Introduction to Electromagnetism
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Lecture - 16
Laplace's and Poisson's Equations for Electrostatic Potential

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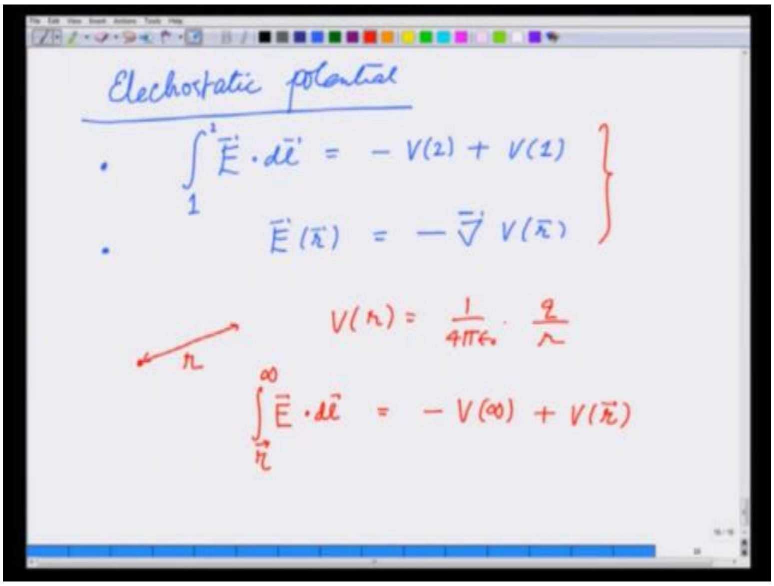

In the previous lecture, we defined electrostatic potential. And what we saw was that E dot dl was equal to minus V 2. We write integral from 1 to 2 – V 2 plus V 1. Its differential form was that electric field; it was equal to minus the gradient of electric potential. Let us see how do we use these equations to get electrostatic potential for a unit charge. You already know the expression that, if I have a unit charge at the origin, then at a distance r from it, potential is given as 1 over 4 pi epsilon 0 q over r. Let us get this using these definitions. So, if I take integral E dot dl going from a point r; let us say r vector to infinity; this would be equal to minus V at infinity plus V at r.

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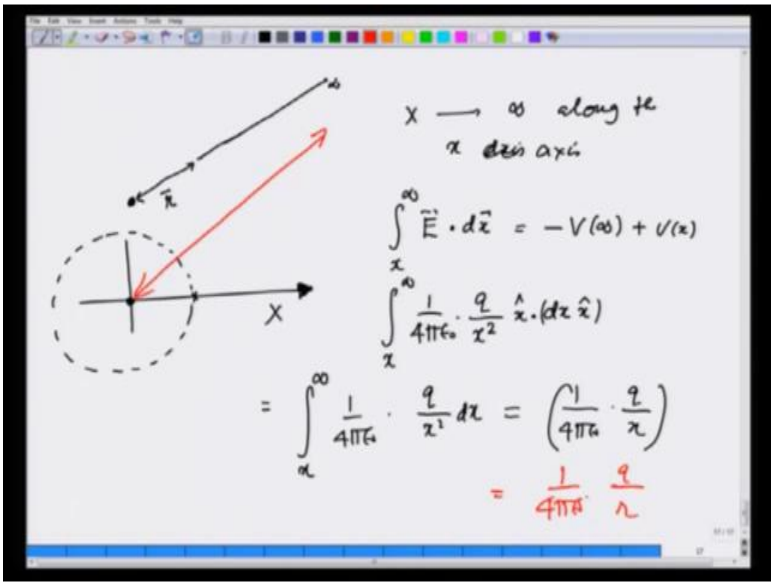

So, now, let me make a picture. This is my point charge and I am going from a distance vector r all the way up to infinity; and I will go along this radial line. If you are not comfortable with radial coordinates, let us say I move along the x-axis. I will talk about the coordinates in next couple of lectures, because just to review spherical and cylindrical coordinates. So, I am moving from a point x to infinity along the x-axis. And then if once I calculate at x, since everything is spherically symmetric, everywhere around at the same distance, the potential would be the same. So, I am calculating integral E dot dx going from distance x to infinity. And this will be given as minus V infinity plus V at x. This I can write as going from x to infinity; E is 1 over 4 pi epsilon 0 q over x square – x unit vector dot dx x unit vector, which is equal to x to infinity; 1 over 4 pi epsilon 0 q over x square dx. And this you see right over here is 1 over 4 pi epsilon 0 q over x; which for a distance r, I can write as q over r. That is by moving radially. I can always take this radius to be along the x-axis.

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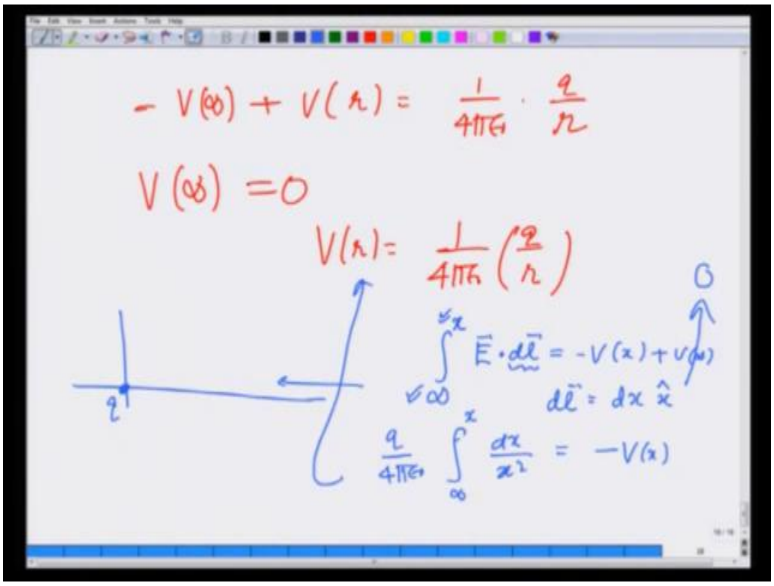

So, what we learn is that, V at infinity plus V at a point r is equal to 1 over 4 pi epsilon 0 q over r. As I said earlier, it is defined only up to a constant. So, I choose a reference point, so that V at infinity we take to be 0. If I take V at infinity to be 0, I define V r as 1 over 4 pi epsilon 0 q over r. And what it means is if I take a charge; bring it to point r from infinity; I will have to do this much work. Sometimes while doing that integral, people get confused.

So, let me just do that also. If I have a charge q at the origin and I am moving bringing a charge from infinity; again, I will have… Now, I am moving from x equal to infinity to a point x E dot dl, which will be equal to V at x with a minus sign plus V at infinity, because point 1 is infinity. I will still take dl to be dx along x, because that I am moving in from infinity to x is actually covered by the limits. Therefore, I do not have to put a minus sign; dl equals minus dx x here. You still take it to be dx x. I am integrating over x. And since x is integrated from infinity to x, that movement in is already taken care of. And this is nothing but again q over 4 pi epsilon 0 dx over x square infinity to x, which is equal to minus V x taking V infinity to be 0. And this gives you the same answer as this. So, this is the potential due to a point charge q at a distance r from it.

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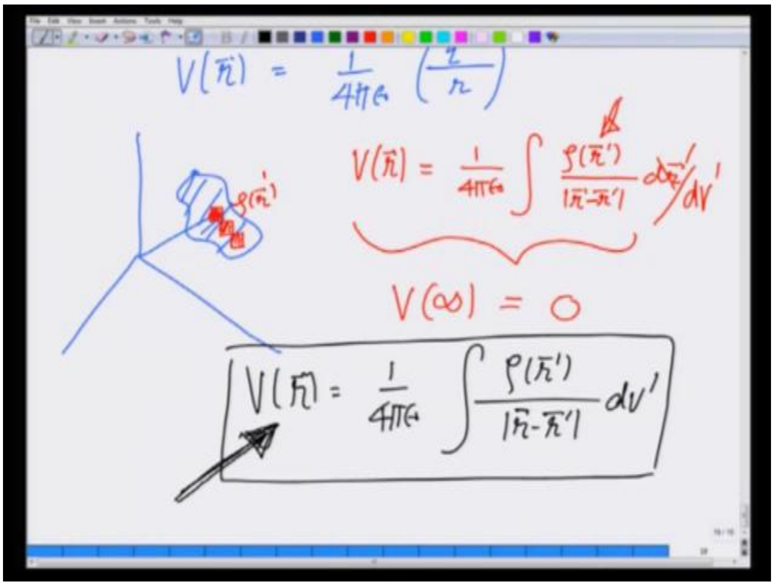

So, we have calculated potential due to a point charge q at a distance r from it. What about a charge distribution? Suppose I am given a charge distribution, so that the density is rho r prime. As we saw in the case of electric field, electric field is superimposed; it follows the principle of superposition. So, if I calculate the work done in that electric field; that work done can also superimposed. So, the potential at point r is going to be potential due to a charge out here added to potential due to charge nearby, potential due to charge nearby. And therefore, this can be written as 1 over 4 pi epsilon 0 integration rho of r prime over r minus r prime dr prime. This is d; I keep writing it like this; this is actually the volume integral dV prime.

And, what this formula is obtained under the condition of that, V at infinity; we are taking to be 0. So, this is in free space. So, in free space, if I have a charge distribution, which has this charge distribution rho r prime; then the potential at r is going to be given by this formula – V r equals 1 over 4 pi epsilon 0 rho r prime over r minus r prime dV prime. Notice that, this is simpler than the corresponding formula for the electric field; where, we had a vector out here. Adding vectors or calculating components is much more… Calculating three components is more difficult than calculating only one quantity. And then if I take the gradient of this, electric field would come out from it. So, we prefer to work with potential rather than electric field. And keep in mind that, this formula is in free space.

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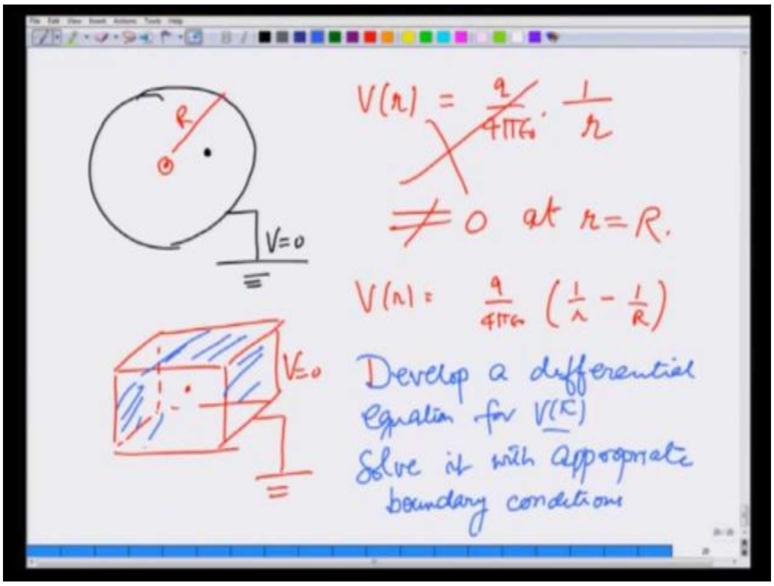

And therefore, now we consider again a situation I took earlier. If I had say a metallic sphere and I grounded it; by grounding, we mean that we made potential equal to 0. Then, if I put a charge somewhere near the surface or even at the centre, and I wrote V r is equal to you q over 4 pi epsilon 0 1 over r. This will not be correct, because this does not give you 0 at the radius at r equals R. You can say ((Refer Time: 08:36)) the way we can define it is that, V r is equal to q over 4 pi epsilon 0 1 over r minus 1 over R. That would be fine.

But, what if I take a different geometry? Suppose I take a box; that means make all the surfaces be equal to 0. Now, put a charge inside. What will be the potential? That will be a difficult question to answer. And therefore, what we need to do is develop. To calculate the potential, we need to develop a differential equation for V r and solve it with appropriate boundary conditions. And that should solve the answer. Suppose I knew the… That should give me the answer. Suppose I knew the differential equation; then I would solve the differential equation with these boundary conditions and that will give me the answer. Of course, I have to prove that, that answer is going to be unique.

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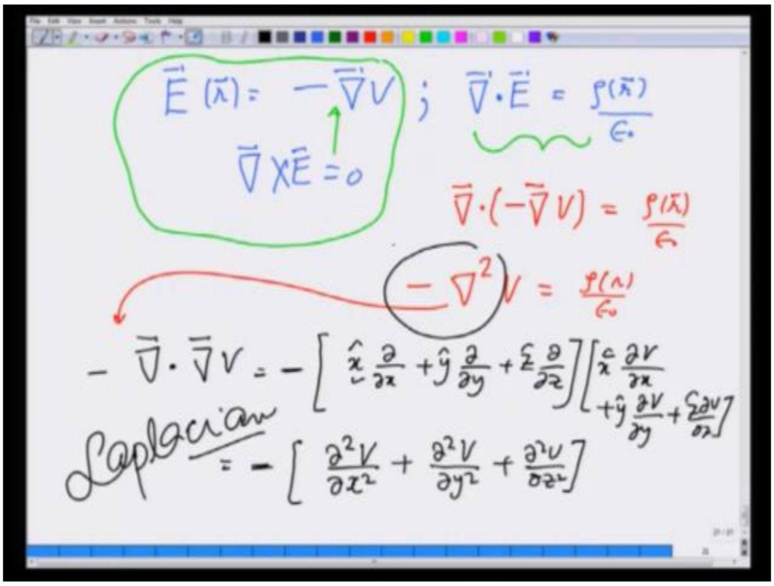

So, let us see what is that differential equation for V. We have E r equals minus grad of V. We also have divergence of E is equal to charge density at that point divided by epsilon 0. We also have curl of E is equal to 0. But, these two equations are one and the same thing because from this follows that, we can define a potential and curl of a gradient is 0. So, the only equation we are left with this is this. And now, we put the formula for V here. It comes out to be rho r over epsilon 0. And that gives me minus – I am going to write this del square – V is equal to rho r over epsilon 0. Let us see what this quantity del square is. So, del dot del V with the minus sign is… The minus sign is there; we do not worry about it – is x d by dx plus y d by dy plus z d by dz acting on x dV by dx plus y dV by dy plus z dV by dz; which is nothing but minus… Now, it comes out to be a scalar function. Second derivative of V with respect to x, because x dot x gives me 1; x dot y gives me 0. So, there are no cross terms – plus d 2 V by dy square plus d 2 V by dz square. This is known as the Laplacian of V. So, this del square term is the Laplacian.

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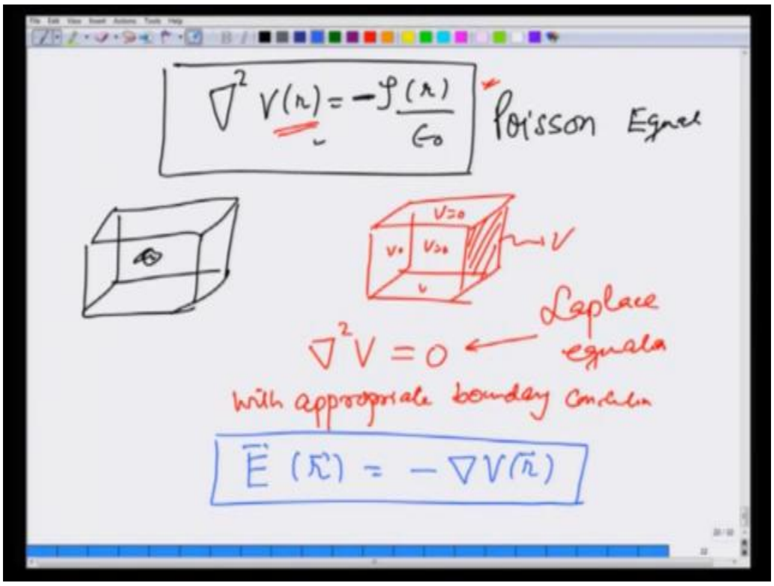

And therefore, the equation – differential equation for V we get is del square of V r is equal to rho r over epsilon 0 with a minus sign in front. This is known as the Poisson equation. So, if I am given some charge density inside a box or some other close volume, some charge density; I solve this equation with the boundary condition and that gives me the answer for v. On the other hand, it could so happen – I may have a situation for example, if I take this box; put this surface at some potential v; and I put all the other surfaces at 0 potential; I will still have electric field inside, because electric field will be coming out of this line. What is the potential inside? Then, I will be solving… There is no charge density inside; delta square V equal to 0 with appropriate boundary conditions. And this is known as Laplace equation.

So, the electrostatic potential either satisfies the Poisson's equation if there is charge density or Laplace's equation or you can just write one equation if rho r equals 0; it goes over to the Laplace's equation. And then solve the appropriate boundary conditions and you get your answer. So, in the future, now on, one would like to solve for V r, because this is easier equation to solve and then get the electric field E r as minus grad of V r. Of course, we know once we solve that for V r, what the interpretation is difference between two V r's is actually the work done in a person taking a unit charge from that point to the next point.