Topology and Condensed Matter Physics

Prof. Saurabh Basu

Department of Physics

Indian Institute of Technology Guwahati

Lecture – 26

Composite Fermions, Hierarchy picture

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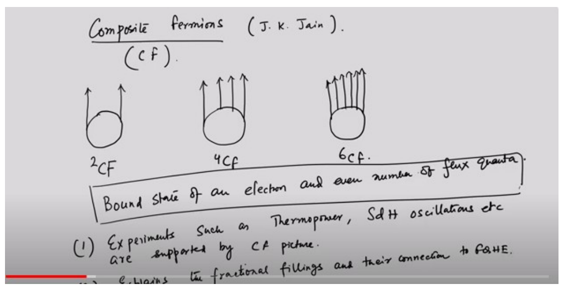

So, we will talk about maintenance. Mainly two concepts before we wind up the discussion on the fractional quantum hall effect. The first one of them is the composite fermions and this was introduced by J.K. Jain, ok. So, this is an alternative approach to understand the fractional quantum hall effect and particularly to understand the even denominator fractions that are observed experimentally whereas the Laughlin states are valid for the odd denominators which are of the form of 1 over m where m being an odd integer.

So, particularly the fractional feeling of nu equal to half was to be explained which is one of the simplest fractions, but of course, this cannot be explained with the concept of the Laughlin wave function, the Laughlin states which are for odd denominator, alright. So, this picture the composite fermion which we will call as CF as abbreviation. So, this is an effective picture it comprises of quasi particles which are the electrons they are carrying even number of flux quantum. So, they absorb the flux quantum from the external field and so they are considered as quasi particles and these composite fermions drastically reduces the complexity of the problem and one actually gets starting from a very strongly interacting fermionic system which is the nature of the quantum hall fluid we get a picture of non-interacting fermions.

And so this was a very nice idea and not only it is an idea, but it also has the implications of relevance on the experiments such as the Shubhnikov-Dihas oscillations and they also verified in thermo power experiments and so on. So, they are of course, ideas and in reality they are no the electrons do not absorb this even number of flux quantum, but it is a picture that aids us in understanding the fractional quantum hall effect, ok. So, let us try to visualize this. So, what happens is that so these are I am drawing an electron to be really large and this is carrying two flux quantum. So, let us call this as 2 CF and here they are 4 flux quantum.

So, let us call this as 4 CF and then there are say even larger number of flux quantum which are say the 6 of them. So, this let us call it as 6 CF, ok. As if it is a bound state of an electron and even number of flux quantum. So, that is the intuitive picture as I told you that it is not a real picture, it really does not happen that these fermions actually absorb even number of flux quantum, but suppose we assume that these are the there is a situation in for the fractional quantum fluid that these fermions have absorbed or they are really carrying this even number of flux quantum. So, of course, the picture changes the intuitively and it gives rise to something that is quite interesting and as I said that it has a relevance or rather it conforms to the experiments that are made or rather they are performed and these picture supports those experiments, ok.

So, two artifacts happen. So, it is experiments, there are others as well, but we have not I have not written them such as thermo power, SDH oscillations, we have discussed this in our earlier lecture Shubhnikov-Diaz oscillations etcetera are supported by the CF picture. Number 2, so it of course, explains the fractional feeling of the Landau levels and basically their and their connection to FQHE.

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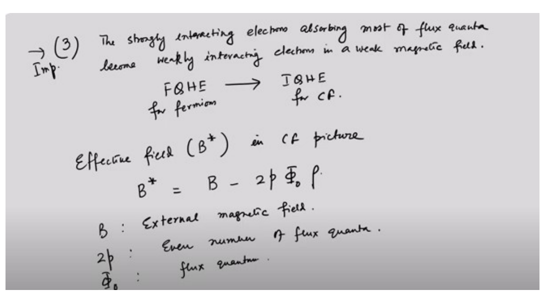

And third and probably the most important thing which I just said is let me write it in the next page 3 and this is quite important. So, this says that the strongly interacting electrons absorbing most of the flux quantum quanta rather become weakly interacting electrons in a weak magnetic field and this tells you that the latter corresponds to the integer quantum

Hall effect which we have seen. So, this picture of FQHE, this goes over to IQHE. So, this is for fermions IQHE for composite fermions, ok. So, this is a big simplification that occurs, ok. So, let us see what happens and how it happens and so on, ok. So, we just said that the electrons actually absorb the flux quanta.So, the effective field is reduced.

$$B^* = B - 2p\phi_0\rho \qquad \text{(Equation 1)}$$

So, let us write down the effective field which we will call it as B star which is actually reduced in the CF picture is given by. So, B star is equal to B which is an external field and then there are these even number of flux quantum which we write as 2 p and then each one carrying a flux phi and the density of the composite fermions. So, B star of course, that is written. So, let me write B which is the external magnetic field, ok. 2 p even number of flux quantum phi 0 of course, that that is a flux quantum which is equal to h over e and rho density of CF.

So, this tells you that the magnetic field, the effective magnetic field or the reduced magnetic field is given by this expression which is this B is the external field and now since these fermions have absorbed most of the flux from the external field. So, the field now remains as this and let us now see that what this means in terms of the electron filling.

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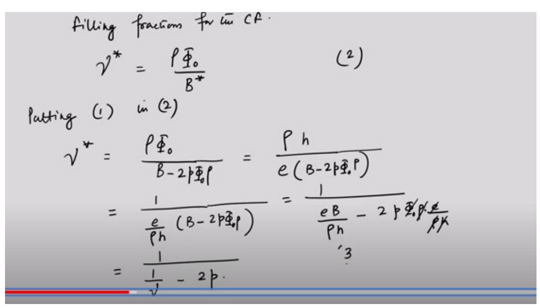

So, the filling fraction for the CF let us call that as nu star in keeping with the B star which is equal to rho phi 0 divided by B star, ok. And this is of course, understood that the B is pointing in the z direction.

$$\nu^* = \frac{\rho\phi_0}{B^*} \qquad \text{(Equation 2)}$$

So, we should also mention that here. So, B is in this direction, the external field is in this direction and let us call this as equation 1 and this as say equation 2. That is the definition of a filling fraction which is rho that is the density of the fermions a composite fermions multiplied by phi 0 divided by the B star. So, a putting 1 in 2 what I mean is put this B star B minus 2 P phi 0 rho in 2, one gets a nu star to be equal to rho phi 0 divided by B minus 2 pi phi 0 into rho, ok. So, I write phi 0 as H over E and this is E and this B minus 2 P phi 0 rho. Now what I do is I take this rho also down and then rho H E to be down.

$$\begin{aligned}\nu^* &= \frac{\rho\phi_0}{B - 2p\phi_0\rho} = \frac{\rho h}{e(B - 2p\phi_0\rho)} \\ &= \frac{1}{\frac{e}{\rho h}(B - 2p\phi_0\rho)} = \frac{1}{\frac{eB}{\rho h} - 2p\phi_0\rho\frac{e}{\rho h}} \\ &= \frac{1}{\frac{1}{nu} - 2p}\end{aligned} \qquad \text{(Equation 3)}$$

So, this is like a 1 divided by E over rho H and a B minus 2 P phi 0 rho and then little bit of algebra. So, I take this in so it is a E B by rho H and a minus 2 P phi 0 rho into E by rho H. So, rho will cancel and E by H is 1 over phi 0. So, that will cancel as well and we are left with EB over rho H which is nothing but the filling fraction of the original fermions, ok. So, this is the inverse of it rather. So, it is 1 over nu minus 2 P and so on. So, this is the thing and so let us call this as equation 3.

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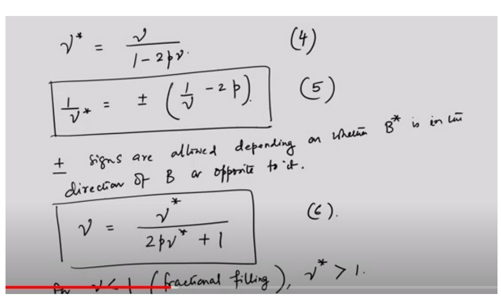

And then we can write down the nu star in terms of this nu to be equal to nu minus 1 minus 2 P nu and let us call this as equation 4. So, this is the expression for the filling of the composite fermions and its relation to the filling of the actual fermions which are there in the system and this P is equal to 1, 2, 3 and 2 P denotes the number of vortices or the number of flux quantum which are actually vortices I will just come to that, ok. So, it can be actually inverted. So, 1 over nu star this is equal to a plus minus.

$$\nu^* = \frac{\nu}{1-2p\nu} \qquad \text{(Equation 4)}$$

$$\frac{1}{\nu^*} = \pm(\frac{1}{\nu} - 2p) \qquad \text{(Equation 5)}$$

So, this is equal to 1 over nu minus 2 P and so this is the filling fraction relationship between the filling fraction and just take a note that we have written both plus and minus signs and they would give rise to a new different fractions. So, to say both signs are actually allowed. So, plus and minus signs are allowed depending on whether B star is in the direction of B or positive 8, ok. So, this is let us call it as 5 is it is same 4 and 5 are same is just written in a different language and this can again be written in another language in which you can see that. So, if you write down the nu in terms of nu star then it becomes equal to a 2 P nu star plus 1 and this is the equivalent expression where nu is the filling of the original problem of the Hall conductance or Hall resistance.

$$\nu = \frac{\nu^*}{2p\nu^*+1} \qquad \text{(Equation 6)}$$

And so, for nu less than 1 which is what we are talking about fractional filling nu star is greater than 1, ok. And when the nu star becomes large it corresponds to the or rather it is an integer, ok.It is equal to 1 or greater than 1 and becomes an integer then it corresponds to the non-degenerate state and the non-degenerate state of course, would correspond to weak interaction or no interaction at all which means that we go to the case of IQHE, ok.

So, let me make this a little more clear because. So, the filling factor is defined in the following way.

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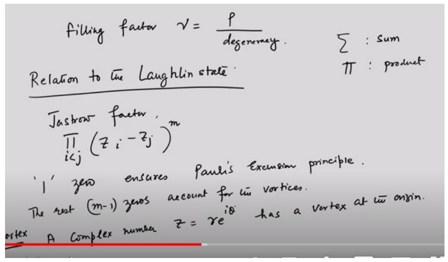

It is also called as filling fraction. So, this is equal to a rho divided by a degeneracy, ok. So, now, if rho that is a density of electrons fermions that becomes equal to the degeneracy which means that the integer number of Landau levels are filled and if it becomes 2 say the rho is twice of the degeneracy which means 2 Landau levels are completely filled and this situation would correspond to exactly what we have studied in the context of IQHE that is integer quantum Hall effect and the system corresponds to non-degenerate states and non-degenerate scenario and that should explain the non-interacting electrons or very weakly interacting electrons which we have studied in the context of IQHE, ok. So, this is the picture that emerges out of this composite fermion picture and let us try to understand what is its relation to the Laughlin state. This is quite a nice explanation.

$$\nu = \frac{\rho}{degeneracy}$$

So, follow me carefully. So, remember the Jastrow factor and the Jastrow factor actually says that, so there is a Zi minus Zj, I might have written Zj and Zk, these are just indices which you can take according to your convenience. So, there is this M and then of course it is multiplied by the Gaussian. Let us look at the Jastrow factor only. You would recognize that for each electron, so Zs are the coordinates of the Zi denote the coordinate of these electrons and this is basically enforces the anti-symmetric property by when M is an odd integer and it also says that if Zi becomes equal to Zj, this is not only 0, but it is 0 M times because of the product that is involved here.

I hope this notation is clear. We write a summation by this and we write a product by this. So, this is a sum and this is a product, ok. So, this product of all i, i is less than j and then it is a 0 of Mth order and that is what comes out from the Jastrow factor and that is what is embedded in the Laughlin state. Now, of course we need one 0 of those M zeros and one 0 we need because of we want to enforce the exclusion principle.

What happens to the M minus one zeros? So, these M minus one zeros account for what are known as vortices. So, let me write as one 0 ensures Pauli's exclusion principle. The rest M minus one zeros account for the vortices and what are the vortices or what is a vortex? A vortex is actually a topological excitation. So, suppose a complex number, a complex number z equal to r e to the power i theta all of you are familiar with this polar representation. It has a vortex at the origin, ok.

$$z = re^{i\theta}$$

So, which means that if a particle encircles this origin it picks up a phase 2 pi. So, that theta actually changes from theta to theta plus 2 pi and the complex number remains unchanged in the sense that because you have a 2 pi i exponential 2 pi i is equal to 1, ok.

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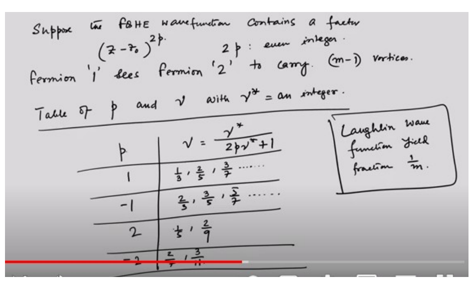

So, that is the meaning and suppose these the FQHE wave function contains a factor z minus z 0 to the power 2 p, ok. We are writing M as 2 p in the sense that these are just to make p to be any integer not only odd integer. So, 2 p will be. So, a p could be any integer. So, that 2 p is even, 2 p is an even integer. So, you see here you have these M minus 1 is an even integer because M is an odd integer. So, M minus 1 is an even integer and so we write that as 2 p. So, this means that a fermion 1 sees fermion 2 to carry 2 p vortices or M minus 1 vortices, ok.

And so basically this means that even number of vortices which is what the this composite fermion picture it proposes. So, let me tabulate a few p and nu, p and nu with nu star to be an integer and I told you that this corresponds to the IQHE when nu star becomes an integer. So, this is a p and a nu which is equal to a nu star and which is equal to a 2 p nu star plus 1 the formula that we have written earlier and let me just show it for p equal to 1. So, this gives rise to one-third, two-fifth and three-seventh and so on, ok. For nu star to be an integer and for p equal to this, p equal to 1 here.

Let us say p equal to minus 1 we get other integers which are two-third, three-fifth and five-seventh and so on and so 2 will give rise to 1 by 5 and 2 by 9 and minus 2 will give rise to 2 by 7 and 3 by 11. So, you see all these fractions have been seen in experiments they have been realized in experiments and that is why just having nu star to be a integer values with p to be plus minus 1 plus minus 2 you can extend it to plus minus 3 etcetera. We are getting all kinds of fractions which were not realized in the Laughlin wave function I mean with the with the Laughlin wave function which is only valid for 1 over m. Else fraction 1 over m where m is a odd integer. So, of course 1 by 3 is in that form, but none of these other 1 by 5 is also of that form, but none other are included in the Laughlin wave function and these are called the non Laughlin states.

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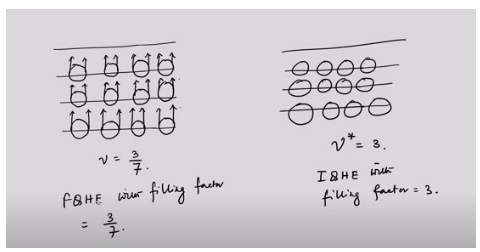

So, let me show you a picture an intuitive picture of this whole thing. So, we will draw so this 1, 2, 3, 4. So, we will have this usual fermions and then the composite fermion picture and where filling fractions will be compared. So, these are these fermions which are carrying these vortices of the flux quantum and this corresponds to the original picture of fermions. This is like this 4 of them in each of the 3 Landau levels and this corresponds to a filling fraction to be nu equal to 3 by 7 ok.

So, this is a fractional quantum Hall effect and when we go to the non interacting picture or the picture of so this ok so this corresponds to nu star equal to 3 ok. So, this is the as a schematic plot this shows i QHE with filling factor equal to 3 and this corresponds to the f QHE with filling factor equal to 3 by 7 ok. So, this is the problem of f QHE becomes a problem of i QHE of the composite fermions ok.

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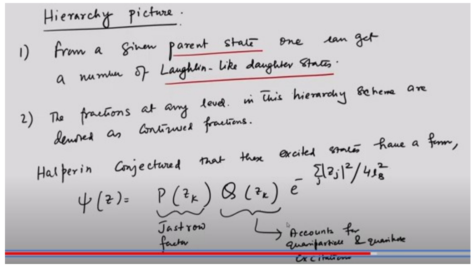

And let us do the last thing in this regard we still are missing a number of fractions ok even though we have been able to go beyond the Laughlin picture and get a lot of other fractions from this composite fermions, but we still are not have not got all the fractions that are experimentally observed ok. And there is a hierarchy picture which again is very subtle the idea is very subtle though getting a lot of other fractions are not too difficult.

So, this relies on the fact that there are other filling fractions that are observed in experiments starting from some parent states one can get Laughlin like daughter states ok. So, from a given parent state so these are Laughlin like daughter states ok so these Laughlin like daughter states and a parent states which are very interesting here and these fractions at any level another important thing. The fractions at any level in this hierarchy scheme are denoted as continued fractions. You might have seen this continued fractions even in school these the fractions carry on and here we will have these different at a given level of the parent state we will have a multiple daughter state emerging out and we will give a large number of these fractions coming out and so on.

$$\psi(z) = P(z_k)Q(z_k)e^{\sum_j |z_j|^2/4l_B^2}$$

It is not that all the fractions are seen in experiments which are predicted from this hierarchy scheme but quite a few of them have been observed ok and it is not that you know at a given Laughlin the daughter states at a given hierarchy of the daughter states there could be many fractions and the probability to observe these fractions in actual experiments it seems to be the same whereas some of them are just not observed at all some of them are of course observed that tells you that this of course doesn't distinguish between one fraction at a given level from another fraction.

Now physical picture is necessary and the physical picture really relies upon not the ground state which we have been talking about so far but it relies on the excited state and so you have a problem of you say a quantum mechanical problem and you have a Hamiltonian and you may be able to solve that Hamiltonian exactly or maybe there are mathematical difficulties or there are computational difficulties and so on but it is very easy to understand that these quantum mechanical Hamiltonian will have a large number of states possible which are energy states that could be degeneracies and so on but forgetting the degeneracies for the moment there are ground state there are first excited state and there are second excited states and so on rarely ever one needs to go very far away from the ground state in condensed matter physics because we always talk about low-lying excitations which are which are the most important thing I mean something that's far away from the ground state may not be taking part in either the thermodynamic or the transport properties at all.

So it's quite important that and we are understanding that the Fermi level lies very close to the ground state and that's why the ground state is the most important thing however in certain scenarios you may have to go beyond the ground state and take into account excitations and these excitations in this fractional quantum Hall fluid are called as quasi particles and quasi holes depending upon the shortage of density or the excess of density. So if you have excess of density of particle like excitations which are like quasi particle excitations and if you have deficit then of course you have hole like or the quasi hole like excitations. So Halperin wrote down so conjectured that these excited states have a form which are given by the psi of Z this is a P of Z K which is our usual Jastrow factor I mean Z K minus Z J product of that and so on so forth and then there is a Q which is again a function of Z K and the ubiquitous these Gaussian that would of course be there. We are familiar with this, this is a Jastrow factor and this accounts for quasi particle and quasi hole excitations and what did he proposed for these Q Z K?

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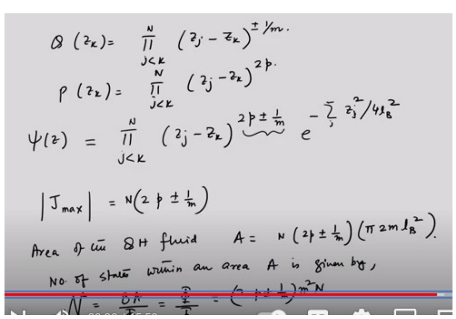

So the Q Z K he proposed to have a form which is equal to it is again a product of J less than K and then it just goes all the way up to M and it is a Z J minus a Z K whole to the power plus minus 1 over M.

$$Q(z_k) = \prod_{j<k}^{N} (z_j - z_k)^{\pm 1/m}$$
$$P(z_k) = \prod_{j<k}^{N} (z_j - z_k)^{2p}$$

We will not go into the details that was the ansatz that was made by him and of course the P Z K that is the Jastrow factor of course has this form J less than K to the power M it is a Z J minus Z K to the power 2 P. So if you combine them then psi of Z so this is the excitation above the Laughlin state and the excitation has been included by these this Q Z K term which of course talks about either particle or hole the excitations of the system. So at the middle of the plateau a quantum hall plateau the density is unifor and so there are number of states that are pinned to a defect or an impurity. Now as the magnetic field is ramped up it is increased beyond a certain value of the magnetic field these fields or these states actually break away from the impurity and they would cause a density imbalance. So if the density becomes larger we have a quasi particle excitation and if the density falls below then it becomes a it becomes a deficit it becomes a quasi hole excitation .

$$\psi(z) = \prod_{j<k}^{N} (z_j - z_k)^{2p \pm 1/m} e^{-\sum_j z_j^2/4l_B^2}$$
$$|J_{max}| = N(2p \pm \frac{1}{m})$$

So this is the idea of this so total psi of Z becomes the product it is J less than K and of course it goes to M and then it is a Z J minus Z K whole to the power 2 P plus minus M 1 over M and then of course that Gaussian thing will of course be there. So it is Z J square divided by 4 L B square. So that is the form of the wave function the excited state wave function M is still an odd integer which is what Laughlin had said. Now of course you can understand that this gives you instead of just M this gives you the total J max. So J max the magnitude of that it becomes equal to 2 P plus minus 1 over M and I will have to multiply it by N because there are N electrons and I am just talking about the magnitude.

$$A = N(2p \pm \frac{1}{m})(\pi 2ml_B^2)$$
$$N = \frac{BA}{\phi_0} = \frac{\phi}{\phi_0} = 2p \pm 1$$

So this is N electrons and 2 P plus minus 1 over M. So this filling fraction corresponding to this scenario can be obtained if we note that the area of the quantum Hall fluid is given by A equal to so N then it is 2 P plus minus 1 over M which is that J and then we multiply it by the pi into 2 M and L B square. So that is the area and the number of states within an area A is given by N curly N. So N is equal to a B A over phi 0 where A is given by this which I just wrote so this is equal to phi over phi 0 this is equal to 2 P plus minus 1 over M and M square N.

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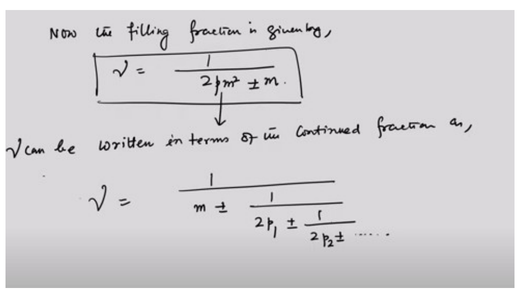

So that is the number of states that you obtain and when we get the number of states the filling fraction is given by this. So nu is equal to 1 by 2 P M square plus minus M that is the filling fraction and this filling fraction can be expressed in terms of a continued fraction but importantly it needs to be noted that there is a term which is this which is in addition to the M term that is there in the denominator.

$$\nu = \frac{1}{2pm^2 \pm m}$$

So the Laughlin state only had this M term because Laughlin states only remained in the ground state. So it is a ground state wave function. So the hierarchy so the continued fraction or rather nu can be written in terms of the continued fraction as. So nu is equal to 1 divided by so this is for for different values of P and different values of M odd integer M and different values of P.

$$\nu = \frac{1}{m \pm \frac{1}{2p_1 \pm \frac{1}{2p_2 \pm \dots}}}$$

So this is equal to 1 plus M plus minus 1 divided by 2 P 1 plus minus 1 divided by 2 P 2 plus minus and so on so forth. So this is the continued fraction that we talked about and so this is like a P is having all these values 1 2 3 etc and M has values 3 5 7 etc and then one can actually form these continued fraction.

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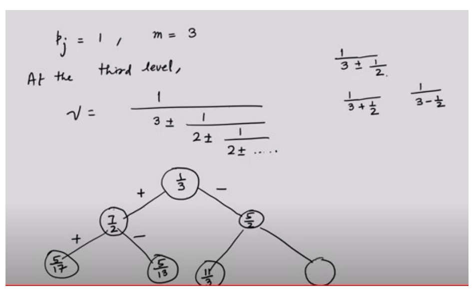

So let me show you one example of this continued fraction. So let's just write for all the P's let's call it a P j to be equal to 1 just to have a simple form so and M let's say equal to 3 that's also simple. So at the third level one has so nu becomes equal to 1 divided by 3 plus minus 1 divided by 2 plus minus 1 divided by 2 plus minus and so on.

So at the third level for corresponding to this thing and I mean till the third level then you can just drop everything out here and then calculate what fraction is coming out from this continued fraction. So let me show you the picture for M equal to 3 and this P j equal to 1. So the picture is like this so we have this so at the third level so this there is 1 so this is the parent state is equal to 1 over 3 so M equal to 1 over 3 and there are two branches that come out from the plus and the minus sign. So let's write the plus branch here on the left and the minus branch on the right. So at the P 1 level so this is P j equal to 1 so P 1 level at the all the P j is equal to 1 but you can have other fractions where P 1 equal to 1 and P 2 equal to say 2 and so on so forth.

So this is equal to 7 by 2 that comes out from the positive branch that is like 1 over 3 and plus so this is how it's coming is that 1 over 3 and then you have a plus and a minus and 1 over 2. So this becomes equal to so there are 1 by 3 plus 1 half that is giving you the 7 by 2 branch and then so that's the plus branch and the minus branch is coming out as 3 minus half and that is that branch gives you 5 by 2. So these are the fractions that are there and let me go to another level so this plus branch here that gives you 5 over 17 and of course there is a 5 over 13 from the minus branch. Now this one will have a plus branch and the minus branch as well and this is coming out as 11 by 3 do it carefully and 7 by 3 coming from the plus branch and the minus branch. So at the third level so starting from 1 Laughlin fraction we get so many non-Laughlin fractions coming out of this the hierarchy picture and it gives you of course a large number of fractions not all of them are seen and at the same level it even though it predicts that the probability of getting this fraction should be same but that does not happen. Thank you.