Introduction to Electromagnetism
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Lecture - 12
Line surface area and volume elements
in Spherical Polar Coordinates

In the previous lecture, we looked at the Line surface area and volume element in Cartesian and cylindrical coordinates. In this lecture, we will focus on a spherical coordinate system.

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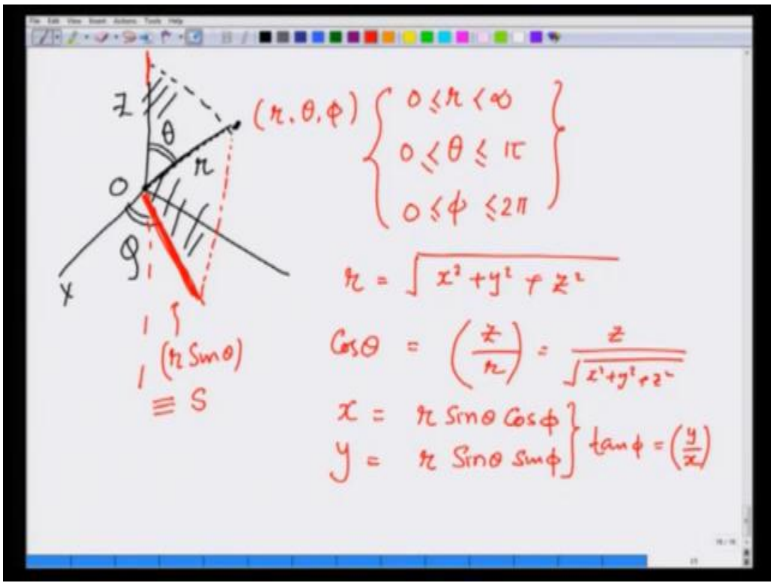

In a spherical coordinate system, a point is specified by it is distance from the origin, which I will call r and if I make the plane containing this point and the z axis. So, this is r, the distance from the origin to the point. In this plane, if I measure the angle from the z axis to this line joining origin and the point, this angle is given as theta and the angle in this plane, this entire plane, this entire plane makes from the x axis is known as phi.

So, this point is specified by r theta and phi, to cover the entire range or entire space, we let r go from 0 to infinity, theta varies from 0 to pi. That means, it varies from plus z to minus z axis and then let phi vary from 0 to 2 pi and that covers the entire space. So, these are going to be the ranges of the variables. Relationship between r theta and phi with Cartesian coordinates is going to be r, you can see immediately is nothing but x square plus y square plus z square.

Theta is the angle between the z axis and r, and therefore you can immediately see that cosine of theta is going to be given by z over r or z over square root of x square plus y square plus z square. Third, this r has a projection in the x y plane, which is this red thick line, which is nothing but r sine of theta. And if you have studied the previous lecture well, this r sine theta is nothing but equivalent to cylindrical coordinate S. And therefore, x coordinate is nothing but r sine theta cosine of phi, y coordinate is nothing but r sine of theta, sine of phi and this immediately tells you tangent of phi is exactly like the cylindrical coordinates y over x.

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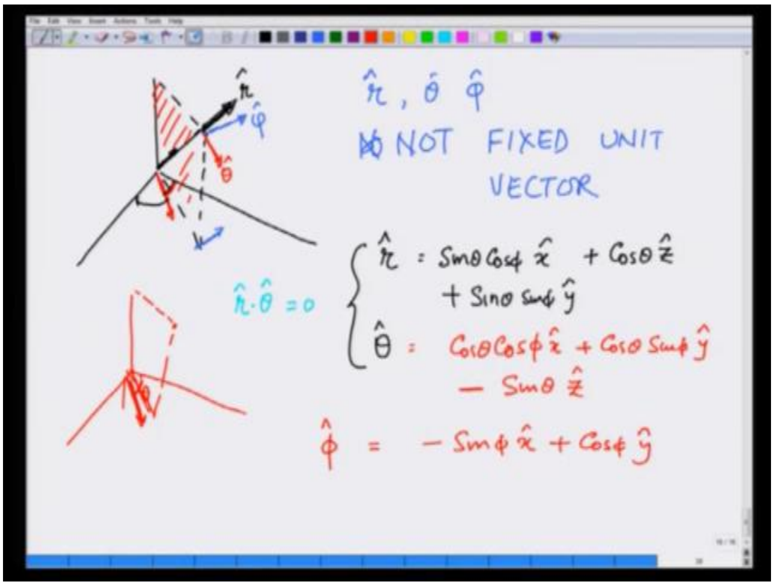

How about the unit vectors? So, I am going to have unit vectors in direction r like this. In direction theta, which is going to be in this plane containing the point on the z axis, it is going to be perpendicular to the r and like this. If I transport it parallel to itself, it is going to be pointing down like this and phi vector is going to be like the cylindrical coordinate phi vector like this, which is perpendicular to r theta and phi.

So, let us write this r unit vector, theta unit vector and phi unit vector and this forms a coordinate system, which is three unit vectors are perpendicular to each other, but none of these vectors r or theta or phi are fix in space. So, they are not fix unit vectors. So, as you change the position, they also change. Let us write them in terms of x y in z unit vectors, r is pointing this way and we have already seen that, if I take the unit vector, it is distance is 1, the z component is nothing but cosine of theta.

So, cosine of theta z plus projection in the x y plane is sine theta, hence it is x component is going to be sine theta cosine phi as we saw in the previous slide. So, x component is sine theta, cosine of phi in the x direction plus y component is going to be sine theta, sine of phi in the y direction, this is r unit vector. How about theta unit vector? Theta unit vector is perpendicular to r, so therefore, it makes an angle and let me make it, show it by red out here or make it separately.

This is my point, theta is pointing down making an angle theta from the x y plane and therefore, in the x y plane, it is component is cosine theta. So, it is going to be cosine of theta and then again that has a component cosine phi in the x direction plus cosine of theta, sine of phi in the y direction. In the z direction, it has negative component. So, it is going to be minus sine of theta times z.

You can already see that, if I take the dot product here, r dot theta, this is 0, this gives you sine theta cos theta sine theta cos theta cosine square phi plus sine square phi minus sine theta and cosine of theta. So, this is going to be 0. So, they are orthogonal to each other, phi unit vector, it is pretty much the same like in the cylindrical coordinates and it is coordinates are going to be therefore, minus sine of phi x plus cosine of phi y. These are the three unit vectors in cylindrical coordinates.

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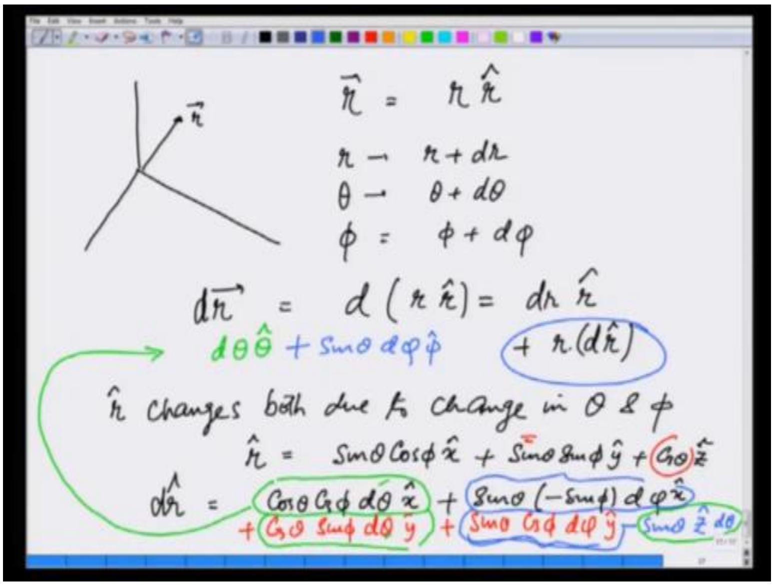

How about line element? First the position vector, position vector r is nothing but r unit vector r; that is the position vector in spherical coordinates. If I am going from one point to the other, I am changing r to r plus d r, I am changing theta to that plus d theta and I am changing phi to phi plus d phi. So, d r is going to be nothing but d of r unit vector r and I said earlier, these are not fixed. So, this is going to be d r in unit vector r direction plus r times change in the r unit vector, r unit vector change comes both r unit vector changes, both due to change in theta and phi.

Remember, what is the r unit vector, r unit vector is nothing but sine theta cos phi x plus sine theta sine phi y plus cos theta z. Therefore, d r unit vector, I can write as cos theta cos phi d theta x plus sine theta minus sine phi d phi x plus the second term cos theta sine phi d theta y plus sine theta cos phi d phi y. This is y plus this change which is minus sine theta z, look at the terms, this one which is and this one.

If I take sine theta out, this becomes minus sine phi x plus cosine phi y and this should be d theta and look at this term encircled in green. And here, if I take d theta common, I get cosine theta cosine phi x plus cosine theta sine phi y minus sine theta z and what is this unit vector, this is theta unit vector. So, I can write this green thing, let me go here, I can write these green things combined together as d theta.

In theta direction and the blue things this one and this one, if I take d phi and sine theta out, you left with minus sine phi x plus cosine phi y which is phi unit vector. So, I can write this as plus sine theta d phi unit vector put these things here.

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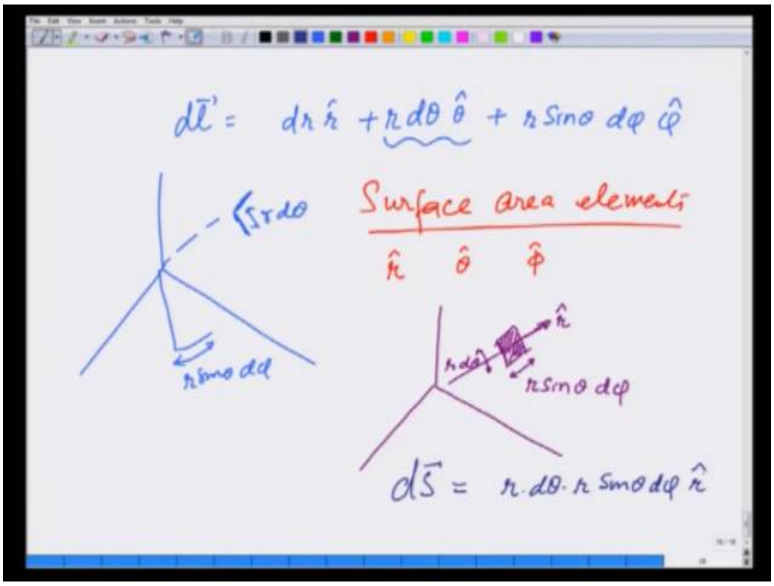

So, what is the line element that I get? I get the line element d l is equal to d r in r direction plus d theta times r in theta direction plus r sine theta d phi in phi direction. That makes perfect sense, because if I look at the coordinate system, if I move in r direction, this distance is d r, if I change theta, this distance becomes r d theta. So, that is this line element and if I change phi, the distance I cover in phi is going to be r sine theta d phi. So, this is a line element.

Next let us look at surface elements, surface area elements perpendicular to r perpendicular to theta and perpendicular to phi let us look at perpendicular to r first. This element is going to look something like this, where this distance, this is unit vector r is perpendicular to r and this distance is going to be distance moved in the x y plane. So, this is going to be r sine theta d phi and this distance is going to be due to change in theta. So, this is going to be r d theta. So, the surface element, I can write in this form is d s is equal to r d theta times r sine theta d phi in r direction, let us go to the next page.

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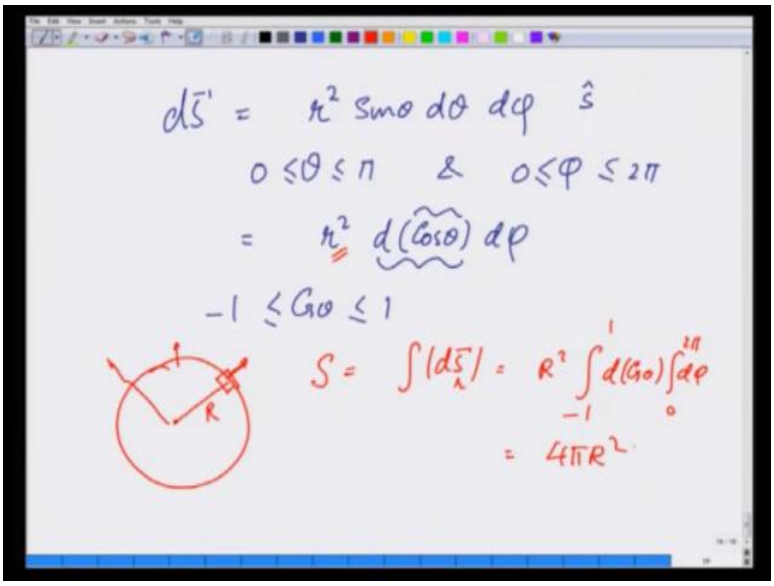

So, d s in r direction is r square sine theta, let me see, if I wrote d theta back there is a d theta here. So, r sine theta d theta d phi and it is in S direction, where theta varies from 0 to pi and phi varies from 0 to 2 phi, 2 pi. Sometimes, this is also written as r square d cosine of theta d phi, where d cosine theta is integral over, you know the now I can take cosine theta as the variable.

So, cosine theta varies from minus 1 to 1, very easy to see that this is correct is, if I take a sphere of radius R, centre dot the origin, an element all area elements are perpendicular to S, because area is perpendicular to the radius. So, the total area is going to be integral magnitude of d s in the r direction which is going to be r is fixed, so R square d cos theta from minus 1 to 1 d phi from 0 to 2 pi and that comes out to be 4 pi R square.

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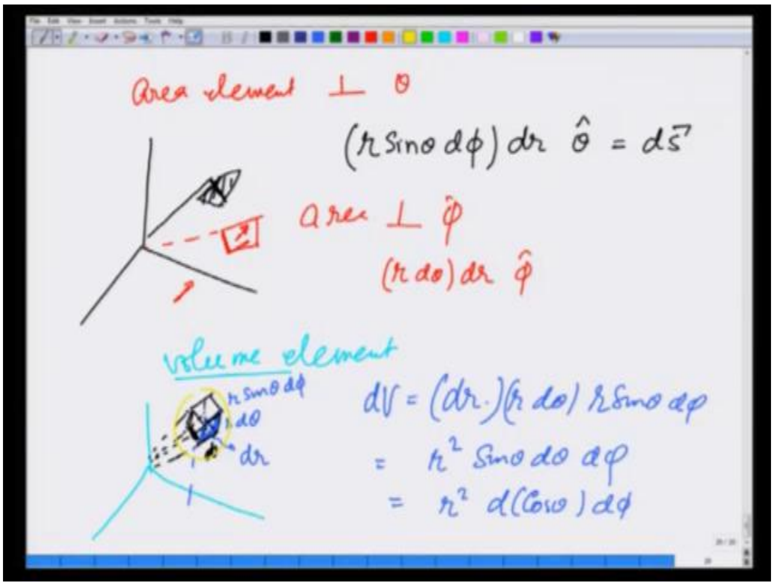

How about area element perpendicular to theta, let us look at that, if I am taking an area element perpendicular to theta, the distances covered are going to be in the r direction and in phi direction. In the phi direction, the distance cover is going to be r sine theta d phi, in the r direction, distance covered is going to be d r and this is going to be area element in S direction.

And area element perpendicular to phi; that is going to be here, perpendicular to phi and that is going to be again using a coordinate systems in r theta and phi will have a length element perpendicular to phi in theta direction, which is going to be r d theta. So, here will be r d theta and 1 in r direction and this unit vector is in the phi direction. So, r d theta times d r in phi direction.

How about the volume element? Let me first make one volume element and you will see, what it looks like, here in phi direction. So, this is in the r direction, this distance this is like projecting like this on to the centre. So, this distance shown by green or shown by red is d r. This distance is moving in theta direction is r d theta and the distance perpendicular to both is going to be r sine theta d phi.

So, these three are the small distances or small height, length and width of the volume element and therefore, the volume element is nothing but d r times r d theta times r sine theta d phi, which is equal to r square sine theta d theta, d phi is sometime also written as r square d cosine of theta d phi. To understand this figure, you have to imagine a bit or look at a corner of the room, and see how you can you know think of that as the origin and how you can think of theta phi an r direction.

So, this is the three line elements area elements and volume elements in it is spherical coordinates. Using these one can derive expressions for curl by the definition of curl, divergence by the definition of divergence in these coordinate systems also. So, we have derived in the 2 lectures, the line elements, area elements and volume elements in Cartesian, cylindrical and spherical coordinate systems. Using these and using the Stokes theorem and divergence theorem or their definition, we can derive the expressions for the gradient divergence and curl also in these coordinate systems.