Topology and Condensed Matter Physics
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Lecture – 27

Topological Consideration of FQHE

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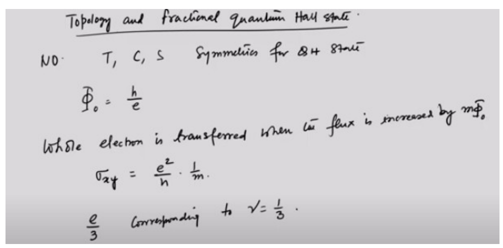

Welcome to the class. So today we are going to discuss topology and the fractional quantum Hall state. The discussion is not complete but I tried to keep it at a level that is in sync with the rest of the material that have been taught throughout the course. So we had a discussion on the fractional quantum Hall effect which we have seen that invoking the electron-electron interaction is of supreme importance to understanding the formation of the plateaus at the fractional values. So let me now ask a question that what will happen to topology in presence of interactions, electron-electron interactions or some other interactions.Particularly we are interested in the electron-electron interactions.

Now it's easy to say that topology is a robust phenomena and as long as the parent symmetries such as the time reversal, charge conjugation and the chiral symmetries are preserved, the topological properties or the topological phases are going to be protected. In fact you can think of including interaction in any of the problems that we have done so far in this course but here we are only interested in talking about the quantum Hall state and the electron-electron interactions they are present there and the effect of the interactions on the topological properties. Now just to remind you that the quantum Hall state does not have any of the TCS symmetries. So this and thus they belong to A class and it's not easy to talk about a topological invariant but we know that the plateaus are the topological invariants in this case and let us try to understand that how the fractional quantum Hall states give rise to topological considerations that we are interested in.

$$\phi_0 = \frac{h}{e}$$

So this is just to give you a overview of fractional quantum Hall effect. As the magnetic flux is increased from 0 to Phi 0 where Phi 0 is a flux quantum and Phi 0 is nothing but equal to H over E. Then in this language of the Corbino disk which had been told categorically that this is a the quantum Hall effect can be seen as a pump where as you change the magnetic field from 0 to Phi 0 and then to 2 Phi 0 etc. there are electrons that are going to be transported from or pumped from the inner edge of the disk to the outer edge and in the case of fractional quantum Hall effect whole electron is only transferred when the flux is increased by M Phi 0. So let me write that where that M is an odd denominator fraction.

$$\sigma_{xy} = \frac{e^2}{h}\frac{1}{m}$$

$$\nu = \frac{1}{3}$$

So this tells you that the Hall conductivity is nothing but this E square over H and 1 over M and now these fractional charges can actually be detected in short noise experiments and what short noise experiment does is that it the fluctuations in the charge is due to the granularity of the charge that is discrete nature of the charge that is detected from the fluctuation in the charge and say for a plateau nu equal to one-third this will tell you the short noise experiment will tell you that it the charge carriers actually carry a charge which is given by E over 3. So corresponding to one-third corresponding to this nu equal to one-third. so this is all they have been told in the context of fractional quantum Hall effect we are just trying to bring in the topological aspects to this and there's just one point of view being presented and we can debate whether this is complete but this is what suits the course.

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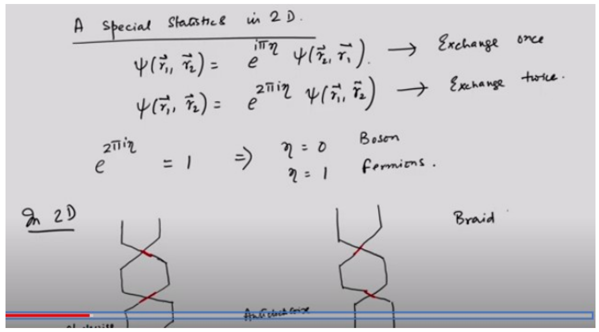

So we are going to talk about a special statistics in 2D. And why do we need a special statistics in 2D and which is distinct than that in 3D that has to be understood. So the exchange of two particles in three dimensions it brings in a phase and that phase can be written as so if I have a psi r1 r2 and I swap these two particles I'm just writing two indices then it brings in a factor of exponential I pi eta and this eta is typically equal to 1 and 0 for fermions and bosons. If you do this twice then it should come back to the same scenario so which means that this is equal to exponential 2 pi eta and psi of r1 r2. So this is exchange once and this is exchange twice. So that tells you that exponential 2 pi I eta has to be equal to 0 so this has to be equal to 1 that tells you that eta equal to 0 or eta equal to 1 both are possible and this is the choice for bosons and this is a choice for fermions. So this is that permutation statistics or exchange statistics that we are familiar with in the context of fermions and bosons.

$$\psi(\vec{r_1},\vec{r_2}) = e^{i\eta\pi}\psi(\vec{r_2},\vec{r_1}) \leftarrow 1\ exchange$$
$$\psi(\vec{r_1},\vec{r_2}) = e^{2i\eta\pi}\psi(\vec{r_2},\vec{r_1}) \leftarrow 2\ exchange$$

But there is a subtle difference in 2D. In 3D the exchanges can be done without crossing each other's path. So the two particles can be smoothly exchanged without getting tangled with each other but in 2D that does not happen it cannot be done without tangling the paths. Let me give you an example. So in 2D let me draw this I will have to draw it carefully.

So this is so it is this part is by a dotted line so let me show the dotted line just to remove any so this is that dotted line which it does not cross I go back to my black color so then and so on. So this is again should be drawn in and I just stop here so this is again in red color . So where the crossing does not occur and so on so forth. So this again here and so on. So this is a clockwise exchange in 2D so this the particles are being exchanged but this exchange cannot take place in 2D without crossing each other's path so they have to cross.

So this is a clockwise exchange let me call it a clockwise and similarly an anticlockwise will look like and so on. So this is that anticlockwise exchange sorry and so on. So all right so this is that exchange that we are talking about and if you recognize this this looks like a braid of the hair so this is called as a braiding or the statistics is called as a braiding statistics and it's also called as a fractional statistics.

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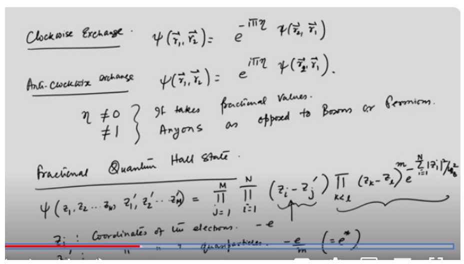

We will not discuss about it too much but it is still important to know that this statistics is quite important in 2D because it gives rise to a value of eta that's neither 0 nor equal to 1 and corresponding to a clockwise exchange we have psi r1 r2 this is equal to exponential minus i pi eta psi of r2 r1 and corresponding to an anticlockwise exchange psi of r1 r2 is equal to exponential i pi eta psi of r2 r1. And here eta is neither equal to 0 nor equal to 1 and it takes fractional values and that's why the particles would be called as anyons as opposed to fermions or bosons.

$$\psi(\vec{r_1},\vec{r_2}) = e^{-i\eta\pi}\psi(\vec{r_2},\vec{r_1}) \leftarrow clockwise\ exchange$$
$$\psi(\vec{r_1},\vec{r_2}) = e^{i\eta\pi}\psi(\vec{r_2},\vec{r_1}) \leftarrow anticlockwise\ exchange$$

Just to rewind eta equal to 0 for bosons eta equal to 1 for fermions but in 2D these are called anyons because it can take this eta can take any value between 0 and 1 any fractional value and that's why they are called anyons and eta takes values that are fractional. So this gives rise to a particular kind of statistics which is called as a fractional statistics or braiding statistics. So we keep this discussion or the deliberation of the statistics to a bare minimum and just tell its significance to the topological properties that we are interested in very brief. So now let's look at the fractional quantum Hall state. So the fractional quantum Hall states are of course incompressible we know that the chemical potential would not arise if you pack more particles or that's if you include more particles the chemical potential remains insensitive and that's how it is incompressible and because it's incompressible the excitations of the system are going to be local excitations and these excitations also have characters as that of the charges which are nothing but the electrons.

Now these excitations actually carry fractional charge which means that these one excited state and another excited state would differ by these fractional charges and these as I said that these excitations would carry charge which are like the electrons themselves but they would carry fractional charges. So this is the only thing that's different here in the case of the fractional quantum Hall state. In the integer quantum Hall state as well they are the states are or the plateaus denote incompressible liquids and there of course the excitations are again local but then these excitations carry integers charges whereas these are fractional charges. So we want to now go out of the ground state that we have studied so far in the context of fractional quantum Hall state or the Laughlin state was predominantly the discussion that we had was about the ground state of the Hall fluid. Now we want to go out and talk about excitations so we are in the excited state and the excited state of this Hall fluid it comprises of excitations which I just said that they are local excitations and how do we now write down or modify the Laughlin state that we had written down earlier and let me write down.

$$\psi(z_1, z_2, \ldots, z_1', z_1', \ldots, z_M') = \prod_{j=1}^{M}\prod_{j=1}^{N}(z_i - z_j')\prod_{k<l}(z_k - z_l)^m e^{-\sum_{i=1}^{N}|z_i|^2/4l_B^2}$$

So these are Z1, Z2 etc are the electrons the coordinates of the electrons and Z1 prime, Z2 prime etc are the coordinates of the excitations. So these are complex space coordinates, coordinates in the complex space and so this is like Zn and just for making it a little distinct we call M excitations. So N denote the particles of the electrons of the system and M denotes the number of quasi particles there in the system and this can be written as J equal to 1 to M this is for the quasi particles and a product I from 1 to N this for the particles and we have an additional Jastrow factor which is like Zi minus Zj prime and we also have the usual K less than L Zk minus Zl prime I mean Zk minus Zl to the power M and exponential the ubiquitous Gaussian which is written as Zi square by 4lb square. This part has already been written down so that's the Laughlin state and now we have added a term which is another term which keeps the quasi particles away from the particles such that the particle density or the quasi particle density say the quasi particle density at the site of the particle is equal to 0. And this Zi denote the coordinates of the electrons and Zj prime denote the coordinates of the quasi particles.

Thus this is actually the excited state and now we need to understand the nature of the state and let me write it down this extra Jastrow term that you see here is it make sure that the these particles or the quasi particle has a charge which is minus E over M which we will call it as E star and these electrons of course, have charge minus E and this makes sure that the particle and the quasi particles do not come together.

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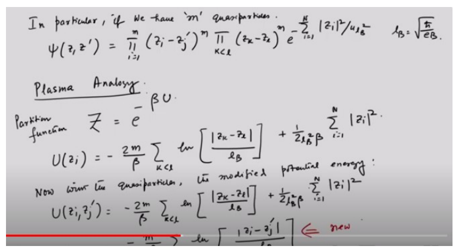

 So, in particular if we have we have M quasi particles I hope you remember that M is actually the magnetic quantum number for the state. So, then we can write down I take a shorthand notation to write this Z and Z prime this is equal to i equal to 1 to M and Zi minus Zj prime to the power M and then of course the usual ground state of Laughlin. E to the power minus i equal to 1 to N and Zi mod square by 4 Lb square Lb being the magnetic length that we have introduced several times earlier it is equal to h cross over Eb. So, it will help us if priori it is not clear, but it will help us if we try to normalize this wave function.

$$\psi(z_1, z_2, \ldots, z_1', z_1', \ldots, z_M') = \prod_{i=1}^{m} (z_i - z_j') \prod_{k<l} (z_k - z_l)^m e^{-\sum_{i=1}^{N} |z_i|^2/4l_B^2}$$

So, now normalizing this wave function is not very easy, but what comes handy is the plasma analogy. So, exploit the plasma analogy and in the plasma analogy the so we will write down the partition functions. So, there is the partition function was written as exponential minus beta U where U is the potential energy of the plasma and of course it is a cold plasma it is not plasma that we are usually familiar with which are at very high temperatures and just to remind us of the potential energy of a plasma that we had written down earlier without this quasi particle term is U now Zi now it is only the electrons is equal to minus 2 M over beta and this is k less than L log of Z k minus Z L divided by Lb and a plus 1 by 2 Lb square beta sum over i equal to 1 to n Zi square and so on.

$$Z = e^{-\beta U}$$

$$U(z_i) = -\frac{2m}{\beta}\sum_{k<l} ln[\frac{|z_k - z_l|}{l_B}] + \frac{1}{2l_B^2\beta}\sum_{i=1}^{N}|z_i|^2$$

So, that is the potential energy and now with the quasi particles the modified potential energy is and that is U Zi and Z j prime and that is equal to minus 2 M over beta k less than L log of Z k minus Z L the same things which we have written here and a plus 2 Lb square beta and i equal to 1 to n Zi square and now we also have a contribution coming from the quasi particles which is this is i less than j this ln Zi minus Z j prime that is a quasi particle term divided by Lb. So, this term is new, new term due to the quasi particles.

$$U(z_i, z_j') = -\frac{2m}{\beta}\sum_{k<l} ln[\frac{|z_k - z_l|}{l_B}] + \frac{1}{2l_B^2\beta}\sum_{i=1}^{N}|z_i|^2 - \frac{m}{\beta}\sum_{i<j} ln[\frac{|z_i - z_j'|}{l_B}]$$

So, once we write down this and let us see how it helps us. So, we will write down this Z I will write it big Z because so that you do not get confused with the coordinate the small z the complex coordinates that we are talking about.

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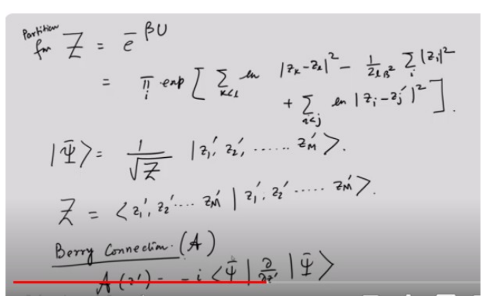

So, this is the partition function. So, just write partition function is equal to exponential minus beta U and this is equal to a product of i and then we have an exponential k less than L and a log of Z k minus Z L mod square minus 1 over 2 Lb square i Zi square and a plus i i less than j log of Zi minus Z j prime square. So, that is the partition function and how this partition function helps is what we are going to see.

$$Z = e^{-\beta U}$$
$$= \prod_i exp[\sum_{k<l} ln|z_k - z_l|^2 - \frac{1}{2l_B^2}\sum_i |z_i|^2 + \sum_{i<j} ln|z_i - z_j'|^2]$$

And so, just going back to what I said earlier that it is important to understand that we have to normalize the wave function and what do we do with the normalized wave function is something that we are going to come we are going to calculate the Berry connection and so, we are just trying to normalize the excited state that we get by including maybe M quasi particles in the system in addition to N particles that are there. So, we can write down this the state as it is like 1 by root over of Z and then we have a Z 1 prime a Z 2 prime and so on and a Z m prime. So, that the Z becomes equal to a Z 1 prime Z 2 prime Z m prime and a Z 1 prime Z 2 prime and a Z m prime. So, that is the partition function. So, this Z will be used in the partition function and we have to calculate the partition function using the plasma analogy.

$$|\psi >= \frac{1}{\sqrt{Z}}|z_1', z_2', \ldots, z_m' >$$
$$Z =< z_1', z_2', \ldots, z_m'|z_1', z_2', , z_m' >$$

And what do we do with that we calculate the Berry connection remember the Berry connection is analogous to the vector potential. So, in general there are two ways to get the topological invariant one is that you can calculate the Berry curvature and integrate it over the Brillouin zone or you can calculate the Berry connection and take a closed loop integral in the Brillouin zone. But of course, in this particular case we do not have a Brillouin zone because there is no symmetry being present in the system. But we will still calculate the Berry connection by using these as the parameters that is the these complex coordinates as a parameter. So, a this Berry connection is written with a with curly a say.

So, this is a Z prime this is equal to the definition is equal to a psi and a del del Z prime and a psi where Z prime the complex coordinates are the parameters that are used here.

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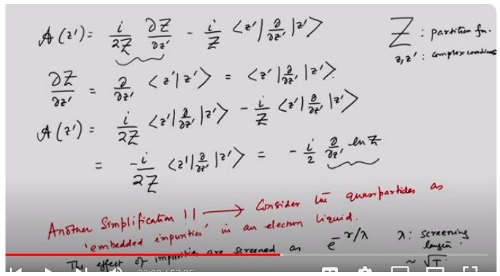

Alright so, we have to calculate the Berr connection and the Berry connection takes the form that this is equal to a Z prime this is equal to i over 2 Z del Z by del Z prime the one that is in the denominator or rather the what you take a derivative with respect to is the complex coordinate that is here. So, it is i over Z I am trying to write it as big as I can here and this is equal to as Z prime del del Z prime and a Z prime. So, just to make sure that this is partition function Z Z prime which are smaller complex coordinates. Then this is equal to so, this is what you get if you apply this psi this psi is this written as this the quasi particle wave function and you calculate the quasi particle or rather the Berry connection from the quasi particle wave function like this.

$$A(z') = \frac{i}{2Z}\frac{\delta Z}{\delta z'} - \frac{i}{Z} < z'|\frac{\delta}{\delta z'}|z' >$$
$$\frac{\delta Z}{\delta z'} = \frac{delta}{\delta z'} < z'|z' > = < z'|\frac{\delta}{\delta z'}|z' >$$

So, a step would bring you to this form. So, this is the form of the Berry connection that we have to calculate that is we have to take a derivative of the partition function with respect to Z prime. So, del Z del Z prime this is equal to a del del Z prime and a Z prime a Z prime because we have said that this capital Z is nothing, but this Z prime and the Z prime inner product of that and this is equal to a Z prime del del Z prime and a Z prime. So, the first term can be written as or rather the both the terms together can be written as i over 2 Z Z prime del del Z prime and Z prime minus i over Z Z prime del del Z prime and Z prime. So, this to put together will give you a minus i by 2 Z Z prime and del del Z prime and Z prime and this is nothing, but minus i over 2 del del Z prime of log of Z and

I hope you understand that log of Z is nothing, but the free energy of the plasma that we are talking about.

$$A(z') = \frac{i}{2Z} < z'|\frac{\delta}{\delta z'}|z' > - \frac{i}{Z} < z'|\frac{\delta}{\delta z'}|z' >$$
$$- \frac{i}{2Z} < z'|\frac{\delta}{\delta z'}|z' >= -i/2\frac{\delta}{\delta z'}lnZ$$

However, calculating this quantity is not so easy it is quite difficult to calculate this quantity, but one can get another layer of simplification if you recognize that these quasi particles can actually be taken or considered as impurities. So, we are talking about impurities in an electron liquid. So, it is like a gelium model because these quasi particles when they act like impurities if the effects of these impurities or these are charged impurities. So, this charged impurities the effect of them or rather the electrostatic potential due to these impurities are going to be screened away by the charges by enveloping them around them and the charge will electrostatic potential will fall off as exponential minus R by lambda. So, we will write as another simplification.

So, consider the quasi particles as embedded impurities in an electron liquid. This brings us to a picture which is known as a gelium model and these the impurities are going to be screened like the effect of impurities R by lambda where lambda is called as a screening length or Debye screening length. It is usually called as a Debye screening length and it in actual systems I mean they are they go as a root over of T where T is the temperature, but of course, here we are talking about plasma which is at very low temperature. So, let us not worry about this dependence on the temperature, but let us keep this in mind that the electrostatic potential really gets screened off. So, the effects of the impurities or rather these quasi particles cannot be felt at a distance which is larger than lambda.

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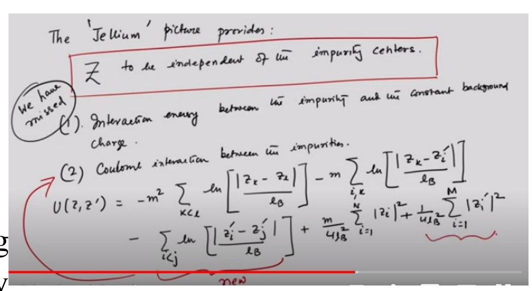

So, Debye screening length. So, that is really a simplification in the sense what will happen is. So, these the picture others this called as let us call it as a gelium picture. The gelium picture decides the partition function Z to be independent of the impurity centers ok. So, this is an important very important simplification that occurs which means that this will not have any effect on the impurity centers or these the Z prime these variables because their effects are going to be screened by the charges or the carriers of the system. It also brings in some subtle additional complications which we are going to deal with.

The complications that it brings is that it ignores so far we have ignored two important things one is the interaction in this picture in this gelium picture in which. So, it is like a plum pudding kind of model in which these the impurities are like a plums which are put in the pudding and this plums the effect of the plums are screened by the charges that constitute the pudding. So, overall the system maintains a charge neutrality. So, there is an interaction energy between the impurity and the constant background charge.

So, we have missed. So, these are the two things we have missed which we have to be included. So, the because of the charge neutrality you need a background positive charge this impurity and the charge that charge has is missed which will have to take into account and suppose we have many impurities in the system then thecoulomb interaction between the impurities. This were missed while writing down the potential energy of the plasma and these have to be taken into account now. So, we will have to write some long expressions, but that cannot be helped. So, this is u z z prime now would consists of this terms which are k less than l this is a z k minus z l divided by l b minus m i and k.

$$U(z,z') = -m^2 \sum_{k<l} ln[\frac{|z_k - z_l|}{l_B}] - m \sum_{i,k} ln[\frac{|z_k - z_i'|^3}{l_B}]$$
$$- \sum_{i<k} ln[\frac{|z_i' - z_j'|}{l_B}] + \frac{m}{4l_B^2} \sum_{i=1} N|z_i|^2 + \frac{1}{4l_B^2} \sum_{i=1} M|z_i'|^2$$

So, this is log of z k minus z i I mean z prime i by l b and then these new term that is going to be put that is i less than j log of z i prime minus z j prime divided by l b that is the potential energy because of the impurities. So, that is a coulomb potential or the which arises because of the coulomb interaction between the impurities and then this is m by 4 l b square k equal to 1 to n z well k for i does not matter i i mod square plus 1 over 4 l b square sum over z i prime square and this is i equal to 1 to m. So, this term is nu which is basically nothing, but this 2 and this term is nu as well and that is basically coming from the interaction energy between the impurity and the constant background charge.

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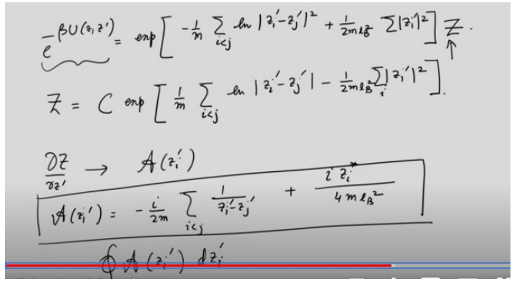

Alright. So, this gives a form for exponential minus beta uzz prime is equal to exponential minus 1 over m i less than j log of z i prime minus z j prime mod square plus 1 over 2 m l b square and a z i prime mod square and then multiplied by z this z we have written earlier ok that is the partition function of the plasma ok.

So,now this simplification that we have written down here that this z has to be independent of the impurity centers now comes into play then now of course, you see that this depends on the impurity centers which are z prime i's and z prime j's. So, this z that you see here has to annihilate the effect of this z prime i and z prime j. So, that tells you that this z has to take a form apart from a constant which is exponential 1 over m i less than j log of z i prime minus z j prime and a minus 1 over 2 m l b square z i prime square and there is a sum over i ok. Just to have cancel the effects of this so that no z prime either z prime i or z prime j or any of the z primes they stay in this quantity which is needed for us to calculate the Berry connection. So, that tells you that if you have this now you can take as del z del z prime ok and then now it is not difficult at all and this is required to calculate this a z i prime or a z prime ok.

$$e^{-\beta U(z,z')} = exp[\frac{-1}{m}\sum_{i<j} ln|z_i' - z_j'|^2 + \frac{1}{2ml_B^2}\sum |z_i'|^2]Z$$

$$Z = Cexp[\frac{1}{m}\sum_{i<j} ln|z_i - z_j| - \frac{1}{2ml_B^2}\sum_i |z_i'|^2]$$

So, a z i prime now becomes equal to minus 1 by i by 2 m 1 over z i prime minus z j prime and so, these are i less than j you see the log gives you a 1 over when you take a derivative with respect to z i it gives you this z i prime minus z j prime plus i z i star and a 4 m l b square you see there is a z i primes mod square.

$$\frac{\delta Z}{\delta z'} \to A(z_i')$$

$$A(r_i') = \frac{-i}{2m}\sum_{i<j}\frac{1}{z_i' - z_j + \frac{iz^*}{4ml_B^2}}$$

So, if you take a derivative with respect to z i you are left with a z i star. So, that is the z i star. So, this is the Berry connection that we wanted and what do we do with the Berry connection about the topological properties? We will take it in the parameter space we will take a closed contour integral which will give us a topological invariant so, called topological invariant and this topological invariant here turns out to be the charge the fractional charge that we get in this quantum hall effect. So, what we need is the following we need a z i prime a d z i I mean z i prime and then this over a closed contour. So, you can drop the i index and then I can do this integral and this integral will give you.

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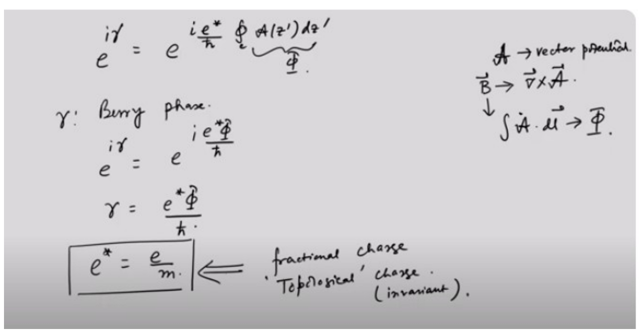

So, we can write down this exponential i gamma is equal to exponential i because your a is like the vector potential which means that the magnetic field is obtained from the curl of the vector potential ok. And this magnetic field is nothing, but the Berry curvature, but here Berry curvature is notof much use because we do not have any crystalline symmetry. For crystalline solids or crystalline systems where there is a band description available then one can calculate this Berry curvature. So, we are calculating the vector potential or the Berry connection.

$$e^{i\gamma} = e^{ie^*/\hbar \oint A(z'dz')}$$

So, this the line integral of this over any closed line will give us this phi ok the flux because a dot d l is nothing, but b dot d s because b is equal to curl a. So, then one gets a flux and that is exactly what one needs here. So, there is a E star over h and this is there is h cross and there is a c and there is a a z prime and d z prime ok. So, this is basically there is a Berry phase. So, gamma is the Berry phase and so if you equate the two and know that this is equal to the flux phi then exponential i gamma is equal to exponential i E star phi by h cross.

$$e^{i\gamma} = e^{\frac{ie^*\phi}{\hbar}}$$
$$\gamma = \frac{e^*\phi}{\hbar}$$

And then of course your gamma comes out to be E star phi over h cross and we know that E star is equal to E over m where m is that fraction say for the nu equal to 1 over 3 m is equal to 3. So, E star is equal to E over m. So, this is the fractional charge here is the topological charge. I mean it is a topological invariant or you can call it a topological charge.

$$\mathrm{e}^* = \frac{e}{m}$$

So, or and this is the topological invariant. So, this is how the topological interpretation or of the quantum Hall fluid can be brought about and in which we see that charge is quantized by these E over m and as that this will lead to the sigma x y quantization which is nothing, but a sigma 0 divided by m and so on and then that gives you these one-third plateau and so on.

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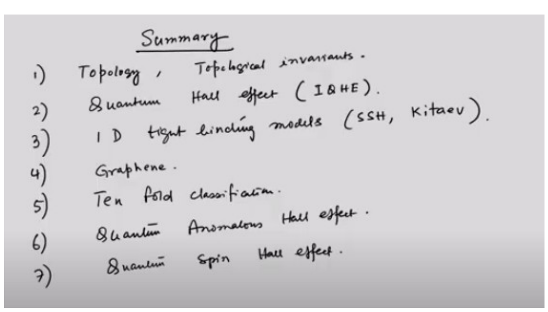

So, this is the topology of the quantum Hall fluid that I thought that is relevant for this to tie up the ends of this discussion on the fractional quantum Hall state which you cannot explain without invoking the interactions. So, in general the classification of matter is laid down by Landau in which he said that with systems with symmetries etcetera you can easily write down an order parameter and a microscopic order parameter for the system and which when either vanishing or diverging would give rise to a phase transition. And now these quantum Hall states as we have aid a number of times that they do not have any symmetries. We are not talking about crystalline systems, we are not talking about the time reversal symmetry which is anyway broken by the external magnetic field.

There is no charge conjugation symmetry, there is no chiral symmetry being present. So, everywhere the box is unticked that is the T, C and S which we have learnt all are unticked here which means that there is no symmetry present and in which case there is according to Landau's classification it is very hard to talk about an order parameter. Instead one can talk about a topological order that is present that characterizes the system and it is the plateaus occurring at fractional values which of course come from the fractional charge that we have just derived. So, there is a limited way of bringing in how the quantum Hall fluid is related to the topological considerations. So, just to rewind a little that we have of course taken help from the plasma analogy and this quantum Hall fluid is indeed a very new form of matter and this plasma really aids us in writing down the potential energy and from there one writes on the partition function of the system and this partition function.

Of course this beta that you have seen or exponential minus beta, beta is really the temperature which comes out from the it depends upon the m the index m which is nothing but the magnetic quantum number of the system and from there we had to undergo another level of simplification where we have seen these excited state comprising of quasi particle as the there are localized impurities that are present at these Z prime locations and because of these structure of the system we can consider that the charges are or other the effects of these charged impurities are screened by the constituent carriers of the system and that is how one achieves the simplification that the partition function would be totally independent of the impurity coordinates or the coordinates of the impurity particles and from there one can calculate the berry connection and take the integral of the berry connection over a closed path in the parameter space that again is is laid down by the Z primes which are the coordinates of the impurity or the quasi particles that are there.

So this in a limited way we establish the connection between topology and fractional quantum Hall state we have done it quite extensively in case of non interacting systems but fractional quantum Hall effect is strongly rather a new kind of system where the electron-electron interaction cannot be ignored and we have shown the importance of topology here as well.

So let me wind up here and say that what we have seen throughout this course. So we have seen we have seen the definitions of topology and how condensed matter physics and topology there is an interplay between these two seemingly different phenomena topology isa branch of mathematics which deals with continuous deformations of the system and is characterized by an invariant which we have seen that it's a the genus of sphere or a genus of an object remains an invariant under this smooth deformation. So it's a topology and topological invariance then we have done quantum Hall effect and to begin with IQHE was done and so this can be explained by a non interacting picture and a 2d electron gas placed in a strong perpendicular magnetic field will show quantized Hall plateaus which actually are nothing but topological invariance coming out which can be seen via the Kubo formula.

So the conductivity expressions will have an integer and that integer is identified with the topological invariant.So the formation of the plateaus and they being resilient to disorder and impurities is not a coincidence but it happens because they are protected by something quite significant. And then we have done 1d tight binding models and have seen topological implications in them. So in particular we have done the SSH model and a Kitaev model and they show a topological to trivial transition and we have also talked about the topological invariant which is here the the winding number. We have done graphene and graphene of course has prospects of topological insulator and actually Holden had shown that if you make the second neighbor hopping to be complex and it has a conditionality then that brings the time reversal symmetry.

So one can actually have quantum Hall effect without the necessity of having an external magnetic field or Landau level. So these are the quantum Hall effect without Landau levels. So graphene could be one of them but of course because of very small spin orbit coupling it is not a candidate for a topological insulator nevertheless graphene was done at length in the course and so on. So then we have looked at a tenfold classification and we have looked at the quantum spin I mean quantum anomalous Hall effect, quantum spin Hall effect. In each of these cases the tenfold classification aids us in understanding the topological invariant depending on the symmetries that are present in the system and the symmetries are categorized as T, C and S where T denotes the time reversal symmetry, C is the charge conjugation symmetry and S is the chiral symmetry which is usually the product of the two.

Then we have done this BHZ model which is a model for the CDTE, HGTE structures. Then we have done fractional quantum Hall effect and showed that how the fractional quantum liquid or which is like cold plasma has a topological implication there is a certain kind of statistics that is valid for the for in 2D which are relevant for this discussion of fractional quantum Hall effect and then briefly we have looked at 3D topological insulators. So I hope you have enjoyed the course you have learnt several things from this. This is the beginning of a research area that is very fertile and every day there are new results coming up, new materials coming up. So all these things that you learn have a lot of significance on the material properties, their transport properties, magnetic properties etc.

And that is how I hope that this course has served the purpose that it was meant for. Thank you very much.