Topology and Condensed Matter Physics
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Lecture – 13

Kubo formula

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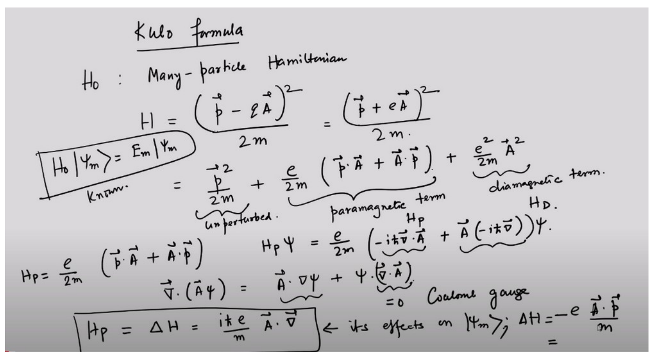

We shall talk about Kubo formula basically the formalism to calculate conductivity and this particularly we keep in mind about the Hall conductivity. So let us see how the Kubo formula is derived and this is derived within linear response theory okay. So this is an example of calculating conductivity within a linear response theory. So the assumption is such that one has many body system so it is not just one particle which we have considered while we consider Schrodinger equation and solved for a single electron say in presence of a magnetic field. So there is a presence of many particles and so we talk about a many particle Hamiltonian. So this is that many particle Hamiltonian. Now why I write it with a 0 is that this does not contain the perturbation term that we are going to talk about. However this can contain inter particle interactions present in the system okay. So that is not an embargo here so H0 may contain inter particle interaction and now we are going to talk about the perturbation that is introduced because of an external field.

Now in this particular case that we have talked about it is the electromagnetic field rather there is an electric field in the longitudinal direction and there is a perpendicular magnetic field being present. However it is enough for us to talk about only one kind of field and that could be say just an electric field but then it can be generalized into both electric and magnetic fields.

So let me write down a general Hamiltonian in presence of an electromagnetic field. We can do a particular case for an electric field later. So we have said this that the Hamiltonian actually comprises of p minus q a whole square over 2m where this a is the vector potential that comes in because of an external field and one can actually talk about the gauge freedom because these a the vector potential can be uncertain by an amount which is delta lambda where lambda is a scalar function and this is perfectly acceptable because even if you add or subtract a gradient of a scalar function b equal to curly is still satisfied. So if we expand this Hamiltonian and if you want to write it specifically for the electron then you introduce this q equal to minus e.

So this is my Hamiltonian.

$$H = \frac{(\vec{p} - q\vec{A})^2}{2m} = \frac{(\vec{p} + e\vec{A})^2}{2m}$$
$$= \frac{vecp^2}{2m} + \frac{e}{m}(\vec{p}.\vec{A} + \vec{A}.\vec{p}) + \frac{e^2}{2m}\vec{A}^2 \qquad \text{(Equation 1)}$$

Now what I am going to do is that I am going to expand this. So this is a p square over 2m and then there is a 1 over 2m and then I have a term I can write it as e over 2m and then there is aterm which is like p dot a and a dot p. Now in general they do not commute that is why we have to write it separately but if you choose a particular gauge called as a Coulomb gauge which we will just see then these two terms can be combined into one term and then of course you have a term which is a square over 2m and e square over 2m into a square and this term is called as the paramagnetic term and this is called as a diamagnetic term. Now it turns out that we will neglect this diamagnetic term and in the parlance of perturbation theory you can say that you know if b is not strong enough then a would also be weak so a term which is of the order of a square can be neglected.

However we have seen that in our case the magnetic field b is not weak at all and then we need to take into account terms such as b square or a square. However it does not contribute to the Hall conductivity even though it contributes to the longitudinal conductivity however since the main focus is on Hall conductivity will drop the diamagnetic term altogether and only worry about the paramagnetic term. This is of course the unperturbed problem which is solved in the sense that these unperturbed problem we know that suppose this and this acts on some wave function psi m will give you a em psi m and that problem is known. So which means that this unperturbed term which is a part of the many particle Hamiltonian is known or at least if you even add inter particle interaction the assumption is that that problem is completely solved and we have to only look at the effect of the paramagnetic term. So this term can be written as so we specialize into this term for the moment which is p dot a plus a dot p and suppose we act so this that the paramagnetic term so let's let me call this as HP and let this call this as HD D for the diamagnetic and P for the paramagnetic.

$$H_p = \frac{e}{2m}(\vec{p}.\vec{A} + \vec{A}.\vec{p})$$
$$H_p\psi = \frac{e}{2m}(-i\hbar\vec{\bigtriangledown}.\vec{A} + \vec{A}(-i\hbar\vec{\bigtriangledown}))\psi$$
$$\vec{\bigtriangledown}.(\vec{A}\psi) = \vec{A}.\bigtriangledown\psi + \psi.(\vec{\bigtriangledown}.\vec{A})$$

So HP is like this so if I want to understand that how do I do a simplification of this I consider an arbitrary state psi and then this will be like e over 2 m now your p dot a is minus ih cross del and dot dotted with a and plus a into minus ih cross del and this acts on psi. So look at this term this term actually looks like a del and a psi where a is the vector potential so this is like a dot del psi and plus a psi equal to psi del dot a.

So there are these two terms that are present now you see that this is actually there are two terms and these term if we consider that a gauge where this equal to 0 this is called as a Coulomb gauge and in this Coulomb gauge it only is left with this term which is same as this term. Okay so with this minus ih cross adjusted there and so these two terms can be combined and one can write down just one term which is so this paramagnetic term which now let me call it as Delta H or H prime whatever you want to call it is that extra term whose effect needs to be you know considered and this is nothing but this is equal to e over ih cross so this is a minus ih cross e over m and a dot del. Okay so this we want to consider its effects on psi m.

$$H_p = \triangle H = \frac{-i\hbar e}{m}\vec{A}.\vec{\bigtriangledown}$$

Okay which is the unperturbed state so we are really doing a perturbation theory but in this case it's slightly different notionally from the perturbation theory that you might have learned in your second course of quantum mechanics or even in the first course of quantum mechanics this is done on a system of many particles. Okay so once we know this so our Hamiltonian has a form which is apart from these constants which we can take into account so the constants are like this, this is nothing but so if you take the minus ih cross this DeltaH takes a form equal to so minus ih cross del so this is e a dot p over m and so the p is the momentum so this is p over m is v and e into v is equal to so we can write this as so there is I think there is a sign that is missed here so this there should not be any sign so there is a sign that comes here and that tells you that this v over m p over m becomes equal to v and then e into v becomes equal to current density so this is actually equal to minus j dot a.

$$\triangle H = -e\frac{\vec{A}.\vec{p}}{m} = -\vec{J}.\vec{A}$$

Okay so we are going to work with this as the perturbation which is minus j dot a and this perturbation the effect of this perturbation to linear order in a is what we actually talk about when we do this okay.

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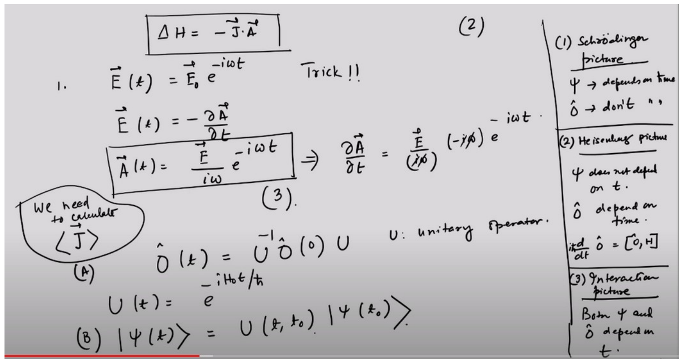

So let me write down the perturbation term and this effect of this would be considered on the conductivity of the problem and in particular we talk about the hall conductivity okay it can also be used in order to calculate the longitudinal conductivity which we are not doing here but can be done and also it is important to remember that you need to take into account the a square term which is nothing but the diamagnetic term here okay. So this is our like our starting point of a perturbation theory in this particular case now in order to proceed further there are a few simplifications or even you can call them as tricks these tricks are being introduced to you know make life simpler and to achieve the goal in a manner that we wish to such as we there is no need but of course we take because these fields are DC fields DC means they are constant fields either the electric field or the magnetic field however we take them to be an alternating field that is we talk about an AC field and with which sort of oscillates with characteristic frequency Omega and as I said that there is no problem if we simply talk about just the electric field even though we know that there is a hall there is a magnetic field there which moves the electrons in the transverse direction and that's how the pile up the charges pile up at the edges which gives rise to hall conductivity however we will simply talk about electric field and do a linear response theory only with the electric field okay.

$$\triangle H = -\vec{j}.\vec{A} \qquad \text{(Equation 2)}$$

So let's consider an electric field of this form so this is your E0 and say exponential minus i Omega t as I said it's a trick and the trick is introduced at this stage for the reason that we are going to talk about a gauge which requires taking a time derivative 1 and secondly of course keeping in mind that we are really talking about DC fields we will put Omega going to 0 so as a limiting case okay.

$$\vec{E}(t) = \vec{E_0}e^{-i\omega t}$$

So this is a particular kind of harmonic dependence on Omega that we have introduced here and what we do is that so your E is obtained from minus delA delt we have taken it's also a minus grad Phi which we have we can scale it to 0 and the time or this from the Maxwell's equation the electric field is obtained from the time derivative of the vector potential. So this is your E of t and it's precisely for this reason that we have taken it is a function of time because your A of t it's really like E divided by I Omega exponential minus i Omega t okay.

$$\vec{E}(t) = -\frac{\delta \vec{A}}{\delta t}$$

$$\vec{A}(t) = \frac{\vec{E}}{i\omega} e^{-i\omega t} \Rightarrow \frac{\delta \vec{A}}{\delta t} = \frac{\vec{E}}{i\omega}(i\omega) e^{-i\omega t}$$

So this is your A and if you take a derivative so take a derivative so you take a del A del t or a da dt that's equivalent so we will have a E over i Omega and we will have a minus i Omega e to the power minus i Omega t so this Omega will cancel and i will also cancel and so on so you get this E equal to minus delA delt that relationship to be valid and that's why so this is the form of the vector potential in terms of the electric field okay.

So now we know or rather we have assumed a form for the vector potential we can put that back into this equation so let me now go one step back and sort of start naming these equations so let's call this as equation 1 and this perturbation term to be equation 2 and let's call this as equation 3 and so on. So we are to find you know the this potential rather these perturbation term which has a form J dot A and A is assumed to have a form like this at the end after do we do all the calculations we will put Omega going to 0 to take the DC limit okay alright.

So let me calculate the current or the expectation value of the current in these perturbed states so these states that includes the perturbation so we calculate J so J is the current density that flows due to this perturbation term that we have so we need to calculate this alright. So this is the job that we have in hand and understand that I remind you of three pictures that in basic quantum mechanics you might have been exposed to one is called as the Schrodinger picture in which you have the wave function that depends upon time but the operators do not so let's call any operator as don't depend upon time okay. So this is the Schrodinger picture and then we have Heisenberg picture where it is just the opposite that is psi does not depend on time however the operators depend upon time just the reverse scenario and if you remember that we have actually calculated the equation of motion by taking a d dt of O equal to OH where O is any operator and H is a Hamiltonian as ih cross I have to write here and so this is the Heisenberg equation of motion.

A third picture which is very important for the many particle system in fact all the many particle system the greens function and all the perturbation theory that we talk about are developed in the interaction picture and both where both psi and O depend on time okay. So this is in a nutshell this is the different pictures of quantum mechanics that one deals with now it is very important to realize that when you calculate the expectation value of an operator which is a physical observable it won't depend upon which picture you adapt to rather it will be independent of the picture and all those results will be same in all the representations.

In any case since I said that the in the interaction picture both psi and H are dependent on time H means the Hamiltonian which is an operator that depend upon time so we will resort to the interaction picture here because you saw that the Hamiltonian or at least the H delta H which we are interested in is time dependent through these A the vector potential.

So we will resort to this picture and of course we will have to evolve the wave function so wave functions are definitely time dependent as the time proceeds or passes by then the wave function evolves as well all right. So any operator let's say we have been writing it as O but let's say any operator in this picture the interaction picture it evolves as some U inverse A of 0 and U which is a unitary operation and U is actually a unitary operator which is written as exponential minus I H naught T over H cross you can set H cross to be equal to 1 otherwise you have to carry it all the way. In any case this is a unitary operation that one can perform on any operator such as the Hamiltonian and in this particular fashion and we will see the time evolution of the operator. So A is the vector potential in this particular case we are not talking about vector potential as an operator this A is okay let me change this then let me go ahead with O because A is already the vector potential and so on.

$$\hat{O}(t) = U^{-1}\hat{O}U$$
$$U(t) = e^{\frac{-iH_0\epsilon}{\hbar}}$$

So this is the O is the an arbitrary operator and this is your the unitary transformation so U is an unitary operator.Okay. So this is my operator and how does the wave function behave this is also standard basically that comes from these interaction picture this evolves as some T to T naught and then psi is T naught. So let us call it as A and let us call this as B so this is how in the interaction picture the operator and the wave function evolve.

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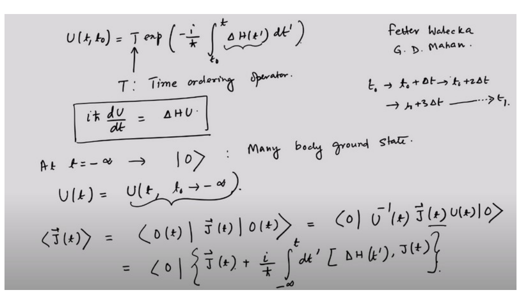

All right so we will have to work in this interaction picture and perform a perturbation theory in order to get the expectation value of an observable. In this particular case the observable is nothing but the current density okay. All right so this one you know the U T of T0 in the interaction picture again is written as minus I by h cross some T0 to T delta of h T prime and a d T prime and so on. Now it's very important to understand in the interaction picture this is actually the time evolution operator invokes the perturbation term or the interaction term. So here you can treat both of them to be with you know interchangeably the what do you want to call them I mean this same as the interaction term in the language of the many particle systems.

$$U(t, t_0) = Texp(\frac{-i}{\hbar}\int_{t_0}^{t} \triangle H(t' dt')$$

Okay. So this is how the unitary operator evolves a system a state psi of T from an initial state T 0 to a final state T. Okay. And that involves the interaction term or the delta H that is here. Okay. There is another quantity called as T this is called as a time ordering operator. Now if you want to know in details you can look at these Fetter Walichka there is a book by Fetter Walichka or there is a book by G. D. Mahan and there are other sources available on the YouTube personally I have a course on these advanced condensed matter physics which deals with the developing the formalism of the perturbation theory from the scratch. So what it does is the following it orders the times you see that there is a T 0 to T and we are writing it as an integral which means that it goes from T0 to T0 plus delta T then it goes to T0 plus 2 delta T and then it goes to T0 plus 3 delta T and so on till it reaches you know the T1. So you need to have a mechanism which orders these from you know the earliest time to the right or left whichever I mean basically to order these times in the sequence that they are you know occurring and why is that important it is important because your delta H at time T1 will not commute with delta H at time 2 in general.

Okay. Otherwise if they commute at all times then there is no problem I mean then the problem is much simpler and because you have a time dependent problem which explicitly depends on time and then you need to keep a track the way they are occurring. So this is that time ordering operator that you come across. Okay. And it of course sort of make sure that you know the ih cross du dt is equal to some deltaH into u. So that's that is the Schrodinger equation for or the equivalent of the Schrodinger equation for the operator.

So that's the time dependent Schrodinger equation for this unitary operator u. Okay. So let us understand that at time t equal to infinity minus infinity sorry at time t equal to minus infinity you had nothing you had only the bare system. Okay. And at time t equal to minus infinity you have switched on the electric field.

$$i\hbar\frac{dU}{dt} = \triangle HU$$

Okay. So the system that you let the system evolve and come to at the present time or at t equal to 0 and so on. At this time the many body ground state let's write this as 0 it's a ground state but nevertheless some many body ground state which may have you know interactions inbuilt into it but of course we do not have any effect of delta H at t equal to minus infinity and we'll call this state as 0 with a ket understanding that 0 is actually a many body ground state. Okay. Fine. So all right so we'll do it step by step.

So u of t it is actually u of t and with a t0 from minus infinity. Okay. So we set t0 the earliest time the most primitive time to be at minus infinity when there was no effect of the perturbation and after that the perturbation is switched on. Okay. So the t0 goes to minus infinity and because now it depends only on one time variable we simply call it as u of t.

$$U(t) = U(t, t \to -\infty)$$

Okay. So you always calculate some expectation value in a perturbation theory by using the different orders of perturbation theory and so on. So we are going to calculate J of t and the J of t can be calculated within this 0 that is many body ground state evolves a J of t and a 0 and t and so on. So this you are calculating the expectation value between the known states. So this 0 is at t equal to minus infinity but that many body ground state would evolve with time that's why we have written 0 of t and we still assume this is a intrinsic assumption of perturbation theory that you still have the 0 that is a ground state to be a valid description of the system. So your the delta H term hasn't taken the system too far away from the ground state so that there is no point in talking about the expectation value with respect to the ground state.

But the perturbation theory intrinsically assumes that it still is a you're not too far away from the ground state and that's why the ground state expectation values can still give you meaningful you know corrections because of the perturbation to the unperturbed energies or other quantities like here we are talking about J average of J.

$$< \vec{J}(t) > = < O(t)|\vec{J}(t)|O(t) > = < O|U^{-1}U^{-1}(t)\vec{J}(t)U(t)|O >$$
$$= < 0|\{\vec{J}(t) + \frac{i}{\hbar}\int_{-\infty}^{t} dt'[\triangle H(t'), J(t)]\}$$

So what is this? So this is equal to the 0 and then you have a U inverse so I'm evolving the state as we have done it here, here U of t now t0 is minus infinity so this and then J of t and then U of t and then a0. So this 0 is that many body ground state at t equal to minus infinity all right. So we will have this as so this is 0 and this is a J of t and plus i over h cross minus infinity to t and then I use a dummy variable dt prime and then I have a commutator which is delta h and so this is at a t prime and then a J of t and so on. So this is the let me use another bracket so that it doesn't look like the commutator bracket. Where does this come from? You have to understand that I have done a simplification or rather skipped one step let me show you that step here.

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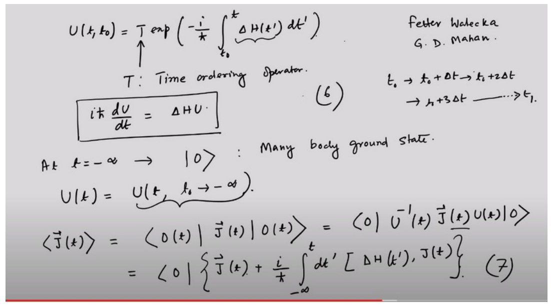

So our U of t is simply equal to now of course a time ordering is of course required at its minus ih cross minus infinity to t h delta h of t prime and dt prime. So this is your that and then I have to calculate this quantity U inverse t Jt and U. So U inverse t J of t and U of t so this is equal to the time this thing and exponential i over h cross minus infinity to t delta delta h t prime and dt prime and now I will write a J of t and then I will write exponential minus i by h cross minus infinity to t delta h of t prime and dt prime. So now what we do is that we make this expansion that exponential minus i by h cross this minus infinity to t and delta h t prime dt prime I write it as 1 minus i by h cross minus infinity to t delta h delta h t prime dt prime and then of the order of delta h square is neglected ok.

$$U(t) = Texp(-\frac{i}{\hbar}\int_{-\infty}^{t} \triangle H(t')dt')$$

$$U^{-1}(t)\vec{J}(t)U(t) = Texp(\frac{i}{\hbar}\int_{-\infty}^{t} \triangle H(t')dt')J(t)Texp(-\frac{i}{\hbar}\int_{-\infty}^{t} \triangle H(t')dt')$$

$$e^{(-\frac{i}{\hbar}\int_{-\infty}^{t} \triangle H(t')dt'} = 1-(-\frac{i}{\hbar}\int_{-\infty}^{t} \triangle H(t')dt' + O(\triangle H^2)$$

So this term is neglected because the intrinsic assumption of a perturbation theory is that we will not of course here we want to see it in the linear response of the system that is to first our power in a how does the system behave or how does the properties of the system behave and also it is true that in keeping with the assumption of perturbation theory we are not considering any higher order ok. So this thing let us call it so this is equation number 3 let us say this is equation number 4 this equation number 5 this is equation number 6 and this is equation number 7 and so we will have maybe this one as equation number 8 this is equation number 9.

So if you put this in equation 8 putting equation 9 in equation 8 so we have a 1 plus because this u inverse is 1 plus i by h cross minus infinity to t delta h t prime dt prime and a J of t ok. J of t is here in equation 8 and then 1 minus i by h cross minus infinity to t delta of h t prime and dt prime ok. Alright so I have two brackets at the two ends and I have a J t in between so I will just do a multiplication so this will be simply a J of t that is 1 J t and 1 and then second term will be i by h cross I have infinity to minus infinity to t delta h t prime J of t ok.

$$(1 + \frac{i}{\hbar}\int_{-\infty}^{t} \triangle H(t')dt')J(t)(1 - \frac{i}{\hbar}\int_{-\infty}^{t} \triangle H(t')dt')$$

$$= J(t) + (\frac{i}{\hbar}\int_{-\infty}^{t} \triangle H(t'))J(t)dt' - J(t)(\frac{i}{\hbar}\int_{-\infty}^{t} \triangle H(t')dt')$$

$$= J(t) + (\frac{i}{\hbar}\int_{-\infty}^{t} [\triangle H'(t), J(t)]dt'$$

And of course you have to write dt prime there you can write it and then the next term will come with a minus sign i by h cross that is 1 Jt and the term that is there you understand that just do this simplification and then you will have so a J of t and a delta h of t prime and a dt prime you can actually keep the J t outside this thing ok. The J of t can be outside and you have a minus infinity to t and so on ok. So this is your term and this can be written as J of t that is the first term and then I have plus i over h cross now you see that this is a delta h J and this is J delta h so this is like a commutator so this is written as a minus infinity to t dot dt prime and this h prime of t delta h of t delta h and a J of t it really does not matter because the integral is over dt prime so it is not touching the t that dependence of J.

So this is the commutator so this is the commutator that I have talked about at the end of the last slide here that commutator ok. So and then there is of course a term which is the J of t which is here as well so and now this J of t is not important for the reason that this J of t is a current density without an external electric field ok which can be there due to a variety of reasons but it is not important for us we are only interested in this commutator and which arises because of the electric field alright.

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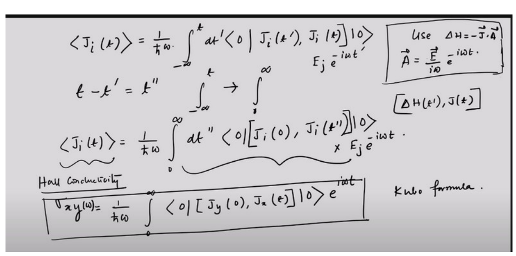

So we got an expression for that for your J now let me get an expression for just a component of J that is maybe Jx or Jy or Jz and so on so forth so this is equal to equal to 1 by h cross omega and I now I put the explicit form I use your delta h equal to minus J dot a and a I use it as what has been written earlier that use a as E over i omega exponential minus i omega t ok alright. So if you do that then the particular component of the current that looks like this and this is equal to 1 over h cross omega and then minus infinity to t and dt prime and a 0 and Jit prime and a Jit and a 0 and the you have to write the electric field so this is Ej exponential minus i omega t prime ok. So the commutator of delta h and its t prime and J at t takes this form because delta h is nothing but J dot a and then you use the relation for a which we have assumed within a certain gauge and so this is equal to a Jit prime and a Jit so this we've taken within the many body ground states and so on ok.

$$\triangle H = -\vec{J}.\vec{A}$$

$$\vec{A} = \frac{\vec{E}}{i\omega} e^{-i\omega t}$$

$$< J_i(t) >= \frac{1}{\hbar\omega} \int_{-\infty}^{t} dt' < 0|[J_i(t'), J_i(t)]|0 > E_j e^{-i\omega t}$$

Now it is easy to see that or rather it doesn't harm us if we assume the system is a time translationally invariant that is we can translate time without any sort of breaking any symmetry so it has time translation symmetry because there is nothing that breaks that time translation symmetry. So what it means is that there are two times t and t prime that you see there so this two times can be converted into one time if there is a time translational symmetry which means a t minus t prime can be written as some t double prime this is perfectly fine and we can in that case your the limit actually changes from minus infinity to t so this is becomes minus infinity to t this limit becomes equal to 0 to infinity ok.

You can trivially see that Jit in that case becomes equal to 1 over h cross omega and from 0 to infinity and we have a dt double prime and this is equal to 0 and Ji 0 so t prime we are setting to 0 and then we are replacing t by t double prime because of the translational invariance and this J it double prime and 0 here and then we will have an exponential so there is an exponential i omega t there inside so this is multiplied by E j exponential minus i omega t ok. So this is a commutator we should not forget that this is a commutator ok.

So Ji at 0 and J i at some t double prime and then you have this thing to be present there so the only you know time dependence is coming as a harmonic function exponential i omega t and so on. So this gives you the linear response of the conductivity so the conductivity is proportional to so remember that relation Ohms law that J equal to sigma E now you have calculated J which is the expectation value of J and this is equal to the sigma is here and this will be an exponential you know a minus i omega t here. So the sigma xy which is a Hall conductivity or the Kubo formula for the Hall conductivity is a sigma xy equal to 1 over h cross omega 0 to infinity and you have a 0 Jy 0 Jx t a commutator 0 and exponential i omega t and so on ok.

$$< J_i(t) >= \frac{1}{\hbar\omega}\int_{-\infty}^{t} dt'' < 0|[J_i(0), J_i(t'')]|0 > E_j e^{-i\omega t}$$

So that's the formula for the Hall conductivity and this is a well-known result so this is the result for the Hall conductivity and which involves only commutators which are J y that's the current density in the y direction and the current density in the x direction at a different time one of the times has been set equal to 0 and this sigma xy is a dynamical conductivity because it depends upon the frequency and of course when we take the DC limit that is we'll put omega equal to 0 this will become the the Hall conductivity that we have seen ok.

$$\sigma_{xy} = \frac{1}{\hbar\omega}\int_{0}^{\infty} < 0|J_y(0), J_x(t)]|0 > e^{i\omega t}$$

This is Kubo formula for Hall conductivity and if you want the longitudinal conductivity that also can be done so you have to have sigma xx where it will be the current operators at two different times but in the same direction so that will give you the the longitudinal conductivity.

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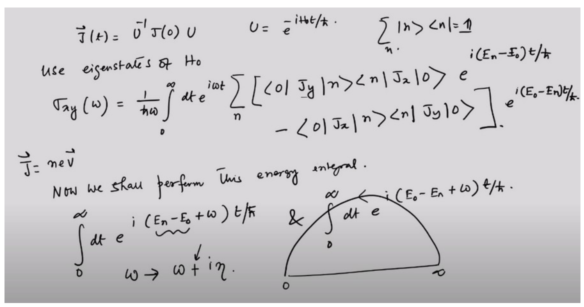

So what we do next is that this is pretty much you know the formula that we wanted to derive but this is not yet in a form that we can use it to compute ok and let us get it into a form that's you know we can compute this and so what we do is we can write down this J of t to be equal to some U inverse J of 0 and U and we have seen this, this is the interaction picture this is how the operators transform and U is nothing but exponential minus i H0 t by H cross ok. Then we can use basically the eigenstates of H0 which we know that's why I want H0 which are say m, n and so on so forth ok. So then we can write down the sigma xy omega by introducing these states the basis states of or the eigenstates of H0 as 1 over H cross omega 0 to infinity and a dt exponential i omega t and I use the completeness of states that is use this n, n that's equal to 1 the sum over n.

$$\vec{J}(t) = U^{-1}J(0)U$$

$$U = e^{-iH_0 t/\hbar}$$

So this is there and then you have a term which is 0 Jy n n Jx 0 and a minus 0 Jx n n Jy 0. So this is the simpler form in the sense that I have used the information that is already inbuilt into the problem that is we know these eigenstates ok. So in a given problem these states will have to be known these n and 0 and m etcetera all these things have to be known ok. And of course we have to write down the energy exponents. So this is equal to En minus E0 t over H cross which is that exponential i omega t and this is also exponential E0 minus En t over H cross ok.

$$\sigma_{xy} = \frac{1}{\hbar\omega}\int_0^{\infty} dte^{i\omega t}\sum_n [< 0|J_y|n >< n|J_x|0 > e^{i(E_n-E_0)t/\hbar} \\ - < 0|j_x|n >< n|J_y|0 >]e^{i(E_n-E_0)t/\hbar}$$

Now off hand it looks like you are calculating the velocity operator because J is nothing but if you write it's n Ev even if you know don't write that n which is an electronic density it's still E into V. So V is the velocity so it's a y component of the velocity between these states of H0 which are known and then x component of velocity and then multiplied by an exponent involving the energies of those states and minus of the Jx and the Jy they are in opposite order or just the reverse order because we are talking about a commutator here ok.

Now we'll perform this energy integral that is and in order to perform the energy integral there are two things that you need to keep in mind ok. Now the energy integral looks like dt exponential i En minus E0 plus omega into t and the other one looks like so 0 to infinity both the cases and dt exponential i E0 minus En plus omega t I mean t and then of course there is a H that one needs to put there omega by H cross is energy omega equal to H cross omega ok. So you have these integrals that you have to perform and these integrals will diverge at infinity because these are exponential integrals.

So the standard technique for calculating this integral such that they don't diverge is by introducing a little bit of damping ok. So let's say that you know these are En minus E0 whenever you have these if you just perform the bare integral it will go to infinity and what you do is that you change omega to omega a plus some i eta or something ok. So once you do that this will not diverge farther because of this small eta and it will go into the complex plane and there is also a reason that you put a plus and not a minus because you are considering t greater than 0 right from 0 to infinity. So there is no negative time so you want to do a contour integral by enclosing it in the upper half plane so that you can use Jordan's lemma. So this integral over the real line is converted into a complex integral by you know sort of trying to close it from the upper half such that in this big circle so this is from 0 to infinity.

$$\omega \rightarrow \omega + i\eta$$

So you close it and it becomes a closed integral and then you can use the residue theorem. Please get used to this complex integrals because they are they are they are ubiquitous in terms of these all these contexts of condensed matter physics, Green's functions and so on ok.

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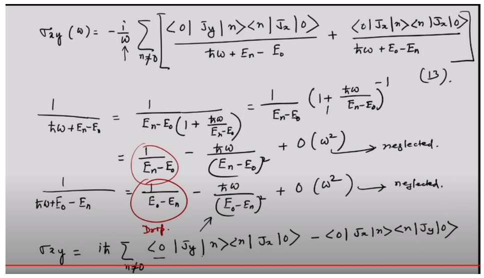

So once you do that then the the sigma xy takes a form which is equal to minus i over omega. So after you do the integral over time from 0 to infinity by using the complex integral now just to remind you so you initially had only over the real line from 0 to infinity. What do you do is that you introduce small imaginary parts so that the pole that is where wherever you have pole of this or rather this he resonance you bump it up by a plus i eta here and enclose it now it encloses the simple pole or whatever is the nature of the pole and then you can use 2 pi i into sum of residues, ok.

$$\sigma_{xy}(\omega) = \frac{-i}{\omega}\sum_{n\neq 0}[\frac{<0|J_y|n><n|J_x|0>}{\hbar\omega + E_n - E_0} + \frac{<0|J_x|n><n|J_x|0>}{\hbar\omega + E_0 - E_n}$$

$$\frac{1}{\hbar\omega + E_n - E_0} = \frac{1}{E_n - E_0(1+\frac{\hbar\omega}{E_n - E_0})} = \frac{1}{E_n - E_0}(1+\frac{\hbar\omega}{E_n - E_0})^{-1}$$

$$= \frac{1}{E_n - E_0} - \frac{\hbar\omega}{(E_n - E_0)^2} + O(\omega^2)$$

And then you perform this integral please go through this text called there is a textbook on complex analysis by Churchill ok. It is a very good book for complex analysis which tells you how to do the complex integrals and so on. So this is equal to that and then of course your n which is this n here this is not same as the as 0 so these are some excited states of the system and that makes sense because for carrying the current it it has to you know visit the excited state because there is a conductivity in the system. So the system would go from the ground state to the excited state and the excited states are assumed to be different than the ground state. So this is equal to 0 jy n and n jx and 0 divided by the energy denominator which is given by En minus E0 and a plus another term. So the term is 0 jx n n jx 0 divided by h cross omega plus E0 minus En, ok.

So you get this energy denominator. Now we are almost there excepting that we have to take the DC limit and DC limit means omega going to 0 and how do we take that? Let us you know sort of write down the denominator as En minus E0, ok and this is equal to you write it as E n minus E0 and you write it as1 plus h cross omega divided by En minus E0 and then you write it as 1 divided by En minus E0 multiplied by 1 plus h cross omega divided by En minus E0 to the power minus 1 and they do a binomial expansion and keep only the term which is with the first term only which is linear in omega. So this is 1 by En minus E0 plus so this is equal to minus h cross omega divided by En minus E0 whole square. So I do a binomial expansion and the minus sign gets replaced here, ok at this place and then you do this and then of course these are terms which are omega square that are neglected.

Similarly you do that for the other term the second term inside the square bracket which is h cross omega plus E0 minus E n and this will be like E0 minus En and minus h cross omega divided by E0 minus En whole square and plus omega square order of omega square which again can be neglected. Now we actually drop this term the first term which is here so these terms we drop, ok. In fact for Hall effect these terms do not make a contribution and you keep the term that is linear in omega because this has no omega part so this is like a DC thing and I mean DC contribution which is not an important thing and in any case that they really do not in a translationally invariant system they do not contribute to the Hall resistivity and so on, ok.

So what you do is you put them there in that equation and then you see very nicely there is a omega in the denominator of equation so I have equation 9 and let us call it as equation 10 and equation 11 which is the Kubo formula and say this is equation 12 again another form of Kubo formula 13 which is another form of Kubo formula we are progressively getting you know better and better in terms of its computability, ok. So now if you keep this term either of the term which are same because there is a h cross omega and there is a square in the denominator and the square means that en minus e0 square is same as e0 minus en square and the omega will cancel with the omega in the denominator of equation 13, ok.

$$\sigma_{xy}(\omega) = \frac{-i}{\omega}\sum_{n\neq 0}[\frac{<0|J_y|n><n|J_x|0> + <0|J_x|n><n|J_x|0>}{\hbar\omega + E_0 - E_n}$$

So once you do this your sigma xy becomes equal to now I do not need actually the omega gets cancelled so it becomes a DC expression and this is equal to ih cross again n not equal to 0 and 0 jy n n jx 0 and minus 0 jx n n jy 0 jy 0 and either you call it en minus e0 whole square or e0 minus en which are same, ok. So this is really the Kubo formula for the Hall conductivity, ok.You can also get the longitudinal conductivity there are many things that we have said which are important that the diamagnetic term should also come in in order to calculate the conductivity the longitudinal conductivity but here nevertheless we are interested in the off diagonal form of conductivity which has this form let us call this as equation 14 and this is what we wanted to calculate and this when you calculate it on a quantum Hall system it gives you a values which are h over e square and there is a integer in the denominator or you can call the so the conductivity so that is a resistivity the conductivity is actually n into e square over h where n is an integer, ok.That is where we stop for now and we will continue from here. Thank you.