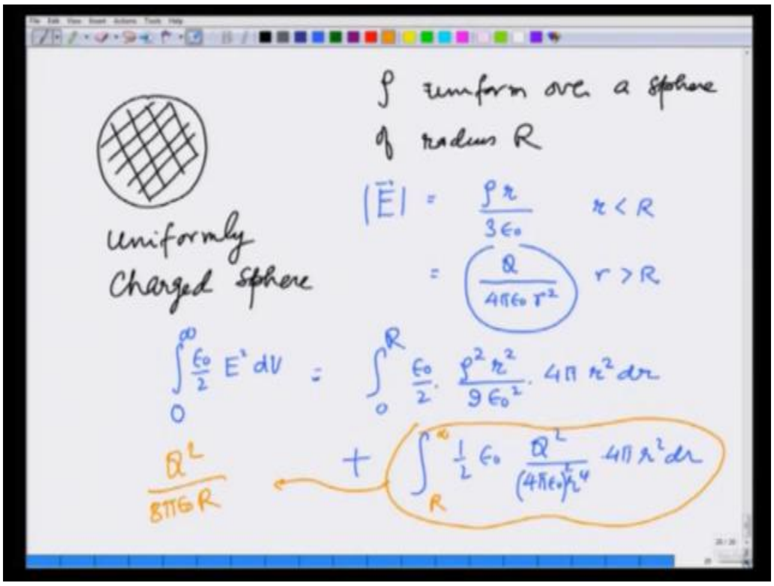

As the final example, I take a problem of uniformly charged sphere; where, the density rho is uniform over a sphere of radius R. This is the problem we solved earlier by calculating one-half v r rho r. In this case, the magnitude of electric field is rho r over 3 epsilon 0 for r less than R; and it is equal to total charge Q over 4 pi epsilon 0 square for r greater than R. Now, when I do the integration epsilon 0 over 2 mod E square dV; now, it will be from 0 to infinity. Remember this part – I have just done in the previous problem – this integration. So, I can write this as 0 to R epsilon 0 over 2 rho square r square over 9 epsilon 0 square times 4 pi r square dr plus integration one-half epsilon 0 Q square over 4 pi epsilon 0 r raised to 4… This is 4 pi epsilon 0 square times 4 pi r square dr. The second integral I just now did in the previous problem. So, this is nothing but Q square over 8 pi epsilon 0 R, because this limit is from R to infinity.

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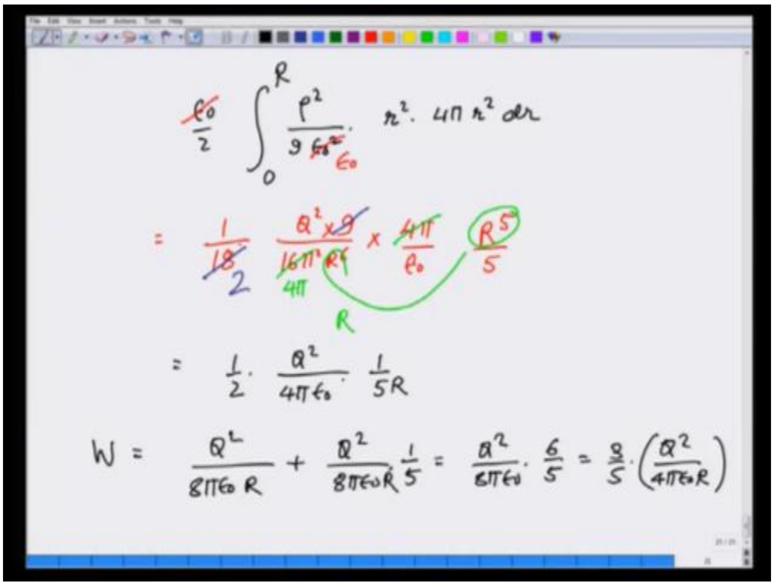

So, it is only the first integral that I am going to calculate now, which is epsilon 0 over 2 integration rho square over 9 epsilon 0 square r square times 4 pi r square dr – 0 to R. Let us again cancel a few terms; this becomes epsilon 0. So, I will get 1 over 18 rho square, which is nothing but Q square over 16 pi square R raised to 6 times 9 times 4 pi over epsilon 0 – R raise to 5 over 5. Let us cancel a few terms here. This nine gives me 2; this 4 pi gives me 4 pi; R raised to 5 with this gives me R.

And therefore, I am left with 1 over 2 Q square over 4 pi epsilon 0 – one-fifth R. This added to the previous expression will give me the total energy and let us do that. So, total energy W is going to be Q square over 8 pi epsilon 0 R plus Q square over 8 pi epsilon 0 – one-fifth, which is nothing but… Or, there is an R also; Q square over 8 pi epsilon 0 – 6 over 5, which is three-fifth Q square over 4 pi epsilon 0 R, which is the same answer as we had obtained earlier. So, what I have tried to show you in this lecture is that, you can think of the energy of a charge distribution as if it is stored in the electric field produced by this with the energy density one-half epsilon 0 E square. And we have seen through three examples that, this gives you the same answer as we obtained earlier with the direct formula.

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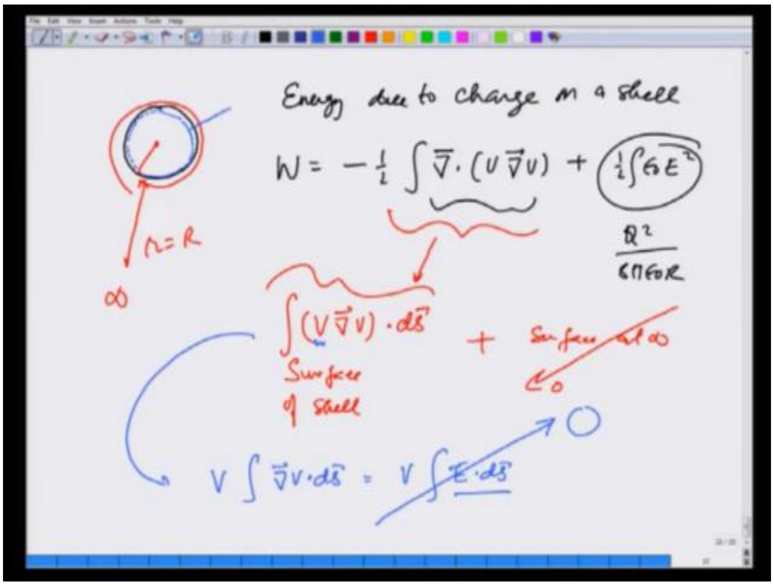

I end this lecture by discussing something which maybe may have bothered you throughout this lecture. And that is when I was calculating the energy due to charge on a shell, remember I wrote the energy to be minus one-half integration grad of – divergence of grad V grad V plus one-half epsilon 0 E square. This term gave me Q square over 8 pi epsilon 0 R. And I said conveniently, this term is 0. But, some of you may have noticed that, I am doing an integration from r equal to the radius up to infinity. So, you may be wondering what happened to this surface term, because this divergence is supposed to give me a surface term; this is a slight subtle point, because this divergence will give me V grad V dot ds over surface of shell plus surface at infinity; and that gives me 0. I will show you now that, this also gives you 0. You see the surface of the shell is slightly inside, because when I am doing this integration, it includes all the charge distribution. So, I am actually doing this integration from this blue surface, which – to infinity; and this blue surface and infinity region in between includes all the charge distribution.

Now, on a shell, the potential is a constant. And therefore, this integration becomes V outside grad V dot ds, which is nothing but V integration E dot ds. But, then this blue shell does not include any charge. And therefore, E dot ds is 0 by Gauss's law. And therefore, this term also vanishes. Think about it; it may have bothered you throughout the lecture; but now I have given you the answer.