Introduction to Electromagnetism
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Lecture - 13
Examples of Application of the Divergence and Stokes' theorems

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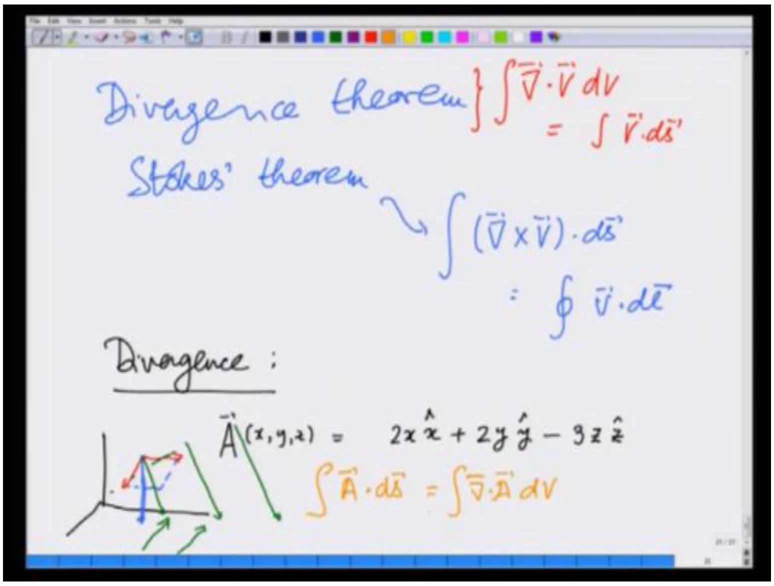

We have learnt about Divergence theorem and Stokes theorem. This relates Divergence of a vector quantity or even better it is volume integral to the surface integral of that vector quantity. Stokes theorem relates the curl of a vector quantity or it is integral over an area to it is line integral over a closed curve and this is over a closed volume or closed surface. In this lecture, we are going to look at two or three examples, so that you get familiar with these concepts.

So, let us first take Divergence theorem, I will work in terms of Cartesian coordinates, and then give you problems for other coordinate systems. Let us take a vector field A, which is given as A x, y, z is equal to 2 x x plus 2 y y minus 3 z z. So, you can see that at any given point, it has components all three components x, y and z. Let us look at a point somewhere here, it has a component in x and y direction, x, y direction which are equal and in the z direction, it has a component which is one and half times, the other three components.

So, you can see that the field is, I make this vector is like this and as you go farther and farther out, it is going to be bigger and bigger and you can make this on all different points. If you are on the negative side, negative, it is going to be, z component is going to be positive. So, you can see that this is kind of diverging, converging or whatever. Now, let us apply Divergence theorem to this. So, what we want to show is that A dot d s is going to be Divergence of A dot the volume element integral.

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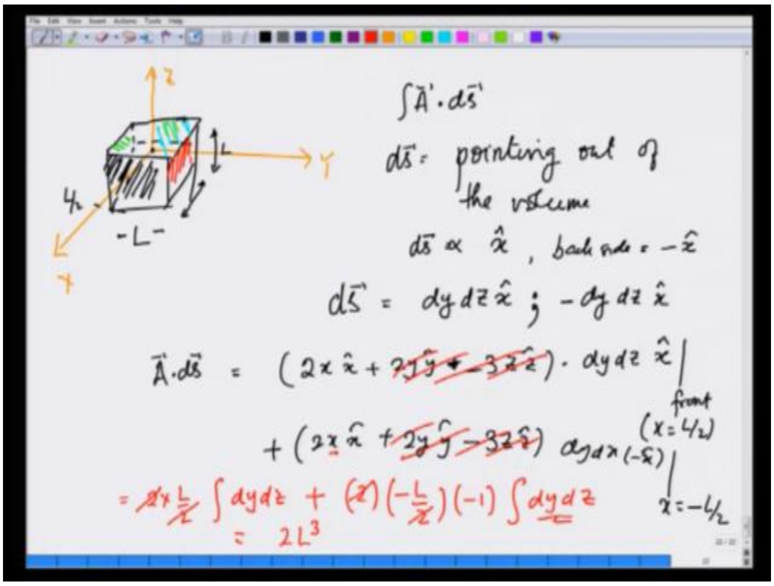

For convenience, I am going to choose a box of size 1, which is centered at the origin, this is a centre. So, this is box of size or let us take a box of size L, it does not a matter. A cube of size L, so that the front surface here this is x direction, this is y direction, this is z direction. The front surface here is at a distance L by 2 from the origin, the back surface will be also at L by 2, the front surface here on the y direction is also at L by 2, the back surface will be at minus L by 2.

And similarly on the z side, the lower surface at minus L by 2 and the upper surface this one is plus L by 2. Let us first calculate the integral A dot d s. Now, when we discuss Divergence theorem, remember d s is taken to be pointing out of the volume. So, on the front surface shown by black, the d s is going to be in the direction of x unit vector.

On at the backside is going to be in the direction minus x and we have already seen that in the x direction d s is d y, d z, x. On the backside is going to be d y, d z, x with a minus sign. So, of for these two surfaces, let us calculate A dot d s, A is nothing but 2 x in x direction plus 2 y in y direction minus 3 z in z direction dot d s will be d y, d z in x direction. For the front surface, where x is equal to L by 2 plus 2 x, x plus 2 y, y minus 3 z, z d y, d z times minus x at x equals minus L by 2; that is a back surface.

So, when you take the dot product, these terms are not going to contribute. So, I am not going to worry about them, x is L by 2. So, what you going to get is 2 times L by 2 times d y, d z, which is going to be nothing but the area of the surface plus 2 times minus L by 2. Because, the back surface x is minus L by 2 and x and minus x give you another minus sign, integral d y, d z, which is also the area, this is nothing but the area of the front surface. And you can immediately see what it gives you this 2 cancels, this 2 cancels and you end up getting 2 L cubed. So, the front surface and the back surface area integral give you 2 L cubed.

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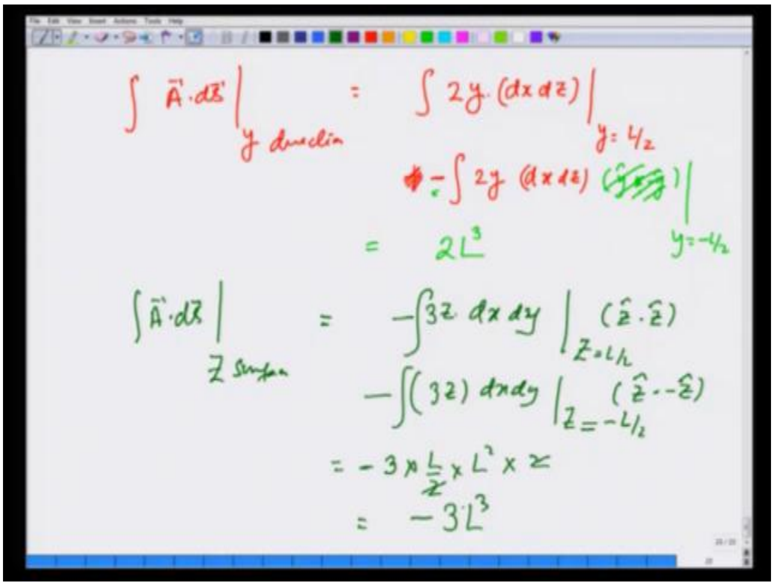

Let us look at the y direction, you have integral for the front and back surface, A dot d s; for the y direction. This is going to be integral 2 y, d x, d z at y equals L by 2 plus 2 y d x d z. Now, this plus going to replace by minus, because the unit vector product out here is going to be y times minus y for the back surface. So, y equals minus L by 2, this is dot product. So, this gave you a minus sign here.

This again will contribute 2 L cubed and how about then finally, A dot d s for the z surfaces, which is nothing but minus 3 z and you are going to have d x, d y. And this is going to get at z equals L by 2 is going to be z dot z minus 3 z d y at z equals minus L by

2 is going to be z dot minus z. So, this gives you 3 times L by 2 times L square times 2, which is with the minus sign here, which is nothing but minus 3 L cubed.

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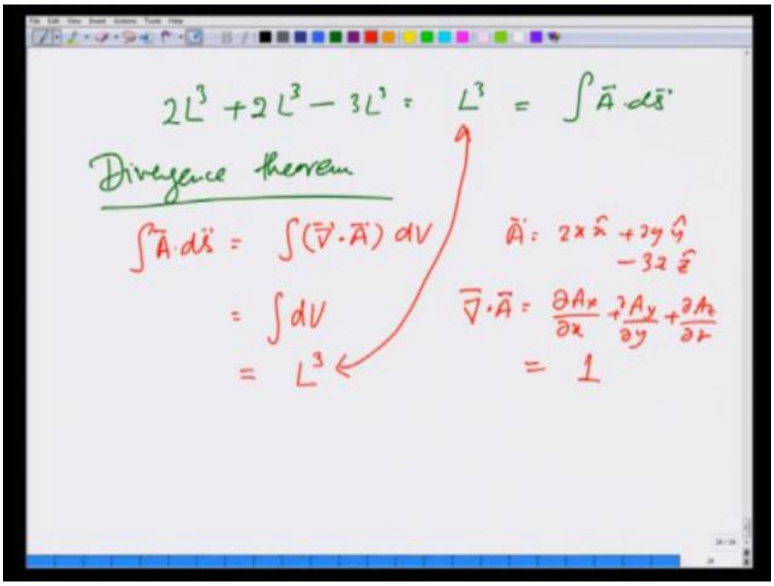

So, if you add all three contributions together, you get 2 L cubed plus 2 L cubed minus 3 L cubed, which is L cubed, which is equal to integration, A dot d s. Let us now apply Divergence theorem and see, if this gives the same answer. According to Divergence theorem, A dot d s is going to be Divergence of A integrated over the volume and A mean 2 x in the x direction plus 2 y in the y direction minus 3 z in the z direction. You can see that Divergence of A, which is partial A x partial x plus partial A y partial y plus partial A z partial z is going to be 1.

This is constant, so I can take this out it is going to be only d v, which is where the d v is the volume integral over the cube, which is L cubed, it gives me the same answer. You can also see the power of Divergence theorem. Whereas, in calculating A dot d s I had to do integral over six surfaces and got an answer of L cubed. In Divergence theorem, I got this answer in two or three lines. So, this is how you can use it powerfully to your advantage.

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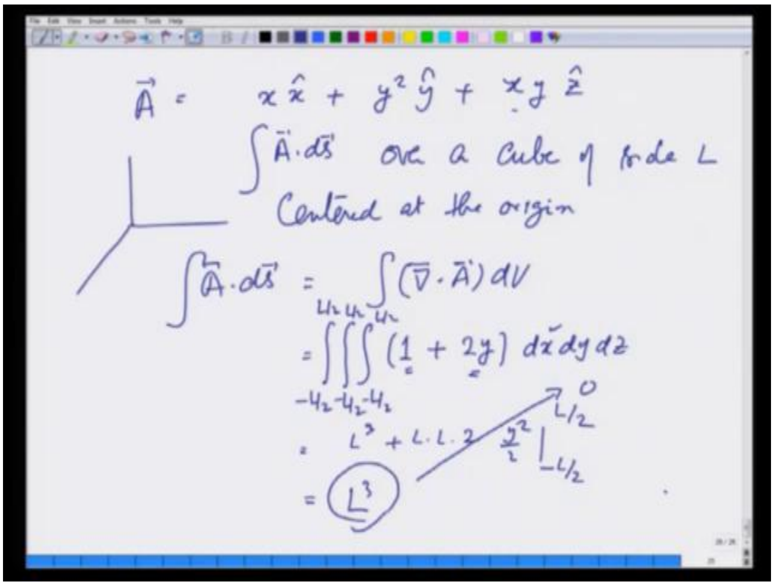

Next, again I take an example for Divergence theorem, where I am going to take A to be equal to x x plus y square y plus x, y, z and again, I want to calculate, it is integral A dot d s over a cube of side L, centered at the origin. So, A dot d s by Divergence theorem is nothing but Divergence of A, d v and you can immediately see Divergence of A is derivative of x component with respect to x which gives me 1 plus derivative of y with respect to y, which gives me 2 y plus derivative of z component z.

Since, that is x y, it does not contribute and I have d x, d y, d z, d x integral, let me write them separately, d x integral is from minus L by 2 to L by 2, d y integral is minus L by 2 to L by 2, d z integral is minus L by 2 to L by 2. First term is a constant, so I get immediately L, L and L contribution from x, y and z. And therefore, I get L cubed from the first term plus for the second term, x again gives me L, z again gives me L, there is a 2 and integral of y gives me y square over 2 minus L by 2 to L by 2. So, this equals L cubed and this term vanishes 0, so this is the answer.

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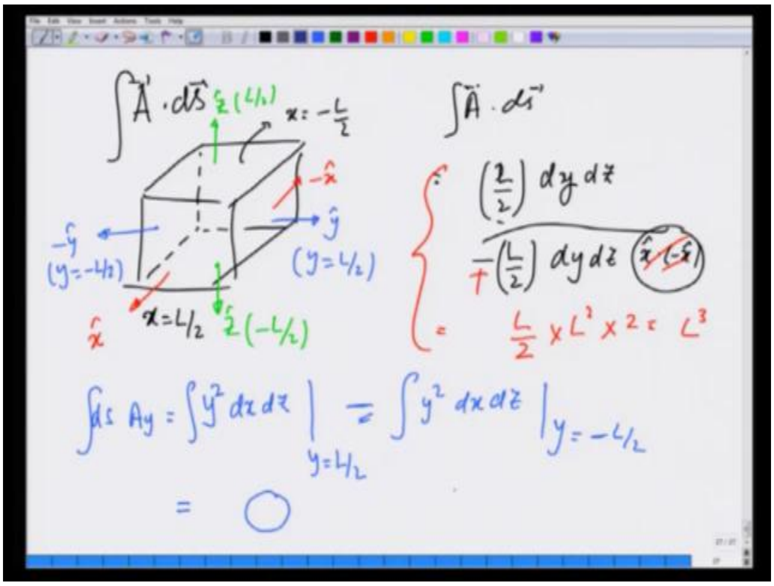

Let us now check the theorem by calculating the surface integral A dot d s explicitly. Remember, now I am doing it over a cube, which is centred at the origin, like the previous example. So, I have x equals L by 2 and backside is x equals minus L by 2 and by 2. The unit vector on the backside is negative x, on the front side is, it is positive x. similarly, on y direction unit vector here is positive y, on this side is minus y at y equals L by 2 and this is at y equals minus L by 2.

And on the top surface, z bottom surface minus unit vector and this is at minus L by 2 and this is at L by 2. So, when I do A dot d s for x equals L by 2, I am going to get L by 2 times d y, d z, because remember A is x. So, L by 2 and for the backside, I am going to get minus L by 2, d y, d z, but then x dot minus x and that will make this sign plus. I can remove this now and this gives me, L by 2 times L square 2 times, so this gives me L cubed. So, this is the x part.

How about the y part, let us do it in blue, y component is y square A y is y square. So, integral d s, A y is going to be y square, d x, d z at y equals L by 2 minus integral y square d x, d z at y equals minus L by 2. This minus sign again arises, because on the backside at y equals minus L by 2, the unit vector is in the minus y direction and this you can see immediately gives me 0.

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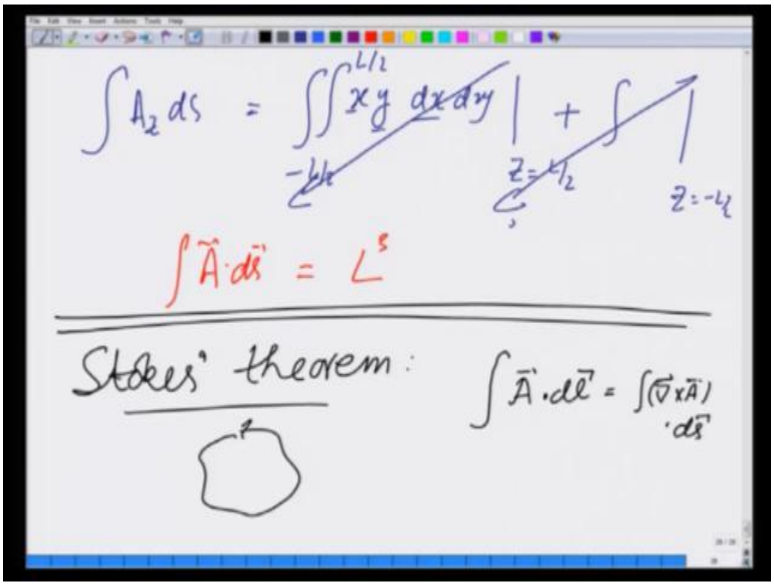

Third term A z, d s gives me x y, d x, d y minus L by 2 to L by 2 at z equals L by 2 plus similar term at z equals minus L by 2, but you see, these integrals x and y integrals are odd and integral is being performed from minus L by 2 to plus L by 2. So, they contribute 0. So, you can see immediately that A dot d s is indeed l cubed and that confirms this Divergence theorem.

Next, I am going to take an example from Stokes theorem, a very simple example, remember what did Stokes theorem say, it said that, if there is vector field A dot d l around a closed path is going to be curl of A dot d s. So, let us take a vector field A, x, y, z, a very simple one, I am going to take this to be 2 y x.

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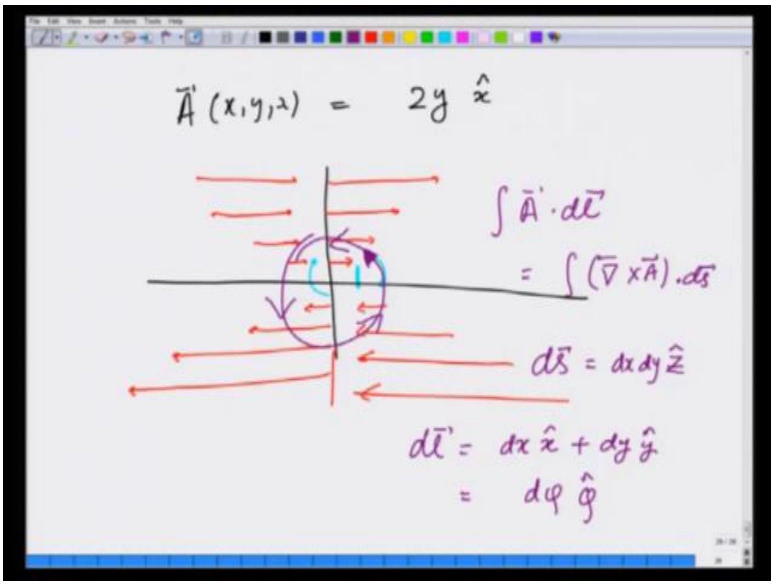

Let us see how it looks, if I plot it is direction is always in the x direction and at any point, it is magnitude is according to y and it keeps on increasing as y increases. On the negative y side, it is going to be like this and you can see that, so all these points is going to be bigger, it depends only on the y coordinate. By definition of curl, you can see this field is curling around or as I said as a test of curl, if I put a small stick in this field and imagine these arrows indicate the velocity of a fluid, you will see that this will tend to rotate this way. So, you can that curl is in the negative z direction.

None the less, let us check Stokes theorem by taking a unit circle and calculate the integral over this unit circle going counter clockwise. And if I do this integral over a unit circle, this should equal curl of A dot d s. For this unit circle, since this in the x y plane, d s is going to be in the z direction, and then I am going to have d x, d y. The d l, since I am going counter clockwise, d l in the Cartesian coordinates is going to be d x x plus d y y. I can also write this as, since this is the unit circle, very simply as just length is r equals 1, so d phi in phi direction, both indicate the same thing.

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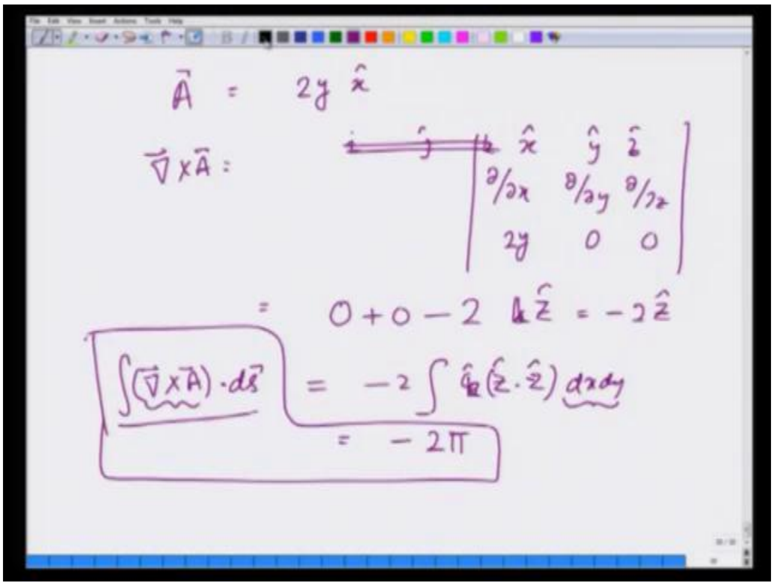

So, let us calculate curl of A, A is given to be 2 y x curl of A is i j, I have been using unit vectors x, y and z; partial x, partial y, partial z, x direction is y, 0 0; this which is equal to x component will give me 0, y component will give me 0 and z component is minus 2. So, curl of A is minus 2 z, so how about this curl of A dot d s, since curl of A is constant, I can take this out from minus 2, I have k z unit vector dot z unit vector, because in the x y plane, they are vector is in the positive z direction.

Because, I am going counter clockwise d x, d y and this gives me the area of the unit circle. And therefore, this is minus pi r square minus 2 pi; that is the integral curl of A dot d s.

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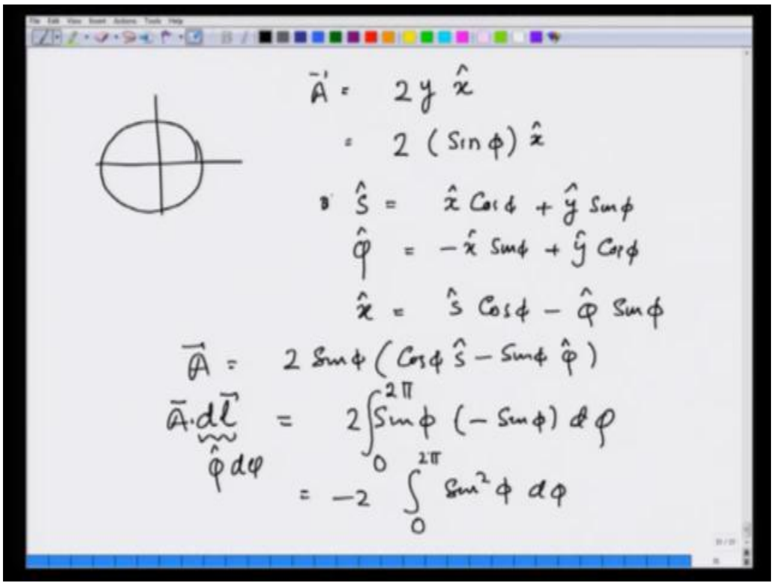

Let us know calculate the line integral, for the line integral, I am doing it over a unit circle, A is given to be 2 y x, which I can write that as 2 y is nothing but sine phi x unit vector. Now, we have already seen that S unit vector in cylindrical coordinates is nothing but x cosine phi plus y sine phi and phi unit vector is equal to minus x sine phi plus y unit vector cosine phi. And therefore, I can write x unit vector as S cos phi minus phi unit vector sine phi.

So, A, I can write as 2 sine phi cosine phi S minus sine phi, phi A dot d l, we have already seen d l is nothing but phi unit vector times d phi, let me check, if I did that, yes I did that indeed here. So, A dot d l is going to be 2 sine phi times minus sine phi d phi integral from 0 to 2 pi, which is minus 2 integral sine square phi d phi 0 to phi, 2 pi and this indeed gives you minus 2 pi.