Introduction to Electromagnetism

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Lecture - 20
Method of images I: point charge in front of a grounded metallic plane – I

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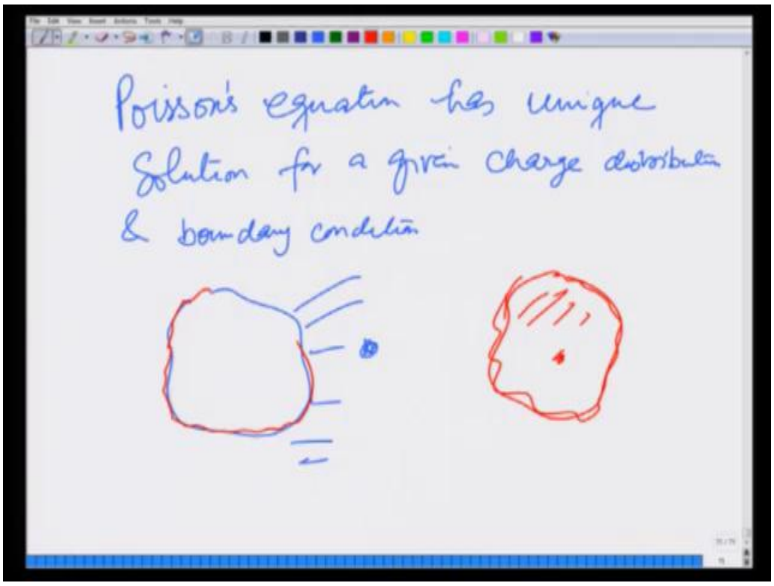

In the previous lecture, while discussing Laplace's and Poisson's equations, what we saw, we saw many things. But, what we saw in particular that Poisson's equation has unique solution for a given charge distribution and boundary condition. What that means is that, if somehow, if I know somebody, who can look through the solution or if I have a computer that give me the solution. If I have been given a boundary and some charges outside, then the volume will be outside or a boundary and charges inside, then the volume will be inside.

The potential is specified on the boundary, the potential in this case is specified on the boundary and charges are given outside. If somehow I can make a solution, I can think of a solution that satisfies that boundary condition and also satisfies that Poisson's equation. Then, I have found these solutions, there can be no other solution, let me make that point again.

If I can find a solution by some method, somebody comes and tells me, you look maybe this works out or computer generates a solution that satisfies that boundary condition and that Poisson's equation. Then, that solution is a unique solution, there can be no other solution, so I found the solution. This is what we use in method of images to solve for the potential in certain specific cases and I am going to take two examples in this lecture.

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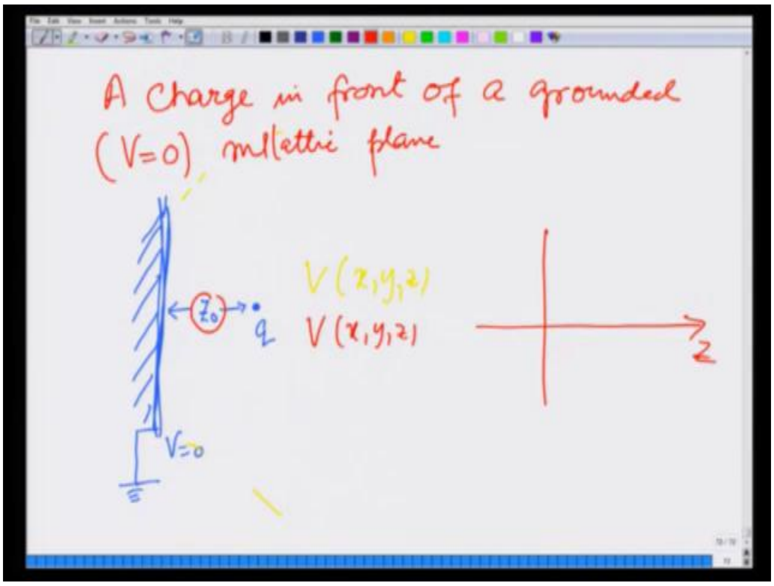

First example is a charge in front of a grounded, what that means is, potential there is 0, metallic plane, plane means of infinite extend. So, let me specify the problem, what the problem is that, I have a plane which has been grounded; that means, potential out here is 0 and I put a charge in front of it, q at a distance, let us say z 0. What is the potential throughout? Notice that, this has been grounded and so this line, this forms one boundary and all over the place up to infinity, now let me just make it, up to infinity, there the potential is 0.

So, I have actually specified potential all over the boundary, I want to know, what is the potential, let us say at V equals x, y and z. Since, this color is not ok, let me write different color x, y and z, I have all ready written this to be z naught out here. So, I am assuming that, this direction is z to make you things visualize things better.

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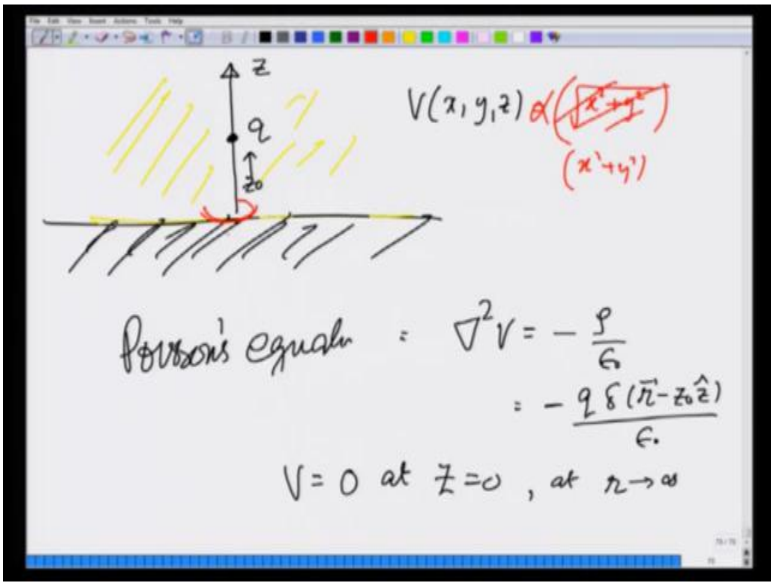

Let me make in the conventional sense, here is my boundary, here is my z axis and here is the charge q at a distance z naught and I want to find potential V x, y, z. In this volume, volume above let me make that with this color, because this specifies the boundary of this volume. Now, inside this volume, the Poisson's equation is del square V equals minus rho over Epsilon 0.

Recall what rho is given for given point charge as, this is given as delta functions of q, delta r minus z naught z over Epsilon 0 and V is 0 at z equal to 0, this is z equals to 0 plane and all throughout outside and at r equal to infinity. I want to find the solution of this. Now, one way of course, is to solve the Poisson's equation, other way can I find the charge configuration, so that this boundary condition is satisfied. If I can do that, then I found my solution, these solutions and that is a trick that we use in method of images.

Now, just a couple of points, one this potential is not going to depend on the where I am on the x, y plane. So, it has cylindrical symmetry. So, it depends only on the distance x square plus y square is proportional to x square plus not proportional to, but depends only on x square plus y square and so obviously is going to depend on z naught.

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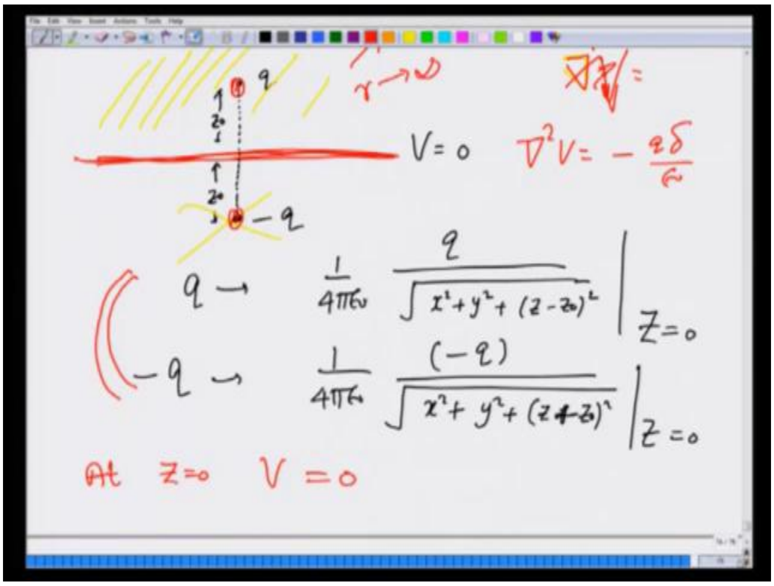

So, now let us look at this plane again, and now we are going to have a moment then that I tell us what method of images looks like. Here is the charge q and I want potential of the boundary to be 0. Now, look at this, if I think of a charge behind this plane, which is minus q exactly at the same distance. Then, q gives me potential on the boundary as 1 over 4 pi Epsilon 0 q over square root of x square plus y square, because the charge is at x and y equal to 0 plus z minus z naught square at z equals to 0.

Because, this boundary that z equal to 0 and minus q gives me a potential 1 over 4 pi Epsilon 0 minus q over square root of x square plus y square plus z plus z plus minus z naught 0 square at z equal to 0. So, although the two give different potentials at different points, but at z equals 0, the two potentials are equal and opposite in sign and therefore, at this immediately tells me that at z equal to 0, V 0.

If at z equals 0, V equal to 0; that means, these two charges plus q and minus q, give me the boundary condition; that is required. Obviously, far away at r tending to infinity both charges give me 0 potentials. So, r equals infinity condition anyway satisfied, the beauty is that even at z equal to 0, the boundary condition is satisfies. Therefore, if I look in the upper region, which excludes this charge, then Poisson's equation in the upper region is del square V is equal to minus q appropriate delta function over Epsilon 0.

So, as far the upper region is concerned, the potential which is a combination of these two satisfies the same Poisson's equation as it satisfies only for the upper charge and the two together give me the same boundary condition.

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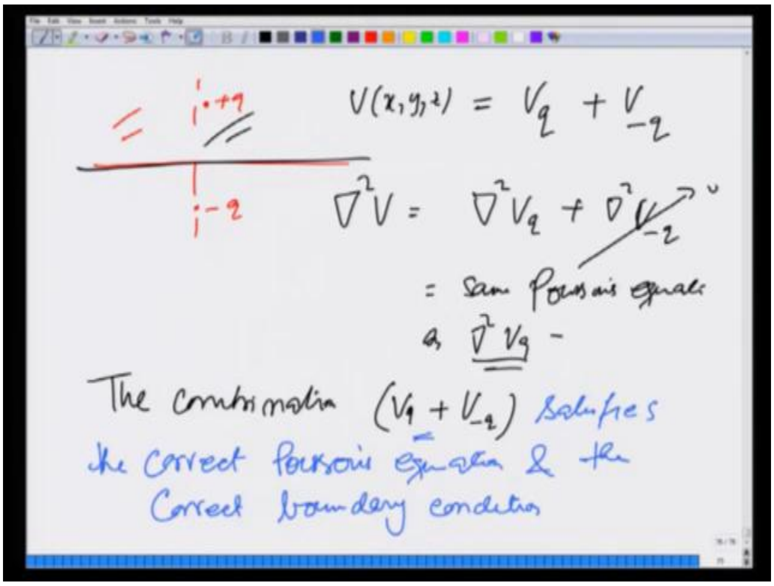

That means, in the upper region, if I write the potential, so let us take plus q here, minus q here. In the upper region, if I write the potential V x, y, z as potential due to q plus potential due to minus q, then del square V is equal to del square V q plus del square V minus q, but for the upper region, this is 0. Because delta function gives me 0 in that region is satisfies the same Poisson's equation as del square V q, V z is and they also satisfy the boundary condition.

So, the combination V q plus V minus q satisfies the correct Poisson's equation and the correct boundary condition. What that means is that, this is a solution that satisfies both and this is, these the solution then by uniqueness theorem.

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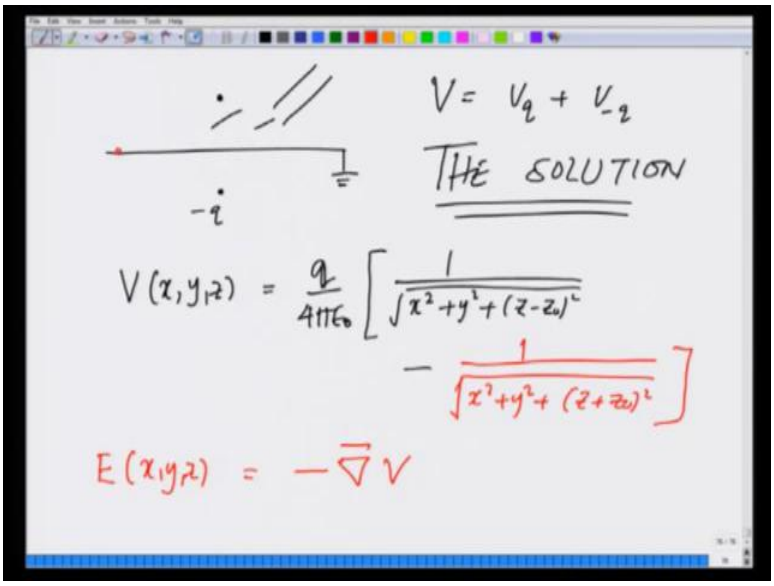

So, we found the solution; that means, for a charge in front of a grounded metallic plane or grounded plane, if I put a minus charge here and write the potential in this region as summation of V due to q plus V due to minus q, this is the solution. And this is the idea behind method of images that you have found a solution by putting some image charge somewhere, you found the solution.

And therefore, I can write potential V x, y, z as equal to 1 over 4 pi Epsilon 0, q is the same for both. I can even write q outside 1 over x square plus y square plus z minus z naught square root minus and thus is due to image 1 over square root of x square plus y square plus z plus z naught whole square, this is the potential. Once I found the potential rest everything follows, if I want to find the electric field E x, y and z; this comes out to be as minus gradient of V. If I want to find the charge on the surface, I will find the electric field here and by Gauss's law related to the surface charge.

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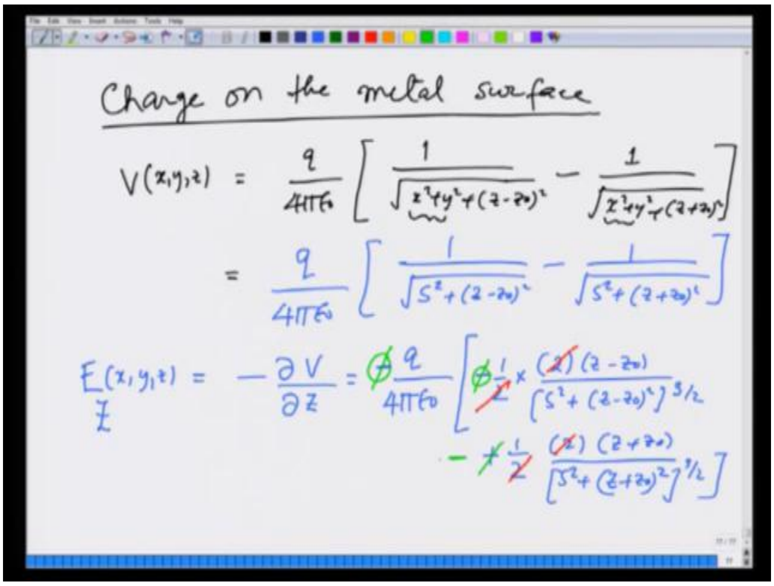

So, let us do that, let us using this find the charge on the metal surface. So, what we want to do is, we found V, which is equal to x, y, z which is equal to q over 4 pi Epsilon 0, 1 over square root of x square plus y square plus z minus z 0 square minus 1 over square root of x square plus y square plus z plus z 0 square. Then, the electric field or let us write this slightly difference, because this come in the combination of x square plus y square, which using cylindrical coordinates, I can write as s square.

So, I will write this as q over 4 pi Epsilon 0, 1 over square root of s square plus z minus z 0 square minus 1 over square root of s square plus z plus z 0 square. Electric field E x, y, z and let us find this z component only is minus, you will see in a minute, why I am doing that, d v over d z is going to be q over 4 pi Epsilon 0. This gives me minus 1 half times that is the minus sign outside, 2 z minus z 0 divided by s square plus z minus z 0 square raise to 3 by 2 minus, minus plus 1 half 2 z plus z 0 over s square plus z plus z 0 square raise to 3 by 2, this 2 cancels.

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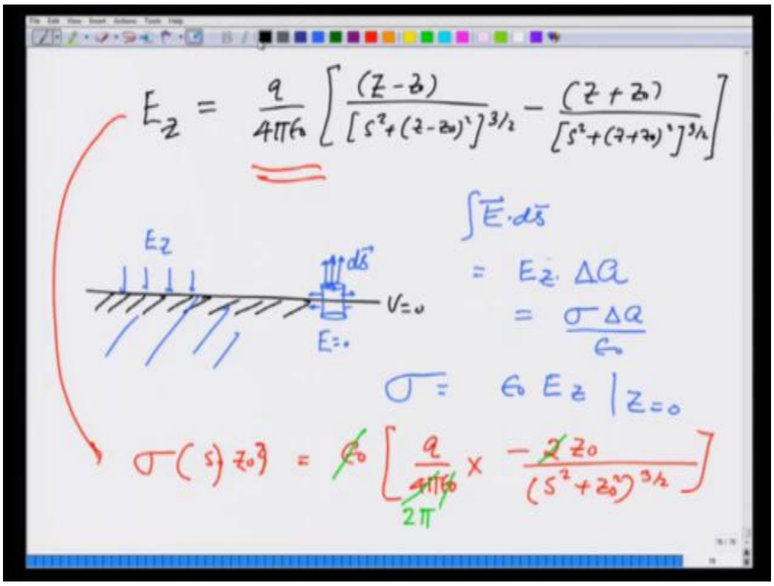

So, I end up getting, I can also remove this minus sign, this minus sign and put a minus sign here. So, I end up getting E z as equal to q over 4 pi Epsilon 0, z minus z 0 over S square plus z minus z 0's square raise to 3 by 2 minus z plus z 0 over s square plus z plus z 0 square raise to 3 by 2. Why I am calculating E z only because, that is required, that is the only thing that is required for the charge.

Since, this is an equipotential surface, the grad we have already discuss the gradient is going to be perpendicular here, and therefore near the surface the field is only z component. Let us see, what is the charge density, if I take a small box here, then the area integral, area on this side of the box is going this way, field insides since this is a metal, field inside is 0, this side area is perpendicular to the fields.

So, therefore, the this will not contribute, so I get E dot d s as E z times this is a small area, let us called it delta a and this should be equal to charge enclose sigma delta a over Epsilon 0. And therefore, sigma is equal to Epsilon 0 E z at z equals to 0, putting it here, gives me sigma at distance s or x y for charge at z naught is equal to Epsilon 0 times this at z equal to 0.

So, q over 4 pi Epsilon 0 times, you can calculate this very easily, it comes out to be minus 2 z naught over s square plus z naught square raise to 3 by 2. So, now, this Epsilon 0 cancels, this 2 gives me 2 pi here.

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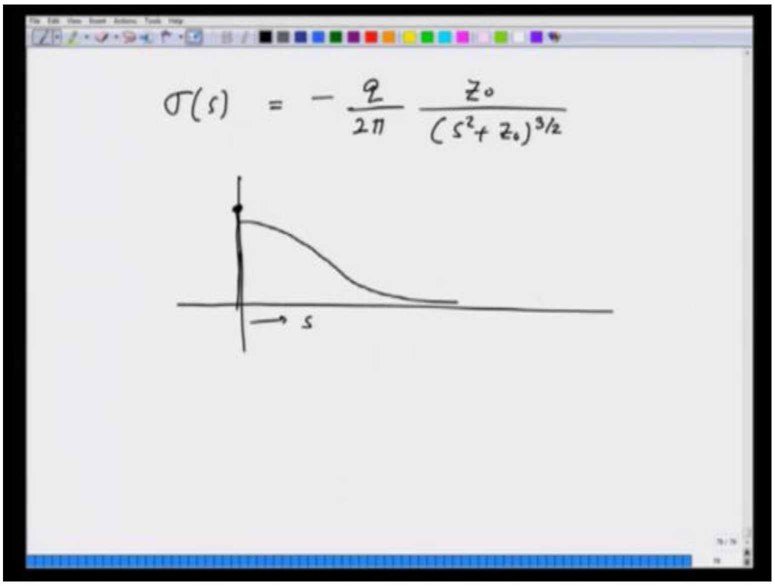

And therefore, you can see that charge on the surface is sigma S is equal to minus q over 2 pi z 0 over s square plus z 0 square raise to 3 by 2. The charge as you go away from the position of the charge, this is distance s, it starts going down. So, charge density goes down.