Topology and Condensed Matter Physics

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Lecture – 04

Ten Fold Classification

Today, we will talk about tenfold Classification of materials basically these are done for the insulators and the superconductors and these tenfold classification dates back to 1996-97 by Altland and Zane Bower. In fact, you can see a very nice lecture by Altland in topological matter school in 2017 and so, we will not get too much into details of this classification, but we will tell you the basic things based on the discussion that we had so far on these time reversal and particle hole and other symmetries that we talked about.

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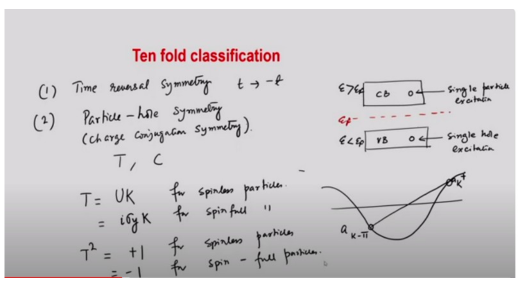

So, we will start with this two main symmetries that we have learned one is called as a time reversal symmetry which of course, means that t going to minus t and so on. And what it means is that if a particle is made to move with a velocity v in the opposite direction that is with a velocity minus v whether the physics remains invariant or whether the Hamiltonian remains invariant and we have elaborately seen the consequences of such time reversal symmetry being present or absent especially if they are present then one has this Cramer's doublets which we have done. Now also we have looked at the particle hole symmetry and this is has been talked about when we discussed a Kitaev chain and so on.

So, whether a particle becoming a hole and so on whether that has a bearing on the Hamiltonian and if it does then of course, there are these energy E corresponding to a particle will have an energy minus E corresponding to a hole and so on so forth. So, if you want a sort of pictorial representation of that so, let me sort of draw a conduction band and the valence band. So, this is the C B and the vB and we create a hole here. So, there is a single hole excitation here and this will correspond to a single particle excitation here ok.

And of course, this is your Fermi energy let me draw it with a color. So, this is Fermi energy and this is for E greater than E f and this for E less than E f and so on ok. So, this is really the particle hole symmetry that we talk about. So, there is a corresponding to an energy minus E for a hole excitation there will be a particle excitation at an energy plus E and so on ok. And in terms of a band picture we can do it suppose we take a cosine band and the we fix the Fermi level somewhere here and then the particle hole symmetry would correspond to something like this.

So, there is a k minus pi will correspond to for a particle will correspond to a hole at these k. So, say this is some a k minus pi and this will be some a k dagger and so on. So, this is the you know the hole and this is the particle and so on ok. So, we have also learned that the time reversal symmetry which we write it by t and the particle hole symmetry which you will write it by c. And the reason that we write it by c usually because traditionally this particle hole symmetry is denoted as a charge conjugation symmetry ok.

It is called a CCS and I believe that the people who work in high energy physics they use this term more often than the people in condensed matter physics. So, this is called charge conjugation symmetry and that is why it is written with ac. So, this t and c are the ones that we are going to talk about and we have in various situations we have talked about how t acts on a Hamiltonian which is of the form of a Dirac Hamiltonian say d dot sigma and then how t acts on each of the components of d and how the Hamiltonian gets transformed. You have done that elaborately in the context of when we did the SSH model and the Kitayev model. So, this t has a form which is we know that it has a form like a k conjugation it is also usually written with a uk where u is a unitary operator and this k is the complex conjugation and it makes it a anti unitary operator and both c is also an anti unitary operator.

$$T = UK \quad for\ spinless\ particles$$
$$T = i\sigma_y K\ for\ spinfull\ particles$$
$$T^2 = +1\ for\ spinless\ particles$$

Now this is for spinless fermions or spinless particles and we have also looked at how in presence of spin this is not enough to represent the symmetry it is written as i sigma y k for the spinful particles and we have seen this earlier. And this sigma y is nothing but the y component of the Pauli matrix and this can be operated on a given term of the Hamiltonian and then term by term we have talked about how these Hamiltonian or rather how these symmetry operation transforms. So now this t is equal to the square of this is equal to plus 1 for spinless particles and so on. I am just trying to remind you of all the properties that we have done and this is equal to minus 1 for spinful particles.

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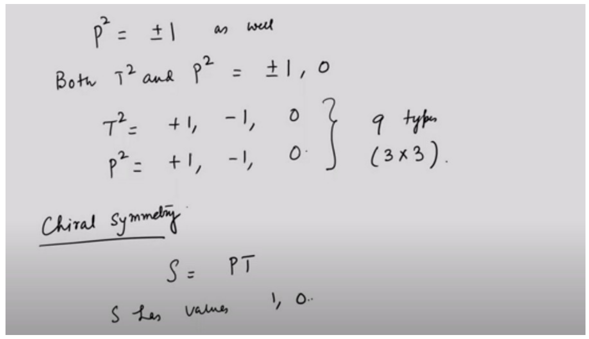

Similarly p also has this so p square is equal to plus 1 and minus 1 as well. So just like whether you do that the particle hole transformation twice it will either give rise to a minus sign or it will give rise to a plus sign that is plus 1 or minus 1 and so both t square and p square have values which we have learned so far is equal to plus minus 1.

$$P^2 = \pm 1$$
$$T^2 \ and \ P^2 = \pm 1, 0$$

Now it is also true that a system not having time reversal symmetry or not having particle hole symmetry should also have these values for this t square and p square and in that case the trivial cases where there is no time reversal symmetry or there is no particle hole symmetry then we write it as t square and p square equal to 0. So now you see that t square has all these three values so it is plus 1 minus 1 and 0 and p square has a plus 1 and minus 1 and 0. So if you put together all the combinations so we can have a consider that t square equal to plus 1 and p square plus 1 minus 1 and 0 and then t square minus 1 and p square equal to plus 1 minus 1 and 0 and similarly for t square equal to 0 p square equal to plus 1 minus 1 and 0. So there are of course 9 sort of these things present I mean 9 types that are present ok.

$$T^2 = +1, -1, 0$$
$$P^2 = +1, -1, 0$$

So 9 types coming from the 3 cross 3 ok. So 9 values for these t square and p square based on these plus 1 minus 1 and 0 they are present in the system but we have talked about a 10 fold symmetry classification of materials and this 10th one is really coming from another symmetry which is called as a chiral symmetry which we have seen as well in certain situations and this chiral symmetry is nothing but like let us write it with s and this s is nothing but it is a product of the p t symmetry and now this s does not have values plus 1 minus 1 and 0 but s has values which are 1 and 0 ok.

$$S = PT$$

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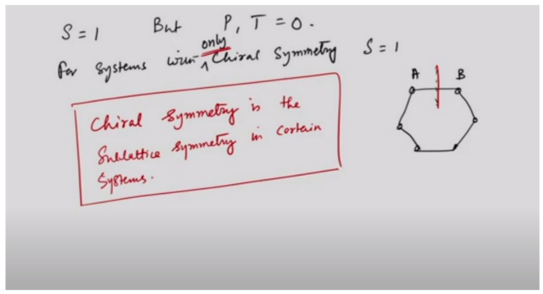

Now it can happen or rather the 10th classification is coming from s equal to 1 but p and t equal to 0 ok. So for systems with chiral symmetry so let me write it with only chiral symmetry ok and this only is important here because t and p are equal to 0. So this so s is equal to 1 so that is the 10th type of system classification that we have ok.

$$P, T = 0$$
$$S = 1$$

So we have all these things all the 9 combinations that we have talked about and the last combination is p equal to t equal to 0 and so on ok. So this chiral is not a trivial symmetry that we can understand just like we have understood the time reversal symmetry or the particle hole symmetry. In fact the chiral symmetry is very apparent in systems with sub lattice structure. For example you have seen a graphene, graphene has a sub lattice structure it consider unit cell consists of 2 atoms there are 2 atoms in a given unit cell and they are called as a and b atoms and there is an inversion symmetry with respect to the bond you know the perpendicular to the bond that connects. So what I am trying to say is these is the structure and so this are so these are a and b so if you draw a line here then you can change or you can invert between you know about this red line and a becomes b and b becomes a.

Now since both a and b correspond to carbon atoms in graphene so that is how it is having a sub lattice symmetry and same that it goes with the SSH model. So the chiral symmetry is a sub lattice symmetry for systems with you know sub lattice. So chiral in certain systems ok and which systems the systems that have these sub lattice structure ok. So it is that is how you know physically you may want to understand how this classification comes about ok.

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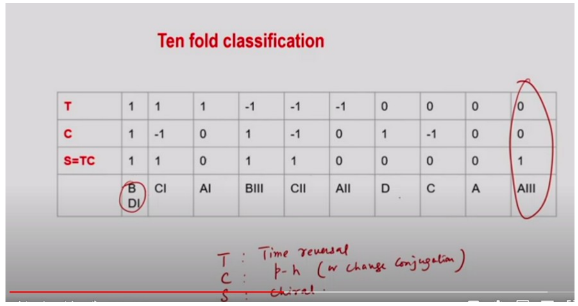

So let me look at the table that sums up this tenfold classification. Now you see that t is the time reversal symmetry c and then s so s is chiral so let me write it t is time reversal once again c is particle hole or charge conjugation so we can write both or charge conjugation and s is chiral ok. And you see that there are these time reversal symmetry being there so the 1 means that it is there so it is either 1 or minus 1 so that is that is how it is classified so 1 1 and 1 so that corresponds to the BDI class. So this BDI that is there then if it is plus 1 minus 1 and 1 it is called ci and then it is ai and then it is b3 c2 a2 d c a a3 and so on. This is the last class the tenth class where t is equal to 0 c is equal to 0 that is these two symmetries are not there there is it is not invariant under these time reversal or the charge conjugation symmetry but still the system has a chiral symmetry and that is called as a a3 class ok. So these are all these names that are you know proposed so there is BDI and a ci or c1 a1 b3 c2 a2 d c a and a3 ok.

So all our materials including the insulators and the superconductors will fall into one of these categories and they are sort of based upon these values that you see appearing at the columns corresponding to T C and S ok. S is nothing but a product of T and C. So this is the tenfold classification of materials of course I am presenting a very concise version of this classification that is only relevant to us and will not go any farther than that ok. So now of course we get this these symmetry classes but then we have to talk about the topological invariants as well.

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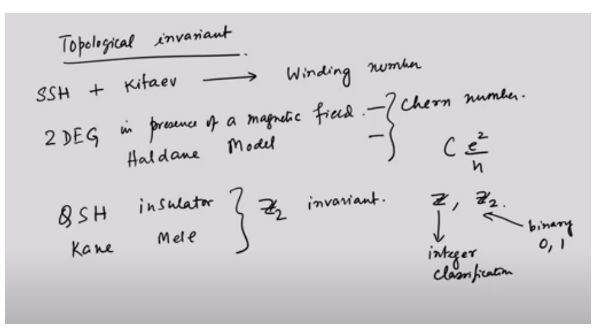

So that we know that which topological invariant we are talking about and we have learnt about a number of topological invariants. So for example in SSH model and Kitaev model we have seen that topological invariant is the winding number ok. There are you know people talk about Zuck phase and so on but all these are related to the winding number. For example even in systems where all of dx dy and dz are present so that winding is a difficult thing to visualize or winding about some particular point in the Brillouin zone is difficult to visualize and then one talks about Berry phase which is nothing but the line integral of the Berry connection you know over the Brillouin zone ok. So it is winding number there so the topological invariant is the winding number. For example we have talked about 2D electron gas in presence of a magnetic field and then and we have talked about this and then the whole conductivity is expressed in terms of this churn number and same that goes with the Holden model. So by and large you know whenever there is a Hall effect quantized Hall effect there has to be a churn number which is the topological invariant.

So the churn number sits right in front of the e square over H and the c takes values which are you know 0 1 and so on ok. Now there are something more to this I mean there are these Z invariant or there are Z2 invariant ok and the Z and Z2 invariants are based upon whether Z invariant the churn number is a Z invariant what it means is that it can take values like 0 1 2 3 4 etcetera and so that is integer classification. So this is an integer classification and this is of course binary 0 and 1 ok. So 0 correspond to the trivial phase and 1 corresponds to the topological phase. So these two are the main topological invariant that one talks about you see Z can take values 0 1 2 3 any value in principle Z2 cannot take any value.

So if you think of winding number then winding number is really a Z invariant because it can take values 1 2 etcetera or 0 1 2 of course and so it depends on how many times you wind the origin as you take a from over the Brillouin zone entire Brillouin zone you have to do it again in order to have another winding and so on so forth ok. The Z2 invariant of course is the quantum spin hall insulator that we have seen recently ok. So this is quantum spin hall insulator Ken Mele model is this corresponds to the Z2 invariant ok. So we need to identify the topological invariant that is important for a particular case and that particular case will be decided by the symmetry of the Hamiltonian that is whether the Hamiltonian has all of time reversal particle hole and chiral symmetry or it does not have 1 of them it does not have 2 of them and so on so forth ok.

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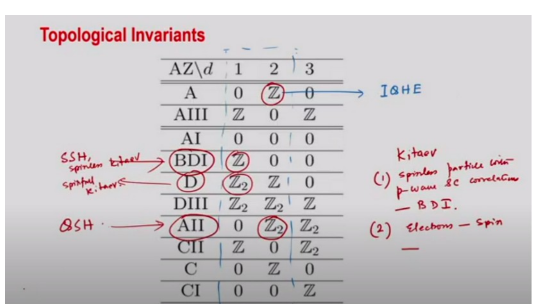

So the topological invariants are classified like this so this is only shown in dimensions 1 2 and 3 of course in this particular case we have looked at only 1 and 2 dimensional materials so far. So this is the our domain so far ok. So the symmetric groups are present here so there is a 3 a 1 b d i and so on so forth there are these the topological invariants are in 1 d 0 which means that there is no topological invariants it is topologically trivial it does not have any topological properties in 2 dimension it has the Z which is a churn number a 3 1 dimension it has Z and 2 dimension it is 0 and so on and then a i b d i and so on so forth ok. So these are the topological invariants and where are we interested in we are interested in say for example we are interested in the IQHE that is integer quantum Hall effect or quantized Hall effect that we have studied and if you look at that that has no symmetry at all so it forms into the A class ok. So it forms into the A class which has no symmetry it has no time reversal symmetry because of the magnetic field being present and it has no charge conjugation symmetry it is a completely disordered material and there will be no chiral symmetry as well. So it forms falls into the A class which is here and these A class if you look at it here then the A class has in 2 d this is the Z invariant and this is your IQHE and the Z invariant is of course identified with the churn number ok.

What about the SSH model? So the SSH model is here so this is the SSH model and the Kitaev P wave chain with spinless fermion so this is SSH let us just write SSH to begin with we will talk about the Kitaev so this is SSH and what is the topological invariant is a winding number which is also a Z invariant just like a churn number ok, and we will just talk about the Kitaev in just a moment. Let us talk about the quantum spin hall so quantum spin hall effect that is the Z 2 invariant you know very well that that is the so there is a Z invariant there is a Z invariant and that is a Z 2 invariant which we have discussed elaborately for a system with time reversal symmetry so that is A 2. So let us just go back and see what A 2 is A 2 has a time reversal equal to minus 1 these are really the t square values ok. So t square values in fact they are written t c but they are the squares of this this is minus 1 because of the reason that this is with a spin so the with spin it is I sigma y into k, k being the complex conjugation and it has no other symmetry there is no particle hole symmetry there is no chiral symmetry and so on. So this falls into the A 2 class and if you go here you see that the A 2 class has a topological invariant which is a Z 2 invariant and that is the QSH ok.

So now let us come to the Kitaev model there is a dichotomy or rather there is an ambiguity present in this Kitaev thing. So Kitaev if we talk about the Kitaev chain with P wave superconducting correlations and we talk about spinless particles suppose we do not talk about electrons at all so we talk about spinless particles with P wave correlations superconducting correlations. Then it falls into the BD 1 class ok it has same symmetries as the Kitaev model and just to go back and forth so it is all of the symmetries being present that is there is a time reversal symmetry there is a charge conjugation symmetry and then there is a chiral symmetry as well ok. So is the same thing as the BDI class and then BDI class will be characterized in one dimension by a Z invariant and which we saw that we have denoted the topological invariant as the winding number for that particular problem ok. But however there is also true that if you talk about electrons and then you have to talk about spin ok and if you need to include spin then you actually can talk about S wave superconductor you do not need to talk about P wave superconductivity you can also talk about P wave superconductivity but you can talk about S wave superconductivity but then you need to make a spin polarization in order to talk about only one you know kind of spin being there and then there are these other things such as spin orbit coupling being present etcetera that is it is usually studied with S wave superconducting correlations.

And then of course because you need a magnetic field there is no time reversal invariance and then you are not really there you are there in this thing where it is only particle hole symmetry because of the P wave superconducting correlations neither chiral symmetry nor time reversal symmetry and that is where you have a D class and the D class is represented by a Z 2 invariant ok. So, this is so I will write SSH and spinless Kitaev and here spinful Kitaev that is a D class but you see that the topological invariant is Z 2 and not equal to you know Z.

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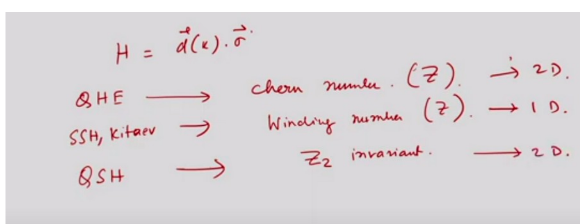

So, these are some of the things that we have you know discussed during the course and in some of the cases there are we can write down the Hamiltonian as say for example H equal to D dot sigma and so on and then you know in 1 D and then sigma denotes a Pauli matrices they do not denote spin but they denote other degrees of freedom that is why sigma is sometimes called a pseudo spinner and in sort of the quantum hall effect we have denoted with the churn number the topological invariant and then for example SSH, Kitaev let us just stick to spinless particles at this moment. So, we have done and then we have done so this Z invariant then these are winding numbers which are also Z invariant and then we have done a quantum spin hall phase which is actually a Z 2 invariant ok. So, this is in 2 D this is in 1 D we have done and this is again in 2 D graphene that we have done ok.

So, there are various such classifications that exist. So, if you look at the table and you know what symmetries are present in a given Hamiltonian then you identify the symmetric class of the material and from this table you can read out the topological invariant and if you know how to calculate it like for all these cases we have shown how to calculate the topological invariant then you are more or less you know that is about the classification of materials ok.

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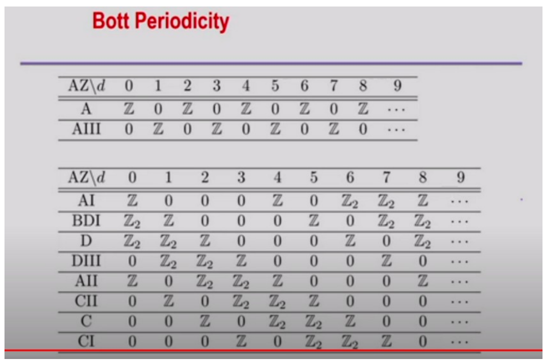

Bott Periodicity

AZ\d	0	1	2	3	4	5	6	7	8	9
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$

AZ\d	0	1	2	3	4	5	6	7	8	9
AI	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\cdots$
AII	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\cdots$
CII	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

One last thing to talk about there is something called a BOT periodicity and you see that it is always written up till 8 of course, we were now talking about 0 1 2 3 or 1 2 3 dimensions now if you write it from 0 to 8 the BOT periodicity says that everything half from the ninth dimension onwards it starts repeating. So, the periodicity is from 0 to 8 this 9 classes is again the same things, but of course, in dimensions which are very large these are of course, you cannot have materials present in this like fourth dimension or fifth dimension, but they are mathematical dimensions which identify the different topological invariants and it is it says that till 8 it sort of it carries on and then of course, everything gets repeated from ninth onwards. So, 9 is same as 0 and 10 is same as 1 and so on so forth ok that is called as a BOT periodicity that is not very important for us, but what is important for us is to have a comprehensive understanding of a certain material and if we can write down a Hamiltonian then we should be able to know the symmetries of the Hamiltonian and from the symmetries just this simple symmetries of T and C and of course, another one which is a multiplication or a product of T and C.

We know that which you know symmetric classification does the material fall in and we can calculate that the topological invariant and that will tell you that whether the material has topological properties or it may not have topological properties it may be a simple band insulator or just a normal superconductor which is well known, but whether you know the edge modes or the surface modes they contribute to the conductivity only or the transport properties only in that case the material is supposed to be topological otherwise it is termed as trivial. Thank you.