Introduction to Electromagnetism
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Lecture - 10
Curl of a Field – II & Stokes' Theorem

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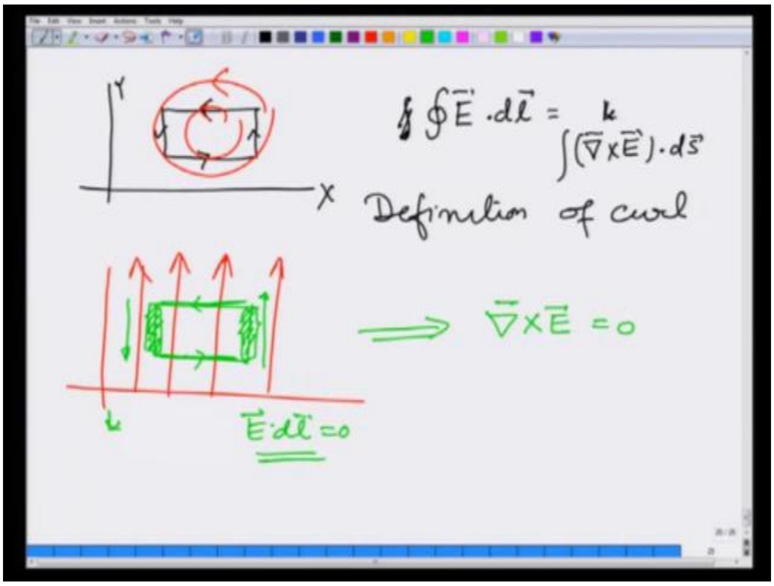

In the previous lecture, we introduced the idea of curl by taking line integral over a path which is rectangular in the x y plane, and we said that, let me write this like this E dot d l over this path. Although, it is very small path, let me write this as an integral was equal to curl of E dot d s, where d s represents this small area, so this it becomes the definition of curl. It is very clear, if there is a vector field that is going around like this, if I take this path, this integral is not going to be zero.

On the other hand, let us look at a vector field which is constant all over and let me take this path here. You will notice that E dot d l over this will give me one value, which will be exactly opposite of it is value over this path. So, going this way I will get one value which is exactly opposite of something going this way. Dot product over this path is zero and the, but the net result is that E dot d l in this case is 0 and this implies, that curl of this vector field is 0 or atleast it is z component is 0.

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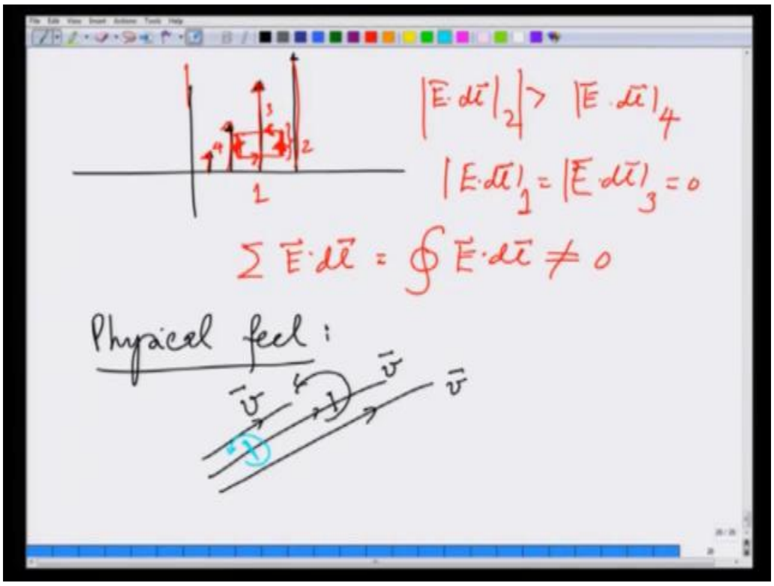

So, very interestingly you notice that, when something swirls around or curls around, it has a curl which is non zero, something which is constant has a zero curl. Let us take another example in the x y plane again. If I have a vector field which is increasing with distance and if I take a path like this in this and again calculate E dot d l. You will notice that on this side, let us name it 1, 2, 3, 4. Contribution on 2 would be larger than the negative contribution on 1, on 4.

Why? Because, as you are going towards the right, the field values increasing, on the other hand on 1 and 3 it is zero. So, E dot d l on 2 is greater in magnitude than E dot d l on 4 and E dot d l on 1 is equal to E dot d l on 3 which is 0. So, again you would notice that if I do E dot d l over the entire path which I will write as the line integral E dot d l is going to be non zero. But, that makes sense, because now you see this field gives you an impression as if it is going to turn around.

It is only when something turns around that as you go away from the center that the speed or the vector increases. All these concepts actually came from fluid dynamics and therefore, a physical feel for curl will arise by the following arguments. Imagine that this vector field that we are talking about is representing the velocity of a fluid flow and put a small stick in it, maybe a toothpick maybe a matchstick. If this tends to turn, so let me make it by different. If I put a small stick and if it tends to turn, then curl is non zero.

If it does not turn, then curl is zero and sense of this turning will also give you the direction of the curl. Again, move your fingers in direction in which it tends to turn, then the thumb gives you the direction of the curl that is a physical feeling of this.

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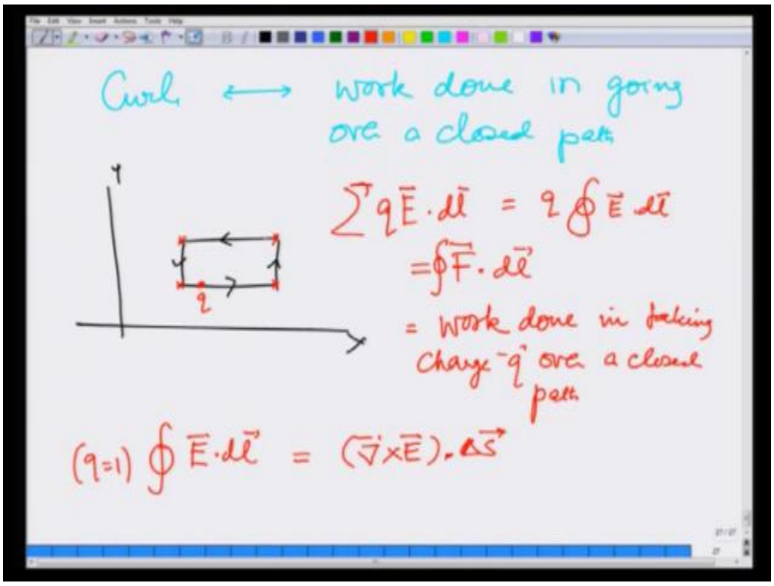

It is also related to… So, curl can also be related to work done in going over a closed path. How is so let us see that. Again go back to x y plane, I will generalize it to 3 or in a minute. Take a charge q and take it around, now E q dot d l summed over which I am writing as q integration E dot d l is nothing but force on the charge dot d l. So, this integral is represents work done in taking charge q over a closed path. So, this we understand E dot d l is work done in taking a unit charge.

Let us write q equals around a closed path and this is related to curl dot delta s. Therefore, if work done is non zero in going around a closed path, a curl is non zero if work done is zero, then curl is zero. Let us see what happens for an electric field. If I have an electric field and if I go from this point to this point and this point to this point, work done maybe the same and we will see that in a minute, what it means.

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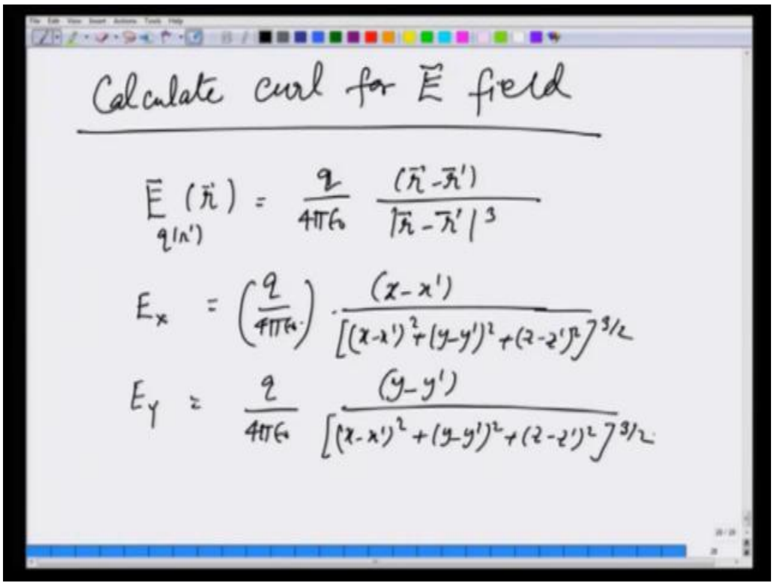

So, let us calculate curl for E field, if I want to calculate curl for E field remember E for a unit charge at r charge at r prime, I have written it earlier. So, I will just write it directly is q over 4 pi epsilon 0 r minus r prime over r minus r prime cubed. So, E x is a constant q over 4 pi epsilon 0 x minus x prime over x minus x prime square plus y minus y prime square plus z minus z prime square raise to 3 by 2.

E y is q over 4 pi epsilon 0 y minus y prime over x minus x prime square plus y minus y prime square plus z minus z prime square raise to 3 by 2. Let us calculate their derivatives.

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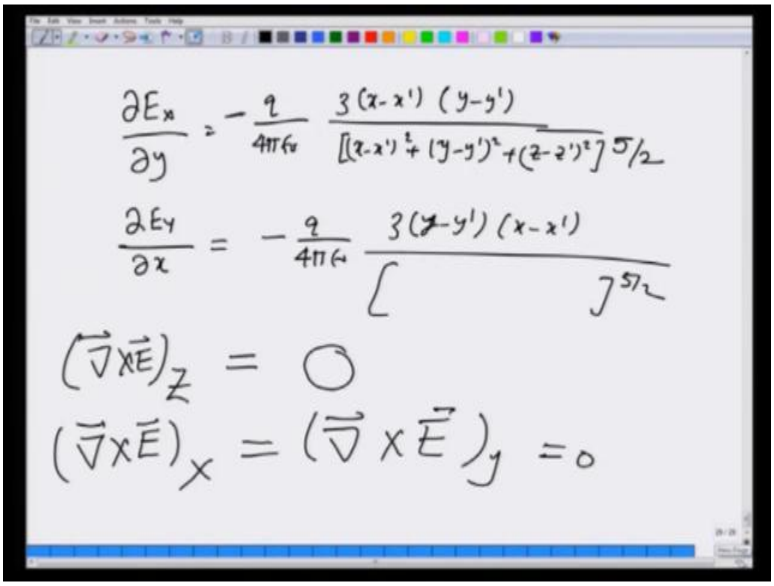

So, partial E x over partial y is going to be q over 4 pi epsilon 0 x minus x prime over x minus x prime square plus y minus y prime square plus z minus z prime square raise to 5 by 2. And her, I am going to have a y minus y prime times 3 with the minus sign. Similarly, E y over E partial x is going to be minus q over 4 pi epsilon 0 3 y minus y prime x minus x prime over this whole thing raise to 5 by 2.

So, curl of E z component comes out to be 0, similarly you can see that curl of E x component is also 0 and so is the y component. You can check that just like I did for other components; that means, in general for a point charge q curl of E is 0.

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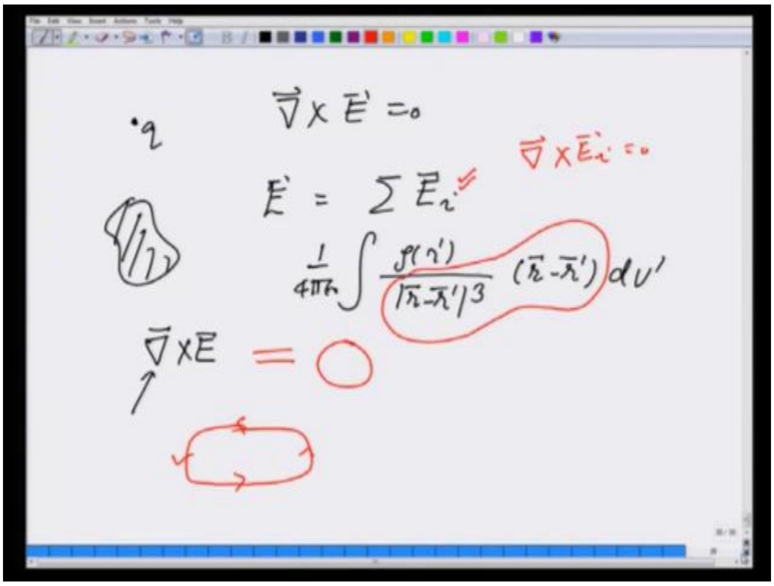

Now, if I have many charges or charge distribution E is nothing but summation of different E's or rho r prime over r minus r prime cubed r minus r prime d v prime 1 over 4 pi epsilon 0. When I take curl which is derivative with respect to x y and z, it is going to act only on this part of the integral and in this case, it gives 0 for individual fields. So, overall you can then see that curl of E comes out to be 0. That means, if I go around any small closed path, network done is going to be zero. How about a larger path? For a larger path, recall the trick we did for diversions.

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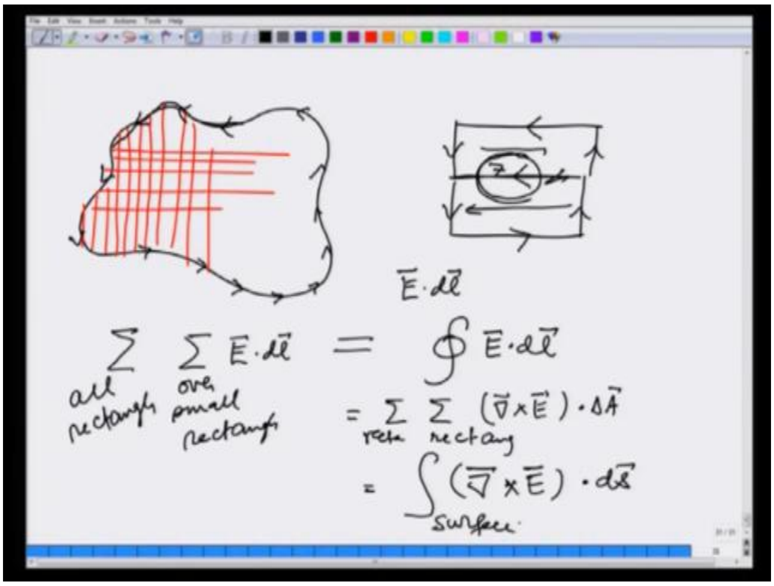

Similarly, here if I take a large path, I can divide this into many, many small rectangles. Let us look at two adjacent rectangles and traverse them counter clockwise. If I do this, you see the E dot d l over this in circle path which is traversed twice once this way, once this way is going to give me zero when I add it up. So, all the adjacent paths are going to give me a zero contribution to the line integral with the result finally, I will have only line integral left which is outside line integral, because these are not adjacent to any other rectangle.

So, summation E dot d l over small rectangles, and then this summed over all rectangles gives me integral E dot d l, where d l is only the outside path. But, this is by definition equal to summation over each rectangle curl of E dot delta A, and then I am summing over all rectangles. So, this gives me nothing but the surface integral curl of E dot d s over the entire surface.

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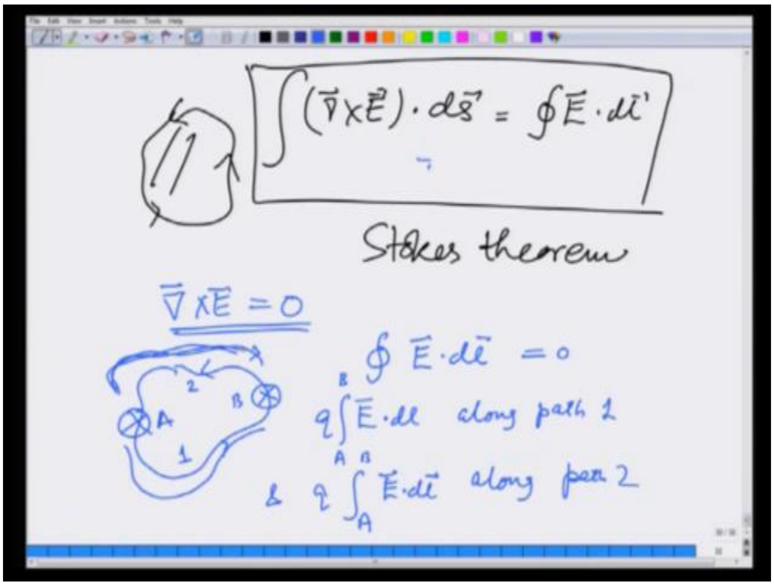

So, what we learn from this is that given a surface curl of E over this entire surface, where the direction of the surface is again given by Right hand rule is going to be nothing but E dot d l on the outside path and this is known as Stokes theorem . So, now, we are ready to understand the physical meaning of this curl. See, earlier I have given it to you it is gives you a sense of if a electric field is turning around, but with this when we saw that E is 0 for an electric field; that means, any path I go around, work done is going to be 0 in taking the charge around.

Let us take two points on this path, let us traverse this path once like this and once like this, from point A to point B. The work done is going to be E q dot d l from A to B, let us call this path 1 let us call a path 2. Along path 1 and q E dot d l A to B along path 2 and I am going from A to B.

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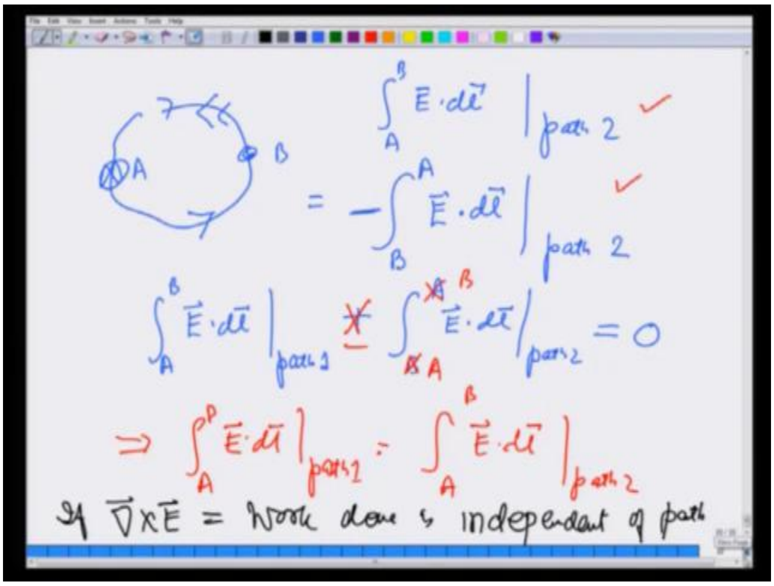

But, notice that A to B along path 2 is same as minus of path B to A d l along path 2. Now, integration A to B E dot d l path 1 plus B to A E dot d l path 2 like this, and then path 2 I am going like this is 0, because curl of E is 0. What is this? This is minus from these two equations from A to B and this implies A to B E dot d l path 1 is equal to integration A to B E dot d l path 2.

Notice that, path 1 and path 2 are arbitrary and therefore, what we conclude from this is that the, if curl is zero work done is independent of path and depends only on the end points. This is the definition of conservative field.

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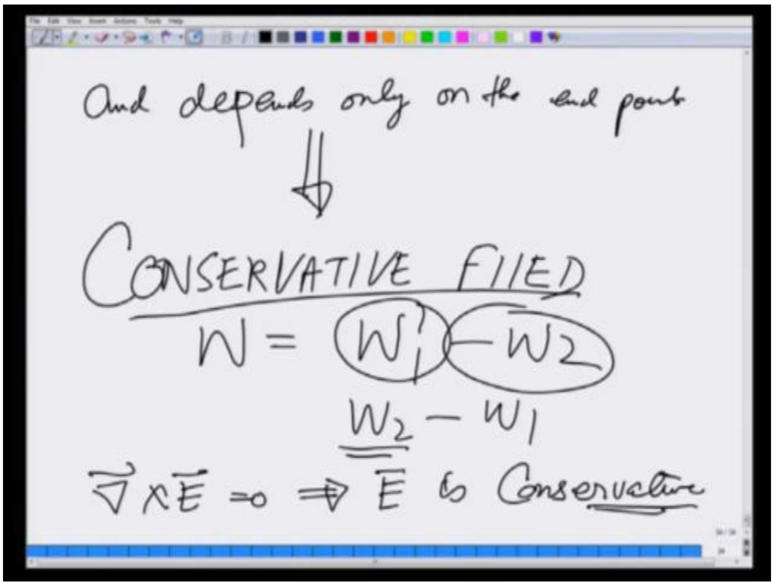

That means, work done can be written as W 1 minus W 2 or W 2 minus W 1 and later we are going to identify these as the potential energy. So, for a conservative field we can identify potential energy, because I can define at each point that between two points what will be the work done, because independent of the path, and therefore curl of E being 0 implies E is conservative.