**Introduction to Electromagnetism**
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**Lecture - 19**
**Uniqueness of the solution of Laplace's and Poisson's equations**

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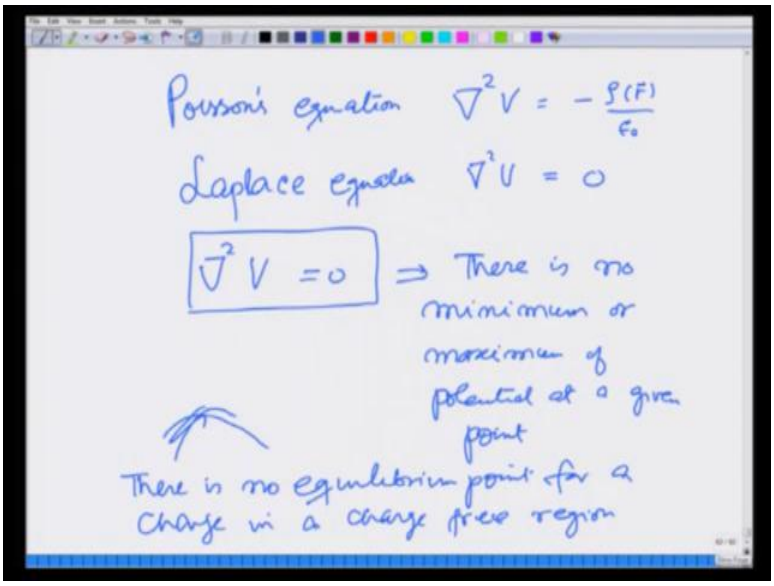


We have learnt that the potential satisfies the Poisson's equation, which is Laplacian of the potential is equal to minus charge density at that point divided by Epsilon 0 or the special cases Laplace's equation which tells me the del square V is 0 in the places, where there is no charge. This has consequences and as we said earlier that we had also said that you know these becomes the differential equations for the potential.

If you want to use these to calculate potential, we should be able to show that these give unique answers given a boundary condition. As for the consequences, I will just mention one thing, if I take del square U V is equal to 0, this implies there is no minimum or maximum of potential at a given point. And therefore, at a given point, if the potential is going to a maxim, through a maximum at one point, it will go through a minimum on the other side.

The one way of good way of visualizing it, look at this part of your hand. In this, if you come from the side of your fingers and thumb, you are hitting a minimum, but if you go from the palm to the other side, you hitting a maximum. So, there is no maximum or minimum here, this is known as a saddle point and this is where del square of this quantity, whatever the height if I calculate is going to be 0. And what it means is, there is no equilibrium point for a charge in a charge free region, let us understand that.

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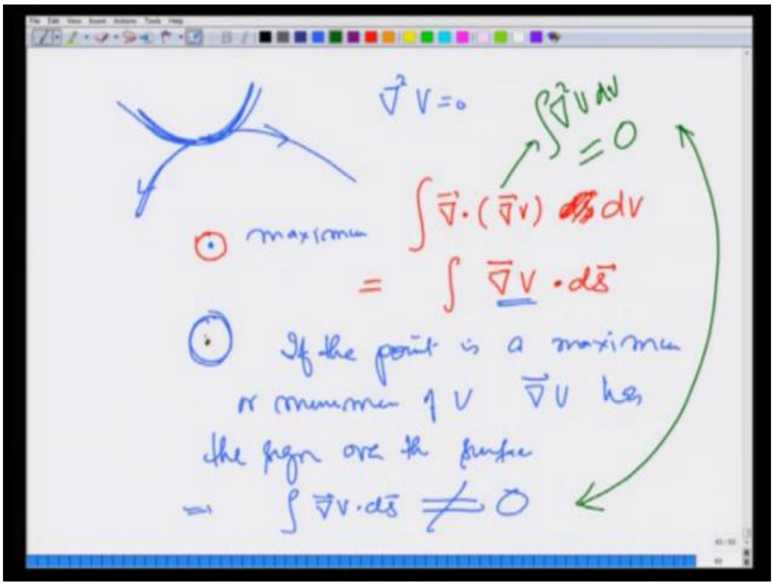

If there is no minimum or maximum; that means the potential is going through a maximum on one side and minimum from the other side. If I put a charge here, it may be stable in this direction from which it is hitting a minimum, on the other side, it will start moving away and this is a consequence of del square V being 0. Mathematical way of seeing this is as follows, suppose there is a point, where there is a maximum.

Then, if I take a very small sphere around it, if I take a very small sphere around it and calculate divergence of grad V over the surface, this is going to be equal to by Divergence theorem, divergence of this grad V over this volume. This is going to be by Divergence theorem, grad of V dot d s over the surface. Since, this is the maximum or the other way, if it is a minimum, then grad of V is going to have the same sign over the entire surface.

So, if the point is a maximum or minimum of V, then grad V has the same sign over the surface and this implies integration grad V dot d s is not equal to 0. On the other hand, look at this, this is del square V, d v and this is 0. So, I am hitting a contradiction and if I am in a contradictory stage, that means, my initial assumption that either I am at maximum or minimum is wrong. So, they can be no maximum or minimum, physical way of understanding this is as follows.

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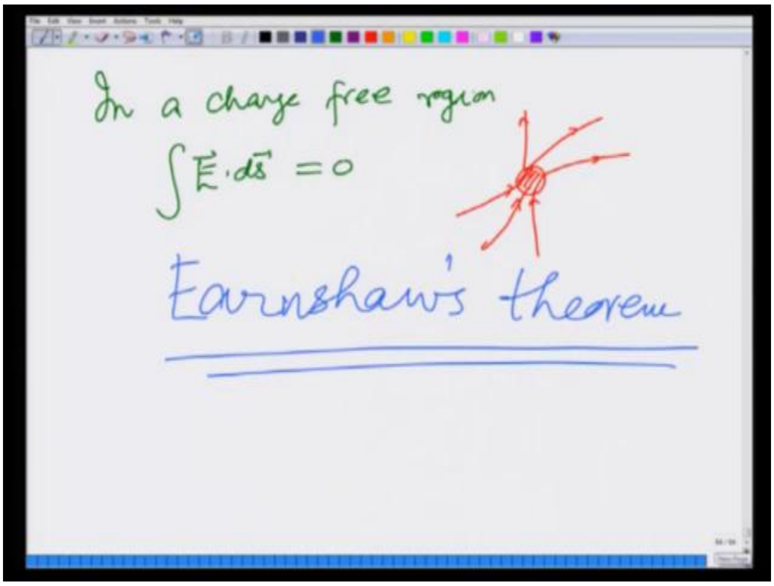


In a charge free region, Gauss's law tells me that E dot d s is equal to 0; that means, in a charge free region, if there is a point, where some electric field lines are coming in, some have to move out. And therefore, a charge put here can never being equilibrium, because whatever lines are coming in they are also going out. So, a charge will tend to move in the direction, where the field lines are going out. This is equivalent to saying del square V 0 and all that is equivalent. This is known by the way as Earns haw's theorem that in a charge free region a charge cannot have an equilibrium point.

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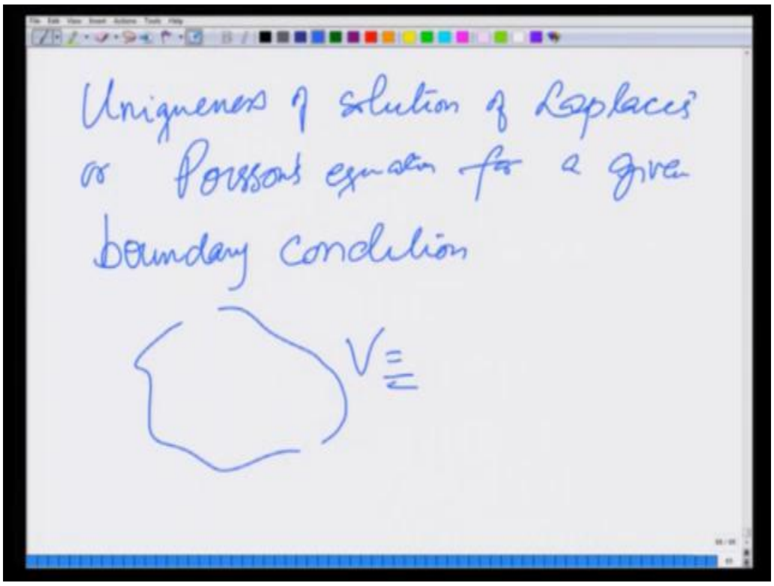


Now, let us go to uniqueness of solution, uniqueness of solution of Laplace's or Poisson's equation for a given boundary condition. What we mean by given boundary condition means that, we have specified that on a given boundary, what is the potential, it is specified.

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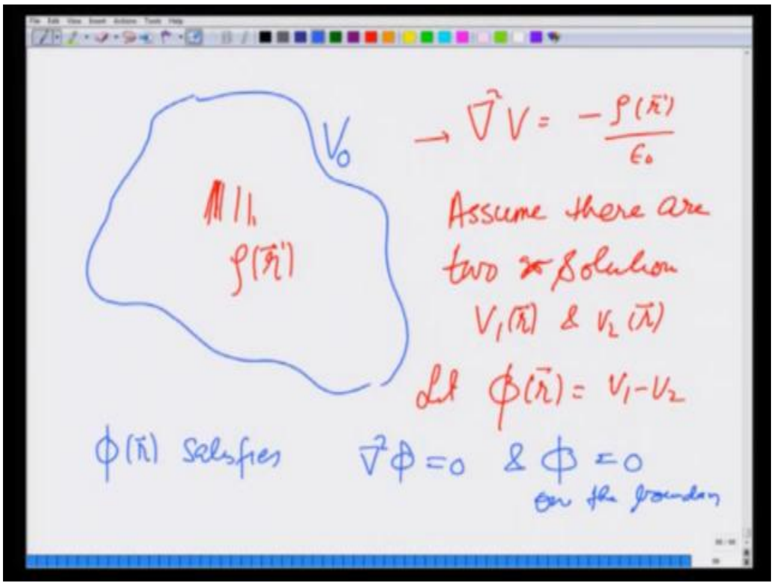


So, let us take a close boundary, it could be at infinity and we have given potential at the boundary, let us call it V naught and let there be some charge inside this boundary given by rho r prime. So, you have given the del square V is equal to minus rho r a given at that r over Epsilon 0. Let us assume there are two solutions V 1 r and V 2 r, both satisfying this equation and both satisfying the same boundary conditions.

And let a function phi r V equal to V 1 minus V 2, then phi r satisfies del square phi is equal to 0 and phi is equal to 0 on the boundary. Why because, phi is the difference between the two potentials where both satisfies the same boundary condition and both satisfies the same Poisson's equation.

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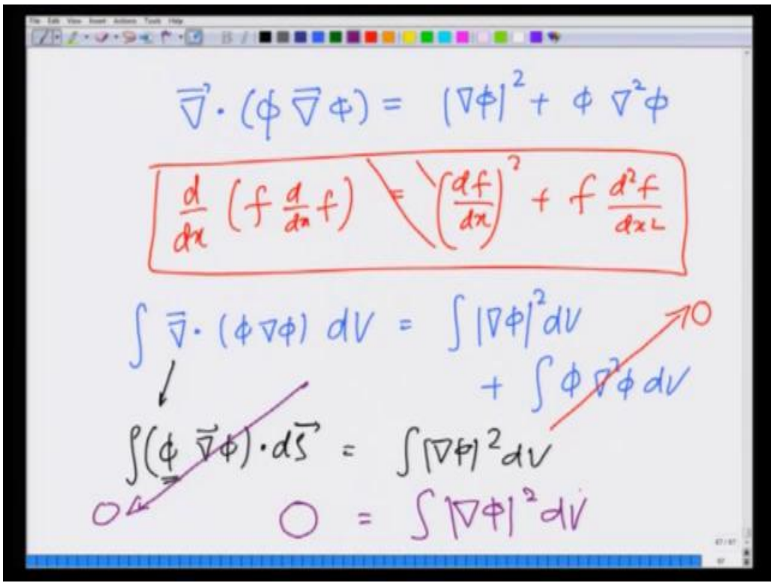


Let us now look at divergence of phi, grad of phi, which you can easily show by taking components is equal to grad phi square plus phi, grad square phi. It is something similar to just to facilitate it, d by d x of a function f d by d x f is going to be equal to d f by d x square plus f d 2 f by d x square. It is similar to that and you can by taking components you can show that, but now I am going to use this to show the uniqueness.

Let us integrate both sides this is divergence of phi, grad phi. So, I am going to integrate this over the volume, this is going to be equal to grad phi mod square over the volume plus phi del square phi over the volume. However, we have seen the del square phi 0, so this terms drops out. On the left hand side, I am going to get by Divergence theorem integration phi, grad phi dot d s over the surface is equal to grad phi square d v. But, we have all ready seen that phi 0 on the surface, and therefore this term also dropped out, as a consequence, what I see is 0 equals grad phi square d v.

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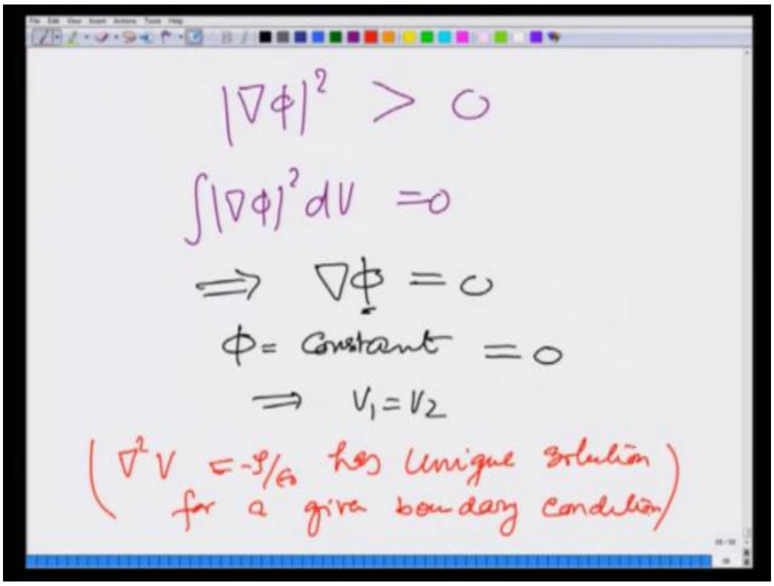


However, grad phi square is always positive is square. So, grad phi square d V is equal to 0, immediately tells you that grad phi is 0, gradient or the derivative of phi 0. So, phi is equal to constant. At most V 1 and V 2 can differ by constant, but since they are the same on the boundary this constant better be 0, so this implies V 1 equals V 2. What that tells me is that and given a charge distribution, the potential, the answer given by solving Poisson’s equations is unique.

They cannot be two different potential satisfying the same Poisson’s equation and same boundary conditions and rho equals 0 is just special case and therefore, Laplace equation also given boundary conditions, gives me the same answer. So, del square V equals rho over Epsilon 0 has unique solution for a given boundary condition.

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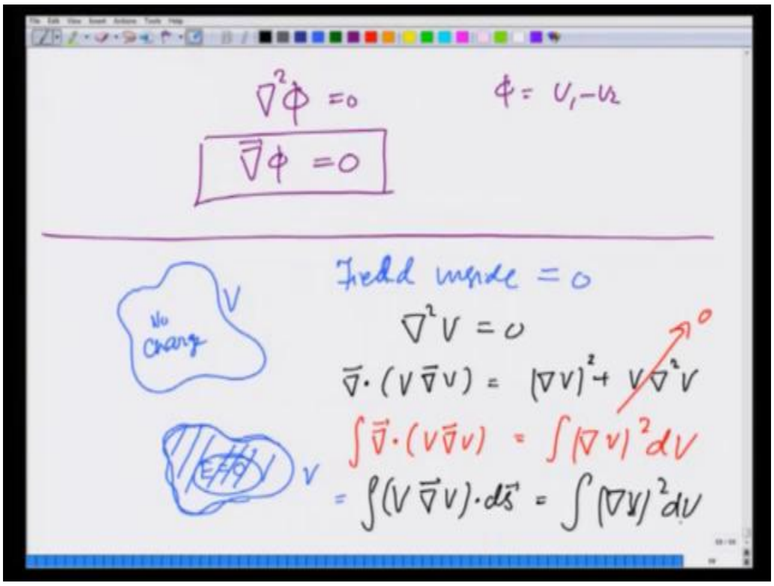


So, let us see how we arrived at this, we arrived at this by looking at a divergence of the difference, where phi is V 1 minus V 2 and this, we used to get that grad phi must be 0. Related physical phenomena is that, if I have a close surface with no charge, no charge inside and V same throughout the boundary, then field inside is equal to 0. This is pretty much like, if I have a metallic sphere, we consider earlier.

But, now let me take a metallic any shape of a metal and I give it some potential that this entire surface becomes an equipotential, because metals, they cannot be any difference between two parts. Otherwise, charge is flow from one side to the other and the field inside the metal is 0, so this is very similar to that that if I create a equipotential, a close volume, who's surface all is at the same potential, then the field inside will be 0. Let us give that before using Divergence theorem.

Now, del square V inside is going to be 0, because there is no charge, again let us take divergence of V, grad of V, which will be grad of V square plus V Laplace in a V. But, since there is no charge inside this fellow is 0. Let us now take volume integrals, divergence of V, grad V is equal to grad of V mod square d over the volume. The left hand side, I can write as integral V, grad V, dotted with surface is equal to integral grad V square over the volume, please do not confuse between this V and this V.

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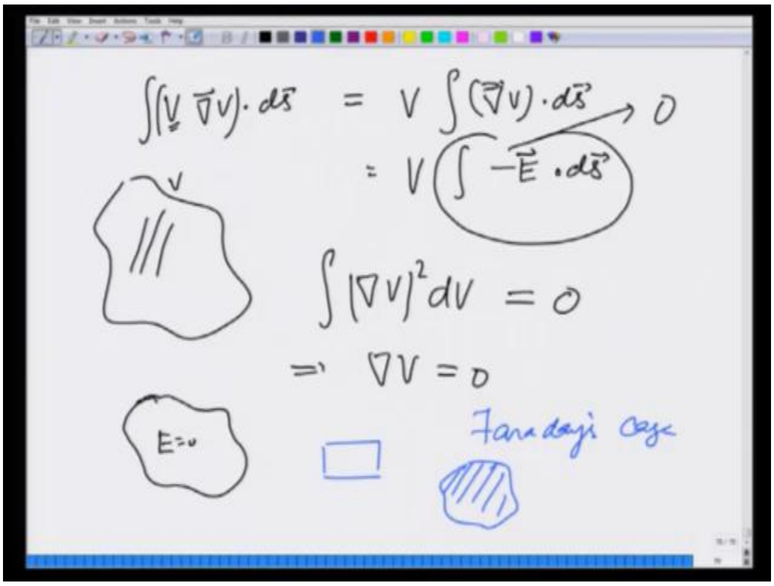


Now, if you look at V, grad V over the surface, we are already saying that the surface has the same potential. Therefore, this v does not have any over the surface, I can write this as, I can take this V out, then it becomes integral grad of V dot d s. What is grad of V, this is V minus E dot d s. Since, there is no charge inside by Gauss's law, this term is 0 and therefore, left inside vanishes and I interpreting integral grad of V mod square d v is equal to and 0, this means grad V must be 0 inside.

So, if I take something with constant potential surface, close surface constant potential over the surface, then field inside will 0. As an example you can do this experiment at home, you take a metallic tiffin box and put you mobile inside, if you call that mobile the call will not go through, because the field inside is 0, everything shielded, the waves cannot penetrate that metallic close surface.

So, this is also used to make something called a Faraday's cage, you take a all most closed metallic surface, all this made out of mesh and put your instruments inside and there not destroyed by any electrical signals from outside. Because, that electrical and no matter what you do outside, the field inside is going to be 0, because for a metallic mesh or metallic surface, the potential all throughout is constant.