Topology and Condensed Matter Physics
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Lecture – 14

Hall quantization and Topological invariant

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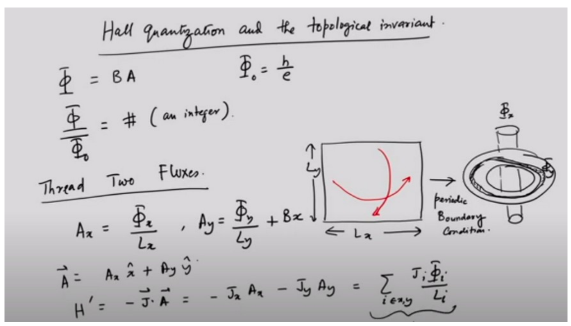

Let me show a very important thing in this discussion particular topic in this discussion which makes the study of quantum Hall effect so interesting. I have mentioned this several times that the system which is a two dimensional electron gas is a dirty system it does not have any translational invariance that is there is no space translational invariance and no time reversal invariance because of the presence of the magnetic field there. However the plateaus are very robust there is so much so that they are used in the metrology of you know fixing the value of the resistance. So what makes this so robust is it really some intrinsic constant that is coming into the picture and because to show that we had introduced various formalism that is related to studying a crystal lattice in presence of a magnetic field. We will continue doing that because we will do graphene but let me take a break for the moment and show you that the Hall plateaus are related to a topological invariant which is called as a Chern number or in general which has a name as TKNN invariant I have mentioned this earlier it is in the name of four people. So that is an invariant that we are going to derive for the Hall conductivity and we have derived the form of the Hall conductivity via the Kubo formula.

So Hall quantization and the Chern number, let me write it instead of writing Chern number let me write topological invariant and in order to do that we will follow a trick and the trick is basically just to suit our requirements you know that if there is a magnetic field there will be a flux of the magnetic field which is nothing but the strength of the field multiplied by the area that it pierces through. So we have talked about such fluxes and we have you know denoted them by this phi which is nothing but B into A, A is the area through which it is threading. We have also seen that you know if this phi by phi0 and this phi0 being H over E which is a quantum if this is a number ok. So an integer say for example, a whole number that is then the system properties will remain invariant.

$$\phi = BA$$
$$\phi_0 = \frac{h}{e}$$
$$\frac{\phi}{\phi_0} = \sharp$$

So as soon as it takes a number such as say 1, 2, 3 the system won't will remain invariant that is the properties of the system will remain invariant. The interesting thing occurs when it is not an integer and a fraction of the form p by q where p and q are co-prime integers. So we want to sort of study the effect of this flux for the quantum Hall system. Now we have done this and it is not that we have not done this we have done this exactly when we derived the Kubo formula ok. But here in order to link it to the topological invariant or which is a churn number which we call it as a churn number let us apply a trick such that we thread not one flux, but there are two fluxes.

So and what I mean by that is the following. So take a square system and of length say Lx and Ly ok. And now you use a periodic boundary condition let me show it with a color that is you fold it in this direction and you fold it in this direction ok. So we just fold them in both the direction in the x and the y direction. So the resultant structure becomes a torus ok, a torus like this which we have seen earlier in the context of what we have called as a Corbino disk or a Corbino ring.

$$A_x = \frac{\phi_x}{L_x}, \quad A_y = \frac{\phi_y}{L_y} + B_x$$

So it becomes like a torus of this form or you have seen donut so it is like that a structure like that. So this is under periodic boundary condition. Now this derivation the way we are following is very typical and only used in a few places particularly you can see this article by David Tong he does that, but actually originally it has been done in a different way which appears in the paper by this T K N N Thaulis, Komoto, Night Angle and Dennis. And then so what we mean by threading two fluxes. So we have Ax which is equal to a phi x by Lx and we have Ay which is equal to a phi y and Ly and a plus a Bx ok.

So that is we have thread two fluxes and they look like as if you know there is a flux that is threading here let us call that as phi x and along this we hread a phi y ok. So phi x is through the opening of the torus and the phi y is in the annular region ok. So there is a phi y flux that is threading this just in keeping with the structure that we have drawn in the vertical which is that phi x ok. Then in that case of course, your A becomes equal to Ax x cap plus Ay y cap ok. And we remind you that the perturbation term which we have used in deriving or rather deducing the Kubo formula is H prime it is equal to minus j dot A and in this particular case.

$$\vec{A} = A_x \hat{x} + A_y \hat{y}$$

$$H' = -\vec{J}.\vec{A} = -j_x A_x - J_y A_y = \sum_{i \epsilon x,y} \frac{J_i \phi_i}{L_i}$$

So it will be like minus jx Ax plus or rather a minus a jy Ay and so on ok. So this is exactly similar to earlier excepting that we are now talking about threading two different fluxes as I said it is just to help us in getting the result. In fact, this will tell you just in a few steps down the line it will tell you that we come very close to the Kubo formula and this is actually aiding that ok. So this can be written as so I which is xy this is equal to a j i of phi i by Li where i is x and y. So phi i is phix by Lx and then phiy by Ly and then this is the one that you have here ok.

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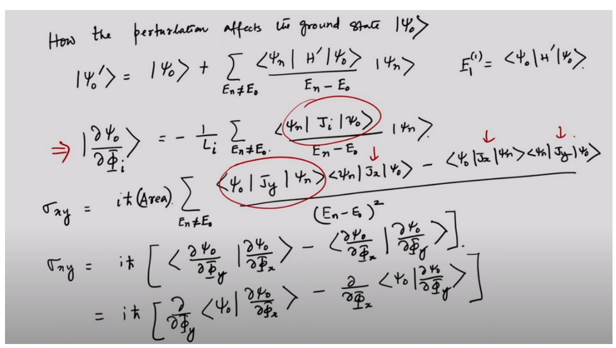

And this is the perturbation of course,and this perturbation we want to see how this perturbation affects the ground state ok. So the idea is to see the perturbation affects the ground state, ground state with a psi 0 and this psi 0 as we have discussed earlier it can be a manybody state by and it can ontain any term excepting the j dot a term which appears because of the inclusion of the magnetic field or the magnetic flux ok. It can also have interaction terms. So we want to see the effect of H prime on this by using a first order perturbation theory ok. So H prime is seen as a perturbation and then we can write down this psi0 prime it is equal to a psi0 in the first order we can write this as some n or m does not matter I mean this n is not equal to psi0.

So this is not the ground state. So it will promote the j dot a term will promote particular state or rather it acts on a ground state and the perturbation promotes it to the system to the excited state. So this is all let me write it as psi n. So this psi n is not equal to psi 0. So this psi n and H prime and then psi 0 and then it is a En minus E0 and what I have done is that I have introduced a completeness of states and that is why I sum over psi n which is not of course, equal to psi0 because if that is the case then the denominator diverges ok.

$$psi_0' >= |\psi_0> + \sum_{E_n \neq E_0} \frac{<\psi_n|H'|\psi_0}{E_n - E_0}|\psi_n>$$

$$E_1' =< \psi_0|H'|\psi_0>$$

$$|\frac{\delta\psi_0}{\delta\phi_i} = -\frac{1}{L_i}\sum_{E_n \neq E_0} \frac{<\psi_n|J_i|\psi_0>}{E_n - E_0}|\psi_n>$$

So we are careful in not dealing with degenerate systems and make sure that this H prime really promotes the system from the ground state to a state which is a low lying one of the low lying excited states ok. So this is the expression for psi 0 prime which is a new ground state because of this perturbation this is the first order correction. Of course the first order correction in energy is given m by so if you want that then the first order correction in energy is simply given by a psi0 H prime and psi 0. We are interested in the state not the energy here ok. So if you consider this psi n how psi n a response to an infinitesimal flux then what we can find out is that we can find out a del psi0 and a del phi ok.

So that is the how a system responds to an infinitesimal flux and that is nothing, but 1 over Li and a sum over this psi n of course, not equal to psi 0 or we can write it in instead of writing it psi n not equal to this thing we can just simply write it En not equal to E0 that is probably more correct way of writing ok. So this so En not equal to E0 and this is equal to a psi n and a J and i of course, and then you have a psi 0 and the En minus E0 and psi n ok. So I have just written H prime as Ji phi i and then taking this derivative of this wave function the ground state wave function for an infinitesimal change in the flux. So you thread the flux and thread it slowly we have discussed this in the context of Corbino ring and what we mean by slow varying of the flux. So we do that and then we land up with this expression for this del psi 0 del phi I which tells you that how the ground state response to the perturbation.

Now just reminding you of the Kubo formula the Kubo formula said that the sigma xy some geometrical factor which is this a let me write a area itself ok because this a already there as a vector potential and then you have these En not equal to E0 and then it is a psi 0 and a Jy and a psi n and a psi n psi n Jx psi 0 and minus psi 0 Jx psi n psi n Jy psi 0 and divided by En minus E0 square you should go and look back the derivation of the Kubo formula and this is exactly what we had written earlier. Now you see that this is this term there which looks like this term here ok and each of these terms I mean this one as well as this one and so on this one and so on. So they look like this so this quantity that is this del psi 0 del phi i actually enters into the Kubo formula and that is why we will write each of these terms in terms of this del psi 0 del phi i ok. So that tells you that my sigma xy can be written as i h cross which is there and then of course there is a del psi 0 del phi y because of this Jy then a del psi 0 del phi x now it is a Jx because of this Jx term and then this is equal to and each one of them is bringing along En minus E0. So that makes it En minus E0 square and of course we know that these En and E0 are different so that the denominator is not allowed to blow up and then so you have a del psi 0 del phi x and a del psi 0 del phi y ok.

$$\sigma_{xy} = i\hbar \sum_{E_n \neq E_0} \frac{<\psi_0|J_y|\psi_n><\psi_n|J_x|\psi_0> - <\psi_0|J_x|\psi_n><\psi_n|J_y|\psi_0>}{(E_n - E_0)^2}$$

$$\sigma_{xy} = i\hbar[< \frac{\delta\psi_0}{\delta\phi_y} | \frac{\delta\psi_0}{\delta\phi_x} > - < \frac{\delta\psi_0}{\delta\phi_x} | \frac{\delta\psi_0}{\delta\phi_y} >]$$

$$= i\hbar[\frac{\delta}{\delta\phi_y} < \varphi_0 | \frac{\delta\psi_0}{\delta\phi_x} > - \frac{\delta}{\delta\phi_x} < \psi_0 | \frac{\delta\psi_0}{\delta\phi_y}$$

So this is your sigma x so which can further be written in a slightly different form in which we do a del del phi y and do a psi 0 and del psi 0 del phi x and minus del del phi x and you have a psi 0 and a del psi 0 del phi y. So this is a form of the Kubo formula and this is what we will you know sort of deal with. So this is a present form of Kubo formula in terms of this present variables where we have introduced two fluxes. Now you see that the why we have introduced two fluxes is because we needed to get this Jx and J y and that is why the J dot a otherwise if it has one component then we will get just you know we will not get a formula like what we have done. I mean there is no restriction on threading a flux as long as the requisition or the parent conditions are being satisfied ok.

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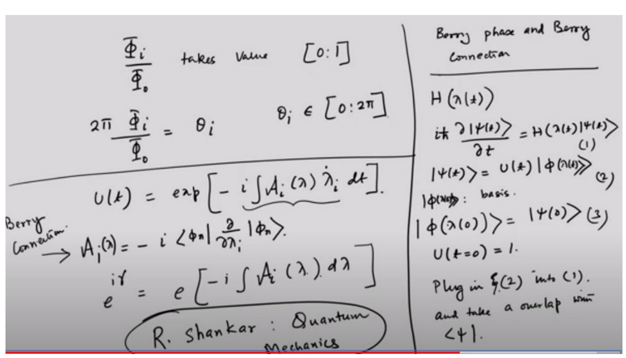

Now let me sort of introduce a variable which is a variable that varies from 0 to 1 and it can take value any fraction. So we can write this as a phi i divided by phi 0 which takes values between in the limit 0 and 1 and because this is what we have discussed that the system actually when you thread the flux slowly each one of the fluxes phi x and phi y are being increased from some 0 to phi 0 and then of course phi 0 to 2 phi 0 and so on. So I am just talking about just you know between 0 and 1 but then of course it takes any integer values I mean it can go from 0 to n. The idea is that in order to you know make more sense let me multiply it by 2 pi such that we find out an angular variable and this is like a phi i by phi 0 is let me call that as an angular variable say let us call it as a theta i so that theta i varies from a 0 to 2 pi ok. I have just changed the condition so that I do not have to talk about any number and I can periodically talk about 0 to 2 pi that is when phi i divided by phi 0 takes some arbitrary values ok.

$$2\pi\frac{\phi_i}{\phi_0} = \theta_i$$

So now let me sort of take a little bit of time off from this and let me introduce quantity called as a Berry phase and Berry connection and we will come back to this in just a while ok. So what is Berry phase? So Berry phase is the simplest demonstration of how you know geometry and topology both can emerge in a quantum system ok. In order to understand it better let me sort of write down a Hamiltonian which depends on a parameter lambda and which is a function of t. It is not only one parameter that it can depend on it can depend on a number of parameters like lambda 1lambda 2 lambda 3 lambda 4 etcetera but we just talking about just one of them without any harming the generality of the discussion and this is a function of t. So lambda is a function of t and so h implicitly is a function of t ok.

$$i\hbar\frac{\delta|\psi(t)>}{\delta t} = H(\lambda(t)|\psi(t))> \qquad \text{(Equation 1)}$$

$$— \psi(t) >= U(t)|\phi(\lambda(t)) > \qquad \text{(Equation 2)}$$

$$|\phi(\lambda(t)) >= |\psi(0) > \qquad \text{(Equation 3)}$$

Now what equation do we have to solve? We have to solve a time dependent Schrodinger equation which is i h cross del psi t del t which is equal to h lambda t psi ok. So this is the equation that we have to solve and the general solution of this is psi of t is equal to some u of t and psi and let us introduce a basis which is say a phi of t ok. So phi of t is the basis for the problem. Now what we can do is that of course this phi of t it depends on both lambda and t. So in fact we will do better justice if we write it as phi lambda of t, but this denotes the basis so this lambda of t.

So this is how the wave function evolves and if we claim that phi of lambda at 0 at t equal to 0 if that is equal to psi at t equal to 0 if this is true then we can fix that u at t equal to 0 is equal to 1 ok. Now let me take this space as well and we will come back to this thing this discussion that we have been doing. So now the whole idea is that we want to find or determine this u of t as this lambda is changed via changing t over a full cycle that is you start from a point and then you come back to a point after a complete rotation and then you ask the question what happens to u of t does it ultimately you know the question is that whether it picks up a phase which is irreducible and it is not the usual dynamical phase that we are aware of and usual dynamical phase is nothing but exponential i e t by h cross.

So does it pick up a phase that is anything more significant and then that does not go away because this dynamical phase does not appear in the probability density because the moment you take the mod square of a wave and the this thing goes away, but however, here it is it is important and let me just write down the solution of this you can follow R. Shankar's quantum mechanics book for a very nice and detailed solution or rather discussion on this Berry phase and so on ok.

So u of t it is just basically this ansatz had to be plugged into this equation. So this is equation 1, this is equation 2 and maybe this is equation 3. So, if you plug in 2 into 1 and then use of course, the condition that then you take a overlap with this and take a overlap with a conjugate psi ok. Now if they do that then u of t comes out to be some exponential minus i and this is written with a curly a and this is a i lambda and so this is a lambda i dot, dot means it is a d lambda i dt and then there is a dt there. So, this phase is not like the dynamical phase where this a i is called as a Berry connection which is defined as a minus.

$$U(t) = exp[-i \int A_i(\lambda)\dot{\lambda}_i dt]$$

$$A_i(\lambda) = -i < \phi_n|\frac{\delta}{\delta\lambda_i}|\phi_n >$$

$$e^{i\gamma} = e[-i \int A_i(\lambda) d\lambda]$$

So, this is a function of lambda and that is how it the time dependence in the Berry connection enters it is of course, a vector quantity it is minus i and you have a phi n del, del lambda i and phi n and so on. So, this is your Berry connection which enters in the integrand and inside the you know exponent of this u of t and this particularly this quantity is called as a Berry phase. So, this e to the power i gamma which is equal to e to the power this minus this a i just make sure that you do not think that this a is the vector potential that is why I am writing it with a curly a this is a lambda and a d lambda. So, this is called as a Berry phase. So, this is related to this the time evolution of the u of t operator and it is called as a Berry phase and that is why you know the Berry phase is a very important quantity it is one of the topological markers of a system a Berry phase that is different than 2 pi will tell you that there is something non trivial going on in the system and I will not prolong this discussion, but as I said that please look at this R Schunker quantum mechanics.

In fact, this Berry connection is like a vector potential actually like a vector potential and if you take the curl of that it gives you a quantity which is called as a Berry curvature and this Berry curvature when you integrate over the entire Brillouin zone that gives you the topological invariant namely the churn number all right.

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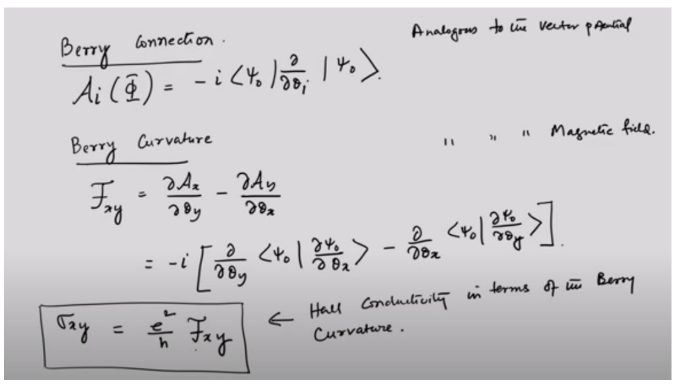

Then of course,we are nearly done we write down the Berry connection for this particular problem which we have been doing with the 2 fluxes. So, the Berry connection is this a i and which is a function of phi now it is that lambda is nothing, but phi here and minus this and a psi0 and del del theta i theta i is a variable angular variable anda psi 0 ok. So, this is called as a Berry connection and from there you can calculate the Berry curvature.So, the Berry connection is analogous to the vector potential ok.

$$A_i(\phi) = -i < \psi_0|\frac{\delta}{\delta\theta_i}|\psi_0 >$$

$$F_{xy} = \frac{\delta A_x}{\delta\theta_y} - \frac{\delta A_y}{\delta\theta_x}$$

$$= -i[\frac{\delta}{\delta\theta_y} < \psi_0|\frac{\delta\psi_0}{\delta\theta_x} > - \frac{\delta}{\delta\theta_x} < \psi_0|\frac{\delta\psi_0}{\delta\theta_y} >]$$

$$\sigma_{xy} = \frac{e^2}{h}F_{xy}$$

And the Berry curvature which is analogous to the magnetic field can be obtained by taking a curl of that and one writes it as a curly f and xy which is equal to del this curly Berry connection by theta y and minus del a y del theta x let us take a curl basically. So, b equal to curl a so, this is like a b which is what we have said and when you do that it becomes something that is familiar. So, it is a del del theta y let me just remind you of this. So, we had this del del phi y which is now a del del theta y and so on and these all these things will be written in the present notation which is del del theta y and a psi 0 and a del psi 0 del theta x and minus del del theta x which is equal to a psi 0 del psi 0 del theta y and then this is what is the Berry curvature. And now if you go back and just take a onto one correspondence with the Kubo formula then you will see that the Kubo formula the conductivity tensor can be written as a minus e square by h and these f of x y.

So, this is Hall conductivity in terms of the Berry curvature. So, this is that formula that the Hall conductivity is expressed in the form of Berry curvature and we have already introduced this k space representation by going to square lattice and writing down you know Bloch's theorem etcetera. So, this will help us actually to calculate the Berry curvature and so on for a conductivity.

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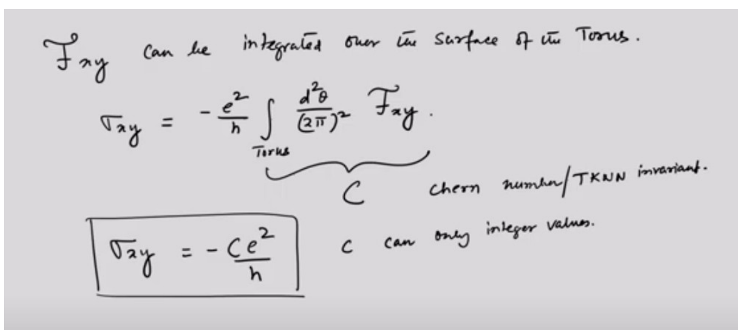

So, this can be these quantity which is a Berry curvature can be integrated to basically show that this is over these over the surface of the torus which is what we have done to the square shaped system as if the 2D electron gas is confined there you do not have to talk about the 2D electron gas it could be any system could be a crystal lattice integrated to the surface of the torus ok. Now again going back to the discussion that we had done at some point of time that this integration of this Berry curvature which is like a Gaussian curvature and when you integrate over the entire system which in this case in this particular case it is the Brillouin zone.

$$\sigma_{xy} = \frac{-e^2}{h} \int \frac{d^2\theta}{(2\pi)^2} F_{xy}$$

So, then this sigma xy for a k space system that is a crystal lattice this surface of the torus is nothing, but the Brillouin zone is what I wanted to say. So, that all our discussions that we are having now would be valid in case of a crystal lattice whether you take it a square lattice or you take it more exotic lattices all these discussions will go through. So, sigma xy is nothing, but a minus e square over h and this is over a torus and you integrate this thing over a torus on this angular variable and this is equal to that ok. So, this is a very important expression and this quantity is called as a churn number. So, which is a topological invariant I mean or we can call it a TKNN invariant ok.

So, that your sigma xy becomes equal to a C with of course, you can either absorb the minus sign or even if you do not it does not matter C e square over h and C can take only integer values. I leave it this discussion at this point, but may come back to this later that why C is necessarily an integer, but of course, the proof has been provided by the experiment. So, the experiment says that the Hall conductivity is quantized in terms of e square over h and this churn number can take values which are 1, 2, 3, 4 and so on and this is called as a Z invariant ok. It can take any value any integer value and it is necessarily an integer and that is the reason that it is so robust. Because of this discussion if you follow right from the beginning that we had done just now you will see that it is completely general and it does not talk about any crystal lattice or 2D electron gas we have just taken a sample and have introduced periodic boundary condition threaded to flux and wrote down a simple perturbation theory in that J dot a minus J dot a term and then we have cast it in the form of Kubo formula and this Kubo formula gives you the Hall conductivity which is some C times e square over h may be minus C times e square over h where C remains an integer because C is an integer we are going to see plateaus in the Hall conductivity necessarily the plateaus would survive ok under this condition alright.

So this is very fundamental in nature and that is why the Hall effect is such an important experiment and it warrants a completely new look at the systems. The system is not yet interacting but if suppose we include the interactions and then more exotic things will happen which we will see in the fractional quantum Hall effect but however at this point it is the C denotes the integer number which corresponds to the plateaus in the integer quantum Hall effect. Thank you.