Nuclear Physics Fundamentals and Application
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Lecture - 18
Shell Model Contd…

We talked about the property shown by nuclear, and these properties suggest that there is some kind of shell structure inside. And there are energy gaps and that decides shell closer, and closes shell nuclei; that means, if proton number or neutron number or both they corresponds to that shell closer then the nucleus is extra stable, the many property suggest this shell structure.

And hence like atomic physics I we tried on single particle potential, where we can fill the energy levels using poly exclusion principle, and then come up with some kind of shell structure as in atomic physics. Now the question was what is to take for that single particle potential, and then I showed you that people tried with infinite square well potential, simple to handle mathematically did not work well, then this harmonic oscillator potential, that also did not work well.

The first three shell closers which we call magic numbers, where reproduced by both of these and beyond that there were problems. So, the both these potentials are unrealistic in the sense that they go to infinity and if that be the potential, then it will be difficult or not possible to take out any nucleon from this, but we do know that yes, by giving appropriate finite energy one can take out a neutron or a proton from a nucleus neutron proton separation energy or proton separation energy.

So, the potential perhaps is the culprit we have to go for more realistic potential. So, people tried this is all late forties. So, people tried different kind of thing finite square well potential was one. Now, the effect of finite potential, finite square well potential is to pull down the energy levels little bit, but the order and the basic structure did not change from infinity square well potential, the levels of infinite square potentials that order of level almost remain same, when one goes to finite square well potential only thing that the values they will come down.

So, that shell closer and those things will almost remain the same. Then much more realistic potential where the edge is not very sharp, that another factor which one can work on finite potential, finite square well potential also the edge sharply falls to 0. So, you have minus v naught to certain distance from the center and then suddenly it become 0. So, that is another unrealistic part in the potential. So, people tried with square well potential, with exponential edges and other things.

And the best one could think of was wood section type of potential the mass distribution in a nucleus goes as that wood section function, it is almost constant and then it gradually falls in a small thickness, so similar function for potential also of course, attractive, so invert that. So, the potential is something like minus v naught almost constant up to some distance. And then in a short distance it will taper of to 0. So, wood section type of potential it is like this.

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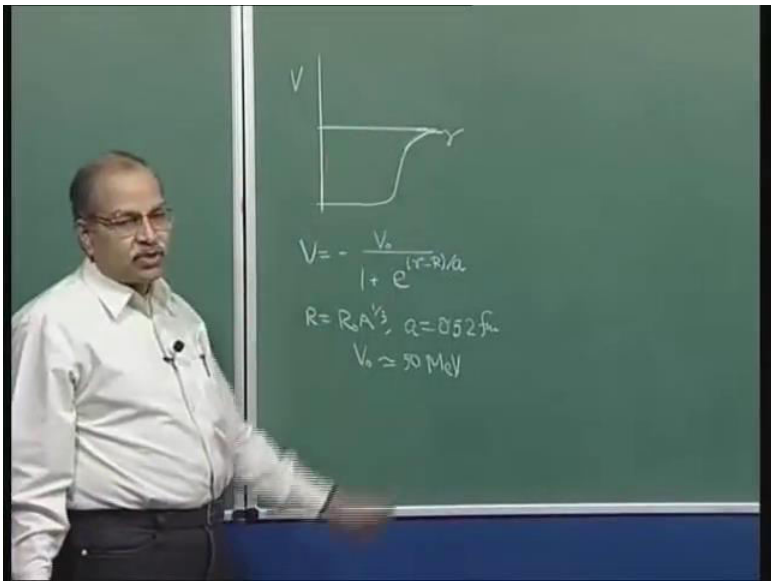

If you have r on this side v on this side it is some minus v naught here, that r is equal to 0 and then it almost remain same up to some capital R and then in a short, thickness it goes to almost 0. Like that is the wood section that you, you note falls to gradually it falls to 0 you are familiar with this we are discussed when we were talking of charge distribution or mass distribution inside the nucleus. If you write a function it would be something like v naught by 1 plus e to the power r minus capital R by a.

So, this minus takes care that it is negative attractive potential and this R 1 can take as R naught A power 1 by 3 and this a is from the charge distribution itself something like 5 2 femtometers and v naught turns out to be around 50 m e b also to get a right kind of separation energies. So, these type of potentials which look the more realistic, these things were also tried, but the results were the same. It does not really change in any significant manner the level ordering or the magic numbers or the shell closers and all that.

So, did not improve much even when we write this kind of potential and solve this Hamiltonian with this kind of potential and write all those energy levels coming in and look for the energy gaps the situation is not significantly improved. So, it is to some extent in sensitive to the kind of potential you are choosing. And then 1949 people brought the suggestion again form idea must have come from atomic physics that tri spin orbit interaction. In hydrogen atom you have all those h alpha lines and all those things and then there is a finest structure, finest structure means the wave lengths are will slightly different and then you have several line in that, so called h alpha line. So, that finest structure was understood finally, using this spin orbit interaction term in the main hydrogen atom Hamiltonian.

So, similar storey here; however there that a spectral this splitting or finest structure was very small, a small fraction of electron volts. So, that is spin orbit interaction could take care of that, here the nuclear energy levels are mega electron volts. And if you want to change the ordering of these measure levels, that spin orbit interaction has to be very, very strong, but never the less people did give it a try and found that yes it works and nothing success like success. So, once that term is added things really fell into order, so what is that? So, in the Hamiltonian whatever, Hamiltonian you write here that Hamiltonian will be there Finite Square, well potential or this wood section potential or harmonic or infinite whatever.

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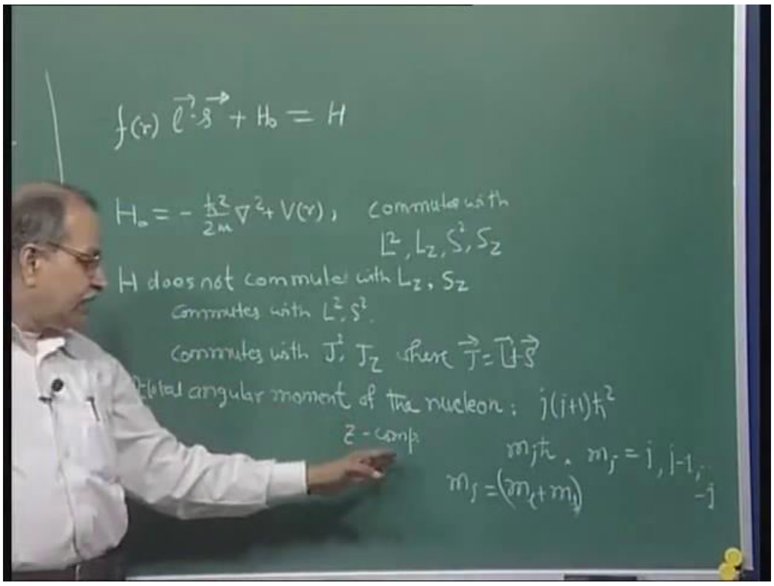

$f(r)\,\vec{l}\cdot\vec{s} + H_0 = H$

$H_0 = -\frac{\hbar^2}{2m}\nabla^2 + V(r)$, commutes with L^2, L_z, S^2, S_z

H does not commute with L_z, S_z

Commutes with L^2, S^2.

Commutes with J^2, J_z where $\vec{J} = \vec{L} + \vec{S}$

Total angular moment of the nucleon: $j(j+1)\hbar^2$

z-comp

$m_j\hbar$. $m_j = j, j-1, \ldots -j$

$m_j = (m_l + m_s)$

So, that potential will be there on top of that, you add extra some function of r and then l dot s. Remember; we are talking of single particle potential. So, this potential is felt by each of the nucleons in the nucleus. So, it is 1 particle. So, that 1 particle has certain l and of course, for 1 particle s we know will be half. So, this term is added to Hamiltonian. So, H is whatever original H was there. So, add this term to write the Hamiltonian. What is effect the first is that this Hamiltonian after introduction of this term.

Does not, commute with l z operator or s z operator. A central potential when you write a central potential some h naught is that kinetic energy term and then v r. So, now, if you this Hamiltonian commutes with, it commutes with l square operator, l z operator, s square operator, s z operator it commutes with all these things that is why you can have definite angular momentum square, definite angular momentum z component, definite spin angular momentum square and definite spin z component. In the energy Eigen functions.

H commutes with this l square means you can have energy Eigen function in which this magnitude square of orbital angular momentum has one single definite value. And similarly, it commutes with l z so; that means, in that energy Eigen function you can have one single unique definite value of this l z and similarly for these 2. Now, when you bringing this l dot s term here, then with this H does not commute, with l z and does not commute with s z. But, it does commute with l square and s square.

So, although you can still talk of p state and d state and f state; that means the small l is definite, small l means the angular momentum square l l plus one H cross square that is definite. So, you still talk of p state and d state, but then you do not talk of what is m l or what is m s, s is any way for a nucleon it is small s is half. So, s square will have half plus half in to half plus one into H cross square that is any way definite, but whether it is a spin up or whether it is spin down that you cannot talk.

It is they get mixed up, it does not have a definite value of s z. So, both spins are mixed up in same state similarly, if small l is equal to 1 if you ask what is m l, we generally say that if l is equal to 1 m l can be 1 0 or minus 1. Now, here it becomes 1 0 and minus 1 it is not define you can have mixing of m l values, in that same energy Eigen function you can have mixed up states where m l is 0, m l is 1, m l is minus 1, everything got mixed up.

So, that is the first thing about this, but then this commutes with J square and J z, where this J is defined by this l dot s plus operator, so you if you define this angular momentum J which is l plus s. So, J z is l z plus s z, z x is l x plus s x J y is l y plus s y and from there you construct J square. So, J square and J z both of them will commute with this Hamiltonian even after inclusion of this l dot s. And hence you can define your state in terms of quantum numbers of these, and you can ask what is the value of small j total angular momentum.

So, this angular momentum, total angular momentum of that one nucleon, of the nucleon not nucleus we are talking of single particle Hamiltonian, single particle functions. So, this potential is to be given to each of the nucleons. So, if we are talking of one nucleon. So, angular momentum of that one nucleon, total angular momentum of that that is the, these are the operators.

So, you will have j j plus 1 h cross square is the square of that total angular momentum, that is this and the z component is m j h cross and m j can have values going from plus j 2 minus j in steps of 1 and this m j can have definite value in the energy Eigen function. This m j will be m l plus m s. Now, because j z is l z plus s z. So, that will be there, but these things can get mixed up you can have several sets of m l and m s giving you the same m j. What happens to energy, what is the change in energy because of this extra term. So, we evaluate this extra term with the state defined by j and m j because these are the definite values, these have the definite values.

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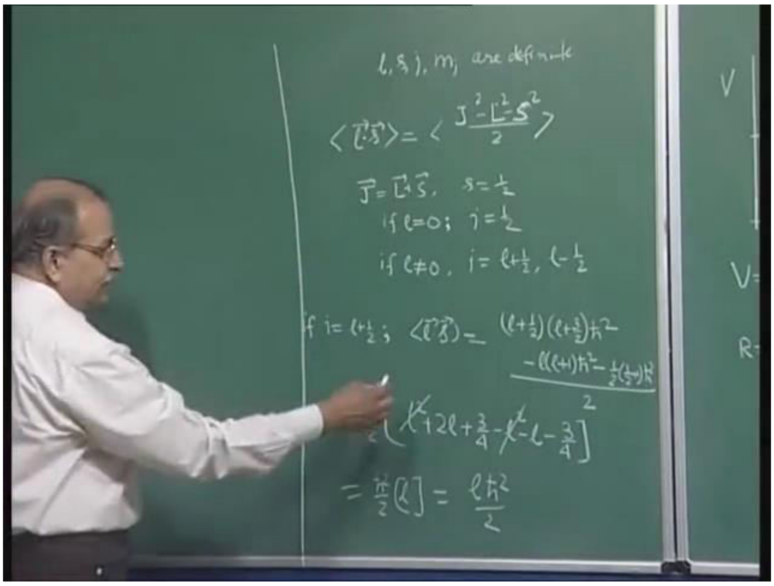

So, j and m j in terms of that; that means, the state we are talking of l is definite, s is definite, j is definite and m j is definite. This is the state we are talking of and of course, that n will be there n equal to 1 2 3 4 will be there. So, in this state if we try to evaluate this l dot s, l dot only this part let us first take this s. So, if you take the value of this l dot s how much this will contribute to the energy. So, this l dot s can be written as j square and minus l square correct right j square minus l square minus s square by 2 that is l dot s and all these are having definite values.

Now, if you are given an l then j and remember s is half j j is l plus s and s is half, l can be 0 1 2 3 4 and so on, but s is always half. So, what are the possible values of this j if l is equal to 0, if l is equal to 0 they same as s. So, j is half and if l is not equal to 0 then this combination gives j as either l plus half or l minus half. If you are adding 2 angular momenta and the corresponding quantum numbers you have then say, j 1 and j 2. So, the final j will go from j 1 plus j 2 to j 1 minus j 2 in steps of 1.

So, here, since that second angular momentum, spin angular momentum has quantum number half. So, it will go from l plus half to l minus half in steps of 1, and there is only 1 step. So, these are 2 possible values now, if j is equal to l plus half then this l dot s you can work out this is j square that is j j plus 1. So, it is l plus half l 3 half times h cross square. And then minus l square is minus l l plus 1 h cross square and minus s square is minus half half plus 1 h cross this is the this numerator and this whole thing divided by 2.

What I have done in place of j I am talking l plus half. So, j square this these are the all operators I should have written capitals j square is j j plus 1 h cross square. Let me write the capitals l dot s vectors is there. So, there is no problem here let me write it j square capital J square to avoid any confusion capital L square and capital S square. Once the vector is there you know it is that operator thing. So, j square is small j small j plus 1 h cross square and small j is this l plus half.

So, it becomes l plus half this is j, this is j plus one times h cross square l square is l l plus 1 h cross square and s square is

Student: minus of divided by 2.

Divided by 2.

So, how much is this work out h cross square by 2 take out, l square plus half plus 3 by 2. So, 2 l plus 3 by 4 this part over, then this part minus l square and minus l and minus 3 by 4. So, this cancels 2 l minus l is l, l l that is all. Next this is 1 j is equal to l plus half.

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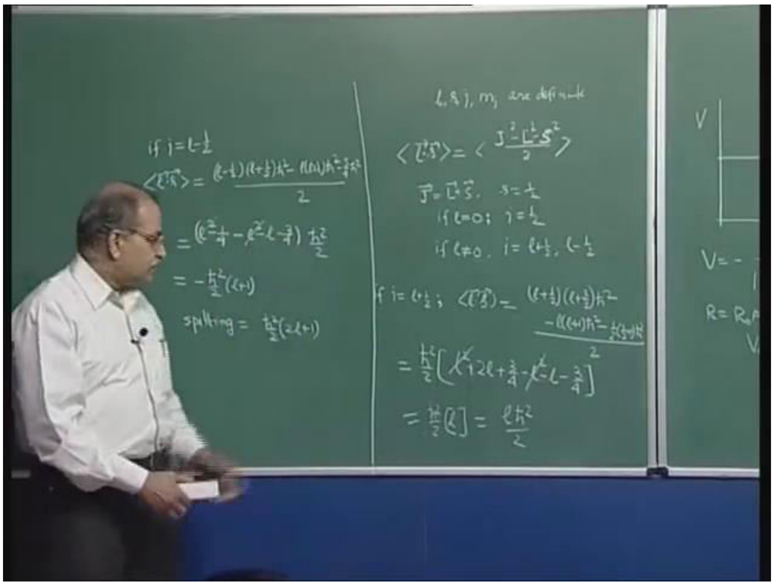

And if j is l minus half, then similarly this l dot s will be l minus half j j plus 1 h cross square. So, l minus half this is j this is j plus 1 h cross square. And minus other things remain the same l l plus 1 h cross square and minus 3 by 4 h cross square and divided by 2 these term remain the same. So, this is l square minus 1 by 4 I will take h cross square out and then you have minus l square and minus l and minus 3 by 4 this whole thing h cross square by 2.

So, this cancels and you have minus l here, and minus 1 here. So, that same l now, splits in 2 parts l is same, but depending on whether j is l plus half or j is l minus half. The energies are different j l plus half that l dot s term will have this value whereas, minus l minus half it will have this value. So, energies will be different f r is there remember, this f r is also there. So, that also has to be taken in to account, but the effect of this is to spilt the energy in 2 parts.

And, so happens all to be consistent with all experiment this f r should finally, give you a negative sign, the magnitudes will come from here, but then this f r is negative. It will give a negative multiplication to this and therefore, this h j equal to l plus half term that energy will go down and this l minus half energy will go up and it will go up by largest slightly, larger amount as compare to the amount by which it is going down. So, some asymmetry, and the splitting will be at this 2 h cross square by 2 2 l plus 1.

Proportional to this f r is there f r is deciding the whole magnitude, but if you just look at the effect of l, larger the l, larger is the splitting s of course, s state l equal to 0 will not split because with l is equal to 0 you have only one possible values of j. So, s states will not spilt, but all other states will spilt p state will spilt by some amount, d state will spilt by a larger amount, f state will split by even larger amount and so on because the splitting is also depending on l here larger l, larger the splitting. So, let me just add this l dot s effect to that harmonic oscillator potential because harmonic oscillator potential is easy to remember than the energy levels. Even though it is not very realistic and this and that, but let just see what happens if I add this to harmonic oscillator energy levels.

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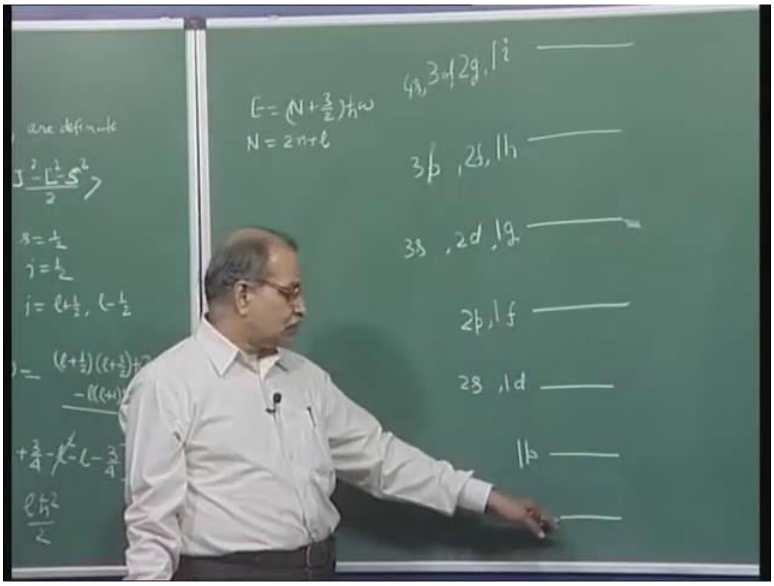

So, energy is n plus 3 by 2 h cross omega and this n is 2 n plus l. So, let us draw the energy levels. So, the lowest will be 1 s then 1 p then 1 d and 2 s, next and 2 p no 3 f will come here, 1 g, 2 d and 3 s then next 1 h, 2 f and 3 p then 1 I, 2 g 3 d 4 s. Now, put the value, put the effect of this l dot s term. Now, this lowest energy of this harmonic oscillator potential corresponds to l equal to 0 s state and capital n is 0, 1 s l is equal to 0 with l equal to 0, what are the values of possible values of j or the 1 value half because if l is equal to 0 you only have 1 value j equal to half and if l is not equal to 0 then you have 2 values l plus half and l minus half.

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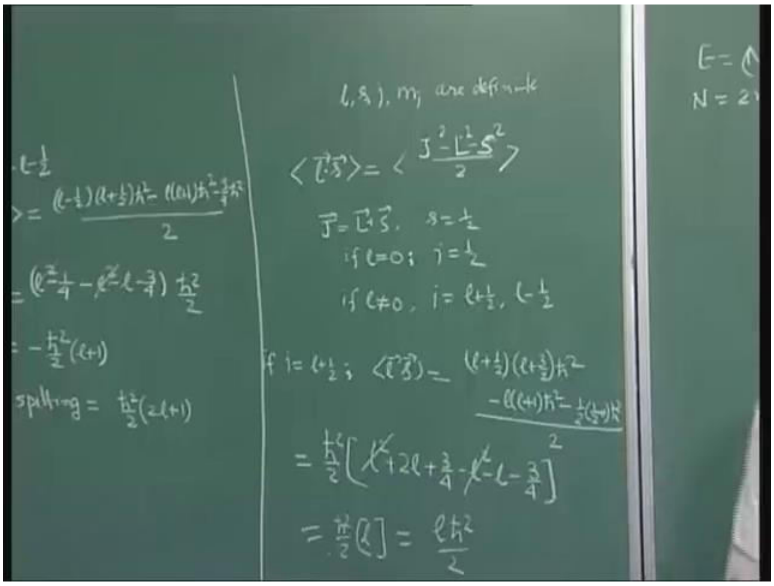

So, with l equal to 0; that means, s state you do not have any splitting it is just 1 level.. So, this 1 s remains 1 s, so it is here.

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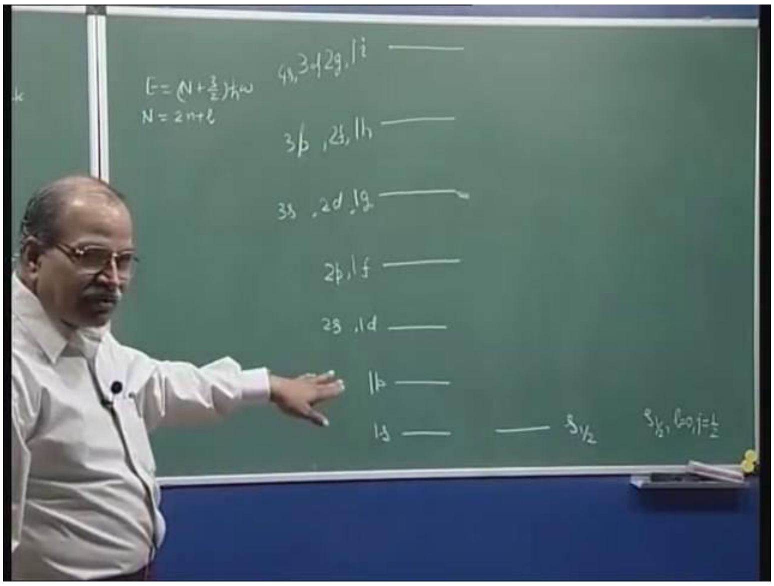

But, what you will write is half here, this half rather this 1 is not very important for me this 1 is already here. So, I just it write it half s half, this half give s you the value of j this is value of j when I write s half; that means, l is equal to 0 and j is equal to half. Then you have next level 1 p. So, 1 p means l is equal to 1, if l is equal to 1 then this spin orbatic coupling will give 2 possible values of j. And what are those 2 possible values 3 by 2 and 1 by 2, l plus half and l minus half. So, you have p 3 by 2 and p half 1 by 2, j is equal to 3 by 2 and j equal to 1 by 2 and these 2 will have different energies.

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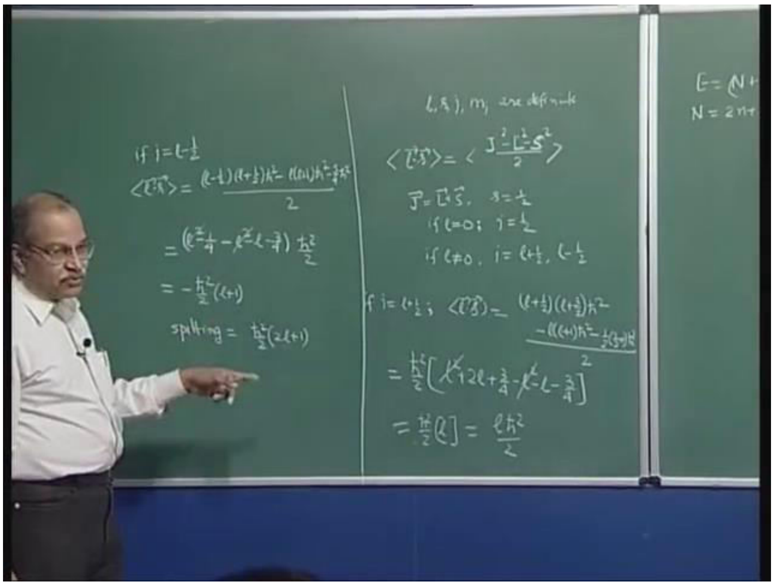

And the this 3 by 2 that is j equal to l plus half, this shift in energy will be this much plus something coming from that f r it has to be energy unit this is not energy unit. What unit is this, what physical quantity this corresponds to l is a number h cross square, angular momentum square h cross a angular momentum, so angular momentum square. So, this is not energy, the energy will come when you add that f r effect also and finally make it energy units.

So, but whatever f r will give you some value this as I said f r will finally, give you a negative multiplication the magnitude will be governed from there also, but with a negative. So, this goes down. So, p 3 half will go down and p half where j is l minus half, where j is l minus half j j is l minus half then this is the shift, this is the shift and you have a minus here 1 minus coming from that f r. So, it will go up p half energy will go up, p 3 half energy will go down this will be the splitting this plus that f r effect. So, let me use this p half will go down.

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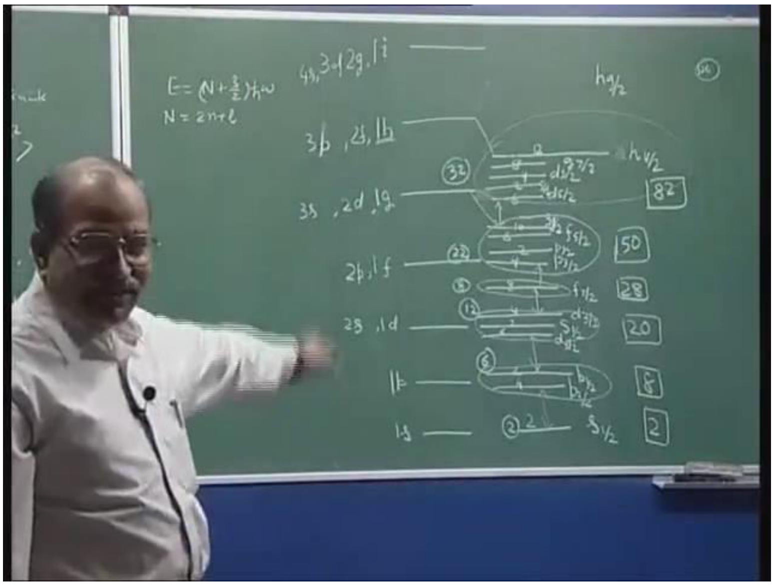

So, p half will come somewhere here p 3 half, p 3 half will come from somewhere here, so p will half will somewhere here. Next is this level, at this energy level ins harmonic oscillator potential, you have 1 d as well as 2 s. So, let us first look at this 2 s will be un split. So, 2 s will just remain 2 s. So, we put 2 s here, this is s. So, s half with s l equal to 0 j is half d, what is l 2? So, j 5 by 2 and 3 by 2 which will go down 5 by 2 will go down. So, 5 by 2 d 5 by 2 will down go down. So, this is d 5 by 2 and d 3 by 2 will go up. So, this is d 3 by 2.

Come here 1 f what is value of l 3 0 1 2 3. So, what is the value of what are possible values of j 7 by 2 and 5 by 2. So, 7 by 5 will come down. So, this is 7 by f 7 by 2. So, this comes down and 5 by 2 will go up. So, 5 by 2 go goes up that is for 1 f then you have this 2 p what happens to 2 p, p is l equal to 1. So, j is 3 by 2 n 1 by 2 3 by 2 goes down and 1 by 2 goes up. So, this is 3 p by 2 and this is p 1 by 2.

1 g there are 1 2 3 4 j 9 by 2 n 7 by 2 n 9 by 2 will come down. So, 9 by 2 will come somewhere down here g 9 by 2 g 7 by 2 will go up 2 d d 5 by 2 and d 3 by 2 and 3 s. So, s half and so on. We can do we will do a similar analysis there, but first look at those lower energy levels. The occupancies how many quantum states are there, these are energy levels at this energy you can still have more than one quantum states. At this energy you had several, at this energy you had several, there some splitting, but even now, it is not that one energy corresponds to one quantum state.

Because, what we have defined this line for example, this line here it says it is d 5 by 2 of course, that 1 d 5 by 2 this 1 is also there, so 1 d 5 by 2. So, that n is fixed n is a n is here 1 d all right. So, that this small n is 0 1 d that n is fixed d means l is fixed l is 2 this 5 by 2 means j is fixed. So, that n is fixed l is fixed that j is fixed s is fixed how much is s half h is half 1 by 2 atoming of 1 nucleon, 1 nucleon experiencing this single particle potential.

So, s is half in this particular 1 j is 5 by 2 and l is 2 and n is 0 1 d, but you still have m j now, talk to what is m j here, and we cannot because all those possible m j s have similar thing energy is not depending on m j energy is depending on j. So, for this d 5 by 2 for example, how many values of m j possible 6 5 by 2, 2 minus 5 by 2 in steps of 1. So, 5 by 2, 3 by 2, 1 by 2 minus 1 by 2 minus 3 by 2 and minus 5 by 2 6.

Just look at this number add 1 here, if it is 5 by 2 it will be 6, 6 values of m j possible 3 by 2 4 3 by 2 1 by 2 minus 1 by 2 and minus 3 by 2 half 2 half and minus half. So, that is how you can count the number of quantum states there and being fermions one quantum state will contain only maximum 1 nucleon, 1 proton or 1 neutron. Proton and neutron can go together in that, but 2 protons cannot go into that 2 neutrons cannot go into that. So, number of quantum states. So, let us make a count 2 here, 4 here, 2 here, 6 here, 2 4 8 4 2 6 10 6 2 4 8 and so on.

Now, where are the energy gaps, the first obvious energy gap is here, and another obvious energy gap is here and here also you can see although I have not drawn to the scale, but still here is a gap, here is a gap and then these are close enough and here is a gap. So, those gaps are to be in shells close that. Now, let us call this as 1 shell and since there is a major energy gap, this is another shell and this is 1 shell because within this the gap is a small and then you have a bigger gap.

Here, within this you have small energy gap. So, this is 1 group and here is relatively bigger energy gap. So, this is a separate group and then here also you have reasonably good that. So, this is 1 single gap here of this, this is 1 group. So, group wise counting 2 here, and how much is this 6. So, this is 6 this group is 2, this group is 6 how much is this group 12. So, this group is 12, this groups this group 22 you work on that first look at this.

So, shell closer, first shell closers are 2, second shell closers are 2 plus 6 that is 8. So, shell closers at 2 then shell closers at 8 no big deal then plus 12 that is 20 here also no big deal because these things even without l dot s we had all these things. But, then now, this is separated and that is coming because of this l dot s thing, other wise it would have been here. So, it has been separated out from this the group coming mainly from here, and the group coming mainly from here, this lies in between.

So, this is that l dot s contribution that it had separated this 1 from this level here and now, it is stands like it is own group. So, shell closers here 28. So, now you can see the effect of this l dot s, l dot is rearranging things and it is giving energy gaps at the right places next to. So, 28 plus 22 50, another magic number coming from experiments remember 2, 8, 20, 28, 50, 82 and 126. So, that 50 is reproduced. Now, let us look at the next one this 7 by 2 this h 1 h how much is l here 0 1 2 3 4 5.

So, 11 by 2 and 9 now 11 by 2 will come down. So, much that it will almost come here rather this g 7 by 2 I can make little bit closer. So, g 7 by 2 is here and this has come down how much what is this 1 h 11 by 2 and 9 by 2 will go up and all that. So, 7 g 7 by 2 is 8 and this is 12 and this forms a group and how much is this group value 6 plus 2 8, 8 plus 4 12, 12 plus 8 20 and 20 plus 12 32. So, this group value is 32. And when you add all these things up to here it is 50 plus 32, that is 82 correct magic number coming.

And similarly, you work out this h 9 by 2 is here from this 1 and this 2 f will split in 2 parts 7 by 2 and 5 by 2 this 3 p will split in 2 parts 3 by 2 and 1 by 2 all these things will be in this group and from this 1 i 13 by 2 and 11 that 13 by 2 will come down and join this group. After this see what happens here the splitting is small. So, it is fine this was a group this is a group. So, it remains almost the same, same story here, here this 1 f the highest l value and with that l plus half j equal to l plus half that comes down.

So, much that it leaves this group, and here a if floating in between, but later it will join the lower group. This 1 g this g 9 by 2 here it has come down. So, much that it is joining this group, these are the levels coming from this group splitting of this group. So, this joins this group and that will happen everywhere, this 1 h 11 by 2. So, it has been pushed down. So, much that it is now, joining these levels which are generally coming from this level, from this level this is 2 d you have this b 3 by 2 and d 5 by 2 and this 3 s.

So, these levels are coming from this group and this h 11 b y 2 has been pushed down. So, much that it is now joining this group. Similarly, here this 1 g was g 9 by 2 was pushed. So, much that it was joining it joins this group which are which is generally coming from here, similar story was here this was also come down this also came down, but not that much it is just in between. So, from it is own group. So, these phenomena of the highest l value and j equal to l plus half that is pushed down to that lower group you add that.

So, you do that and you find that the next shell closer be at 126 that is your task. So, the magic numbers are reproduced very well, once you put this l dot s right on the harmonic oscillator potential I choose harmonic oscillator because I think these energy levels is very simple, but then the order of levels, order of the states this order it is really sensitive to what you are choosing, square well potential or harmonic potential or wood section potential or rounded edge and so on.

So, as far as magic number has are concerned you understand this mechanism and you can work with this harmonic oscillator potential and get those magic numbers. But, if you go, if you use these level diagrams for other properties, like how much is the spin of oxygen nucleus or platinum nucleus or so on. Then these orders of these levels is also important. And there one must use not this harmonic potential something else. So, here the order that you are seeing here 3 by 2 here, 1 by 2 here 5 by 2 here and so on these orders can be different.

The effect; that means, do not take this 3 s 3 up to this up to here it is fine, up to here it is fine, up to here it is fine, up to here also it is fine that ordering will not change whether you take finite square well potential or you take wood section potential or whatever you take up to here this order will be the same beyond this if you go here, here, these orders do not take very seriously because now you have to take more realistic potential.

If you really want to see whether this s half is here and then b 3 half is here or g 7 by 2 comes here these final details if you wish to go in to which you need if you use this scheme to predict various measurable quantities then you will have to look for what happened that is 1. Second thing there are 2 3 things I would like to talk, second thing is this whole energy levels, will depend on or the potential itself that single particle potential itself, the parameters of that will depend on the size of the nucleus.

For example, if you take wood section potential you had that v v naught divided by 1 plus e to the power r minus capital R now, this capital R depends on the size. So, this it is not the same potential for which is applicable to beryllium and which is applicable to phosphorus and which is applicable to zirconium. As a function of this total mass number, the potential itself has to check that happens in atomic physics also if it is hydrogen atom is 1 proton. If it is helium you have 2 protons.

So, potential is different although the form remains the same it is a 1 by r type potential, but then the depth of the potential is different, the energy the value of energy if it is hydrogen atom you know ground state energy is minus 13.6 e v, but if it is helium or and there are 2 electronics, but is to say that I solve that single particle potential and this is 1 s state and this is 2 s state and so on. So, both the electrons of helium are in 1 s state.

But, that 1 s energy is not 13.6 e v, because you have to take 2 protons into account the radius atomic radius is different. So, z a naught becomes a naught divided by z and all those things, if you go for higher atomic numbers for they this then the 1 s energy it is, so high 1 s 2 s these energies are, so high that x rays which are emitted are in kilo electron volt remember that this from the same atomic transactions k alpha it goes from 2 p to 1 s electron d x i 2 p to 1 s and if you k alpha and k alpha energy for iron is 6.4 kilo electron volts.

So; that means, those 1 s, 2 s, 2 p the energy differences are kilo electron volts, is no more minus 13.6 e v. So, the energies the values of energies they are ordering and all these will also depend on the mass of the nucleus, heavier is the mass you will have different, low energy things are more or less settled. If you have light nuclei up to say 20 up to 20 25 20 30 it is almost the same, but then if you go for heavier nuclei then exactly the same thing you cannot crime and say that this is applicable for everything, we will talk from here.