Topology and Condensed Matter Physics

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Lecture – 01

Introduction to Topology

Welcome once again let me get into the course called topology and condensed matter physics and this is Saurabh Basu who would be teaching this course I am from department of physics IIT Guwahati my email address is here.

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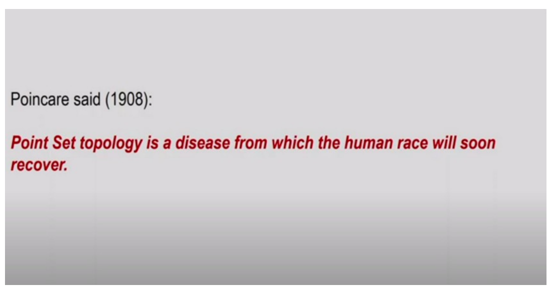

So I start with conjecture by Poincare in made in 1908 and in fact he is the sort of person who initiated the study of topology in different branches of physics and he made this statement is somewhat strange. However there is a conjecture which was later on solved and this conjecture said that the Poincare topology is a disease from which the human race will soon recover. We will not get into what he exactly wanted to mean by this but there are far reaching consequences of this topology in various branches of physics which is what we shall elaborate here. So it is known for a reasonably long time that topology may have significant bearing on classifying systems and the way it occurs is that we know that there are systems which have these singularities that appear in the system and these singularities in certain systems they appear as vortices or vortex and these vortex is something that it is like complex number equal to r exponential i θ. So when theta goes from 0 to 2π these complex number remains unchanged.

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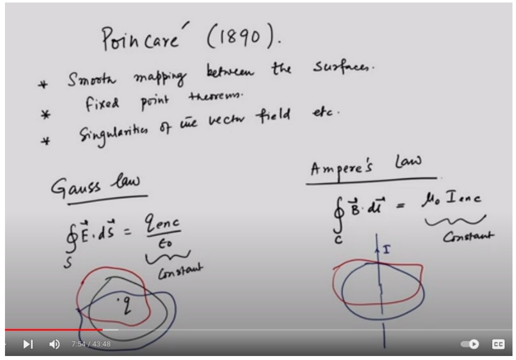

So there is sort of the center of this or the origin of this definition of complex number is where a vortex resides. So the focus is that whether these vortices can be contracted to a point without crossing each other that is without crossing one vortex and this branch of mathematics which now has entered significantly into physics which is called as a homotopy theory and it plays a very important role in algebraic topology. So Poincare started all these studies and so all these things started roughly about the end of the 19th century which is 1890 and so on and so while investigating problems in celestial mechanics which is popularly known as classical mechanics in the modern times. So there are a few things that he noted one is the smooth mapping between the surfaces.

Second is there are these fixed point theorems and then there are the singularities of the vector fields etc. there may be a lot more and so these are really the applications of topology in physics and Poincare so he sort of pointed out that these results of these they can be systematically applied to other mechanical systems, electromagnetic systems, optical systems in particular acoustic systems and so on. Now is this very new is this something that we have never been aware of these applications of topology in physics say for example and it's not that I mean if you really think about this Gauss's law let me write that and Ampere's law in standard electro magnetics. So this was $E.dS$ is equal to Q enclosed by ϵ_0 and it's a closed surface integral and this is the surface is a we sort of call it as S and also $B.dl$ where B is the magnetic field here is the electric field so flux of the electric field and this is like the line integral of the magnetic field is like μ_0 I enclosed I am sort of for convenience I am taking it to be vacuum but then you can write down in matter as well where you to use T and H and so on.

$$\oint_S \vec{E}.d\vec{S} = \frac{q_{enc}}{\epsilon_0}$$

$$\oint_C \vec{B}.d\vec{l} = \mu_0 I_{enc}$$

Now you see that these integrals over these geometric sort of curves so here there is a surface for the Gauss's law and there is a contour in the this Ampere's law and these fluxes and the line integral of the magnetic field gives you a constant so this is just a constant here and this is a constant here so these constants are really the topological invariant which means that we can deform this surface and we still would get the same result.

So, they involve the surface integral and line integrals and as I said that so suppose you have a charge here a Q and you can have a surface which is you know surrounding it enclosing it will be the Q by ε_0 and I can also deform this surface like this it still would be the same integral or rather the resultant will be same of the integral and if I still deform in this particular fashion it will still be same and similarly the line integral of B say we start with this and this contains a current so I will just show that. So this is that it's a current and the magnetic field the line integral of the magnetic field will give you the I enclose so it's μ_0 I and if I now change it to another one but enclosing this will still be the same and so on.

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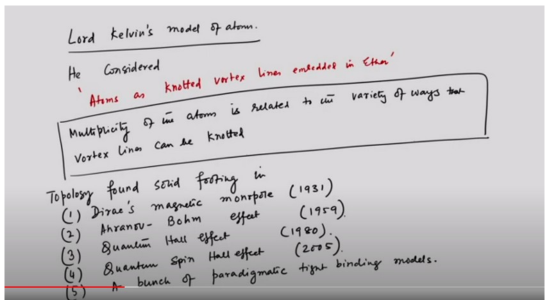

So this is the first realization of topological invariant or the application of topology in the field of physics and so if we continuously deform that these things as long as we do not cross that charge which is like a singularity or a vortex or we don't cross that current current carrying wire we are fine in defining these invariants or rather these equations. And in fact Lord Kelvin in its model for atoms have actually said that these atoms are knotted vortex lines in ether. So this is Lord Kelvin's model of atoms.

He considered atoms as knotted vortex lines embedded in ether. Ether is just a medium and we have seen this in the context of light which eventually Michelson and Morley actually said that there is no medium through which the light travels and that was a famous Michelson and Morley experiment. So this is something similar to that and in particular you know this multiplicity of the atoms which arises because of the number of unpaired electrons and these multiplicity is due to the different ways that you know the vortex lines can be knotted. So multiplicity is related variety of ways that the vortex lines can be knotted which means all these vortex lines can be tied with each other and this is of course not a good description which Lord Rutherford came and you know completely modified it and it's actually the model of atom that is taught in undergraduate as well as even in school levels but however this was the original idea of Lord Kelvin. So there have been bits and pieces of these topological implications coming into various fields of physics. However it found really its you know foothold in a few areas and these areas are So one of them is Dirac's argument or his you know conviction about the magnetic monopole. So let me write Dirac's magnetic monopole.

This was as early as 1931 then came the Aranov-Bohm effect. This was in 1959 and then it was you know one of the main topics of our discussion which is quantum Hall effect which is 1980. So the first one as I said is due to Dirac and then it was Aranov-Bohm. Quantum Hall effect was due to Klitzing and soon after that Gossert, Stormer and Sui they have published the fractional version of it the fractional quantum Hall effect as it's known and then we have these quantum spin Hall effect is around 2005 and you know really a bunch of paradigmatic models or tight binding models. So as a material you know this monopole is not a material neither is Aranov-Bohm effect which are just phenomena. Quantum Hall effect is the first realization that 2D electron gas in presence of a strong magnetic field can be considered as a topological insulator and this was the first realization of quantum or rather topological insulator which was in 1980.

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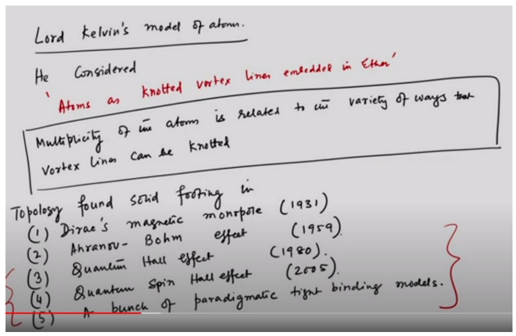

Different kind of topological insulator with spin filtered edge modes etcetera they were discovered in 2005 and they found to have a lot of applications in very interesting phenomena called as spintronics which in principle could replace the electronics and would be a next generation devices communication devices and so on. And we'll talk about you know 3, 4 and as well 5 where we would talk about a number of models which are very simple models yet they capture the topological properties in a very systematic way and we can because of these are tight binding models so they can be written down nicely in case space. So we can calculate the topological invariant and we can show that when the system sort of makes a transition from a topological phase to a trivial phase or vice versa or from one topological phase to a different topological phase that are characterized by different values of the topological invariant. So we'll talk about these 3 in details throughout the course however just to make matters complete let me discuss these Dirac's magnetic monopole and Aronov-Bohm effect in very brief so that one gets an idea that which are as you see that chronologically they occurred much before the quantum hall effect and these arguments are very elegant and they are definitely worth seeing in this context.

However we'll not sort of deliberate on them too much and we'll go on to this actual studies of topology in condensed matter physics whereas these the first two actually correspond to any general physics in particular say quantum physics or electrodynamics say for example.

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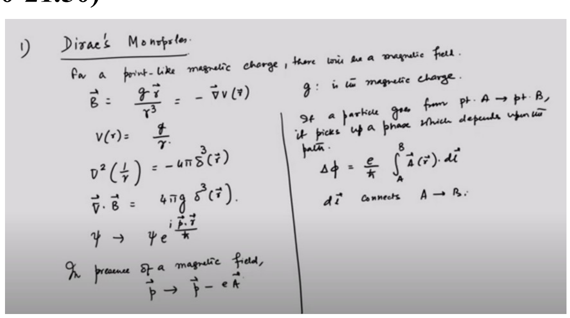

So let's talk about these Dirac monopoles. Okay so suppose there is a monopole that exists okay so it's like a point like magnetic charge and if that would create a magnetic field. So for a point field which is given by *B* equal to let's call it a *g* to be the magnetic charge and let's write down this as *r* by r^3 which would be represented by a potential which is given by this and this is as opposed to a vector potential there is a scalar potential which depends upon the vector *r* and if we want to reduce the complexities then these *r* can simply be the vector *r* can simply be just *r* and not a vector which means that it has no dependence on the angular variables θ and φ okay. So g is the magnetic charge and *V* would be simply equal to so *V* of r will be equal to g over r and then so this is you know because of this identity that the Laplacian of 1 over *r* so if you take the Laplacian of that and this is equal to minus 4 π $\delta^3 r$ and this is a Dirac delta function in three dimension and this is the Laplacian of 1 over *r* this must be a known result in electrodynamics.

$$\vec{B} = \frac{g\vec{r}}{r^3} = -\vec{\nabla}(\vec{r})$$

$$v(r) = \frac{g}{r}$$

$$\nabla^2(\frac{1}{r}) = -\ 4\pi\delta^3(\vec{r})$$

So what it tells you is that these if you take these things if you take the Laplacian that's del square and operate it on 1 over R it will give you 0 contribution everywhere excepting at r equal to 0 okay. So that's why this delta function comes and because we are talking about three dimensions say spherical polar coordinates so it's a δ^3r. So then the magnetic analogue of Gauss's law becomes equal to divergence of *B* is equal to a 4 π g δ^3r okay. So that's like the Gauss's law that we are aware of. Now you know what happens when you have a magnetic field to be present the wave function of the particle that evolves.

$$\vec{\nabla}.\vec{B} = \ 4\pi g\delta^3(\vec{r})$$

So usual you know the wave function sort of can be represented as if *p* is a good quantum number or k is a good quantum number it's represented by this but in presence of a magnetic field *p* is actually p minus e A okay where a is the vector potential corresponding to the field. Now in literature you might see a C below which is actually in Gaussian unit we want to write down in the SI unit. So now because of this there is a phase difference if a particle you know sort of goes from let me write it here itself.

$$\psi \to \psi e^{\frac{i\vec{p}.\vec{r}}{\hbar}}$$

$$\vec{p} \to \vec{p} - e\vec{A}$$

If particle goes from point A to point B it picks up a phase which depends upon the path. So remember if there is no magnetic field then these phase that you see is not a path dependent phase however this picks up a path dependence phase and which this called as a minimal coupling or the piles coupling and this phase is given by e by h cross and a to b a *r.dl* where *dl* is a vector that connects the point a to b. So *dl* connects a to b okay this points okay.

$$\nabla\phi = \frac{e}{\hbar}\int_A^B \vec{A}(\vec{r}).d\vec{l}$$

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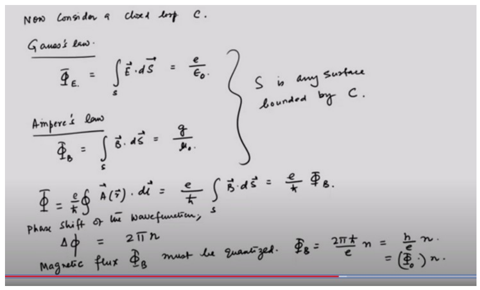

Now consider a closed loop C okay. So we will write down Gauss's law. So the Gauss's law which we have written down just a while back. So this Gauss's law will give us a potential okay this is basically the flux of the electric field. So e dot ds and this is over s and this is nothing but let us just call it as because we are talking about electrons and or the electric charge and this so let us write e over ε_0 and similarly for the Ampere's law which we will write it as phi b this is equal to s and b dot d say for example s again so that is this which is nothing but if there is a monopole then this will be like g over μ_0 okay.

$$\phi_e = \oint \vec{E}.d\vec{s} = \frac{e}{\epsilon_0}$$

So s is any surface which is bounded by this by the C alright so as the particle it is transported around this loop which is this closed loop C then it will sort of this magnetic flux say for example so particularly we are talking about this flux for a closed loop is nothing but it is like e over ħ then it is a r.dl and this is equal to e over ħ we apply stokes theorem and stokes theorem says that a.dl or the line integral of any vector is equal to the curl of that vector and the corresponding surface integral where the surface actually is just like this that is bounded by this closed curve C.

$$\phi_B = \int_S \vec{B}.d\vec{S} = \frac{g}{\mu_0}$$
$$\phi_B = \frac{e}{\hbar}\oint \vec{A}.d\vec{l} = \frac{e}{\hbar}\int_s \vec{B}.d\vec{S} = \frac{e}{\hbar}\phi_B$$

So this is equal to b.ds and this is nothing but a e by ħ and this is a φ which is a b okay. So this is that b.ds which is what we have defined as flux of the magnetic field okay so we just assuming that there is a magnetic monopole that exists and the description of the electric field and the magnetic field are in the same footing and that is why we can write down this. So now talk about the wave function of the particle so the wave function needs to be single valued which means that if you take the wave function over a closed loop then there is no effective phase that it picks up you are not talking about the dynamical phase but over that complete circle it will not pick up any phase.

$$\Delta\phi = 2\pi n$$
$$\phi_B = \frac{2\pi\hbar}{e}n = \frac{h}{e}n$$

So the δφ should be equal to some 2π into n where n is an integer okay. So this tells you that if the change in so this is a phase shift of the wave function and then this is δφ equal to 2πn now that tells you that the magnetic flux is quantized φb must be quantized so that tells you your φb is equal to 2πħ by e into n which is nothing but h over e into n okay and so h over e is nothing but you know that is the flux quantum so this is the flux quantum which is very familiar and multiplied by n okay.

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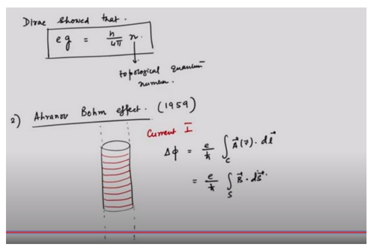

So the magnetic flux will be some integer times the flux quantum and in fact what Dirac showed that this product of the electric charge and the magnetic charge so this product of electric and magnetic charges and so this is equal to h over 4πn and this is the main result that he said that these charges are quantized so the existence of magnetic monopoles would explain the quantization of the electric charge okay and these integer n that you see is a topological quantum number okay. So it counts you know the number of windings around a singular point and the singular point is the point at which the monopole resides so the monopole actually acts like a vertex and you take the wave function or to take the particle around that and it just counts the number of times it sort of winds okay and these are also you know can in a sort of tight binding model we will see how to calculate that and so on in order to calculate the topological quantum number or which is called as a topological invariant okay. So the next is Aronov-Bohm effect let us write S2.

$$eg = \frac{h}{4\pi}n$$

So before this effect was you know first proposed as a thought experiment by Aronov and Bohm and then you know it was actually verified in experiment by Chambers in 1960 so this was in 1959 yeah so just about a year later it has been verified. So the burning question is that we know about fields electric field and magnetic fields and electric field corresponds to a potential such that E can be written as you know mine as grad phi and for a static case the curl of the electric field is equal to 0 and just the other way around the B has also sort of is represented by a vector potential and this has no scalar analog I mean in the sense that B is obtained from a vector potential by taking the curl so B is equal to curl A and this is quite an important thing in the development of these Maxwell's relations and I mean equations say Maxwell's equations and how the wave propagation etc were sort of you know from these written down from these Maxwell's equations. However, Aronov and Bohm argued that it is not the fields that are most important it is the potentials which are phi and A are quite important in fact they are more important than the fields and in fact the all the Maxwell's equations can be equivalently written down in terms of these potentials instead of the fields which is more familiar to anyone you know working on this or studying physics. So in order to you know sort of pin down their views they asked to consider so this is like a solenoid a very long solenoid and so to show that it is long that is like infinite this thing we are showing this and there are these this turn the wires are wound around the solenoid and why we have taken it to be very large is or very long is that we want to eliminate the edge effects. Now inside the solenoid suppose the current is I so current I which is going through the loops and wound around the solenoid and this will produce a very constant or uniform magnetic field inside the solenoid and this magnetic field outside the solenoid is 0 and that is why we have taken we have neglected the edge effects and that is why we are talking about that we are you know talking about a very large solenoid.

$$\Delta\phi = \frac{e}{\hbar}\int_S \vec{A}.d\vec{l} = \frac{e}{\hbar}\int_S \vec{B}.d\vec{S}$$

So inside the magnetic field is nonzero and uniform outside it is equal to 0 but outside it is equal to 0 does not mean the vector potential is equal to 0 in fact the vector potential exists such that the curl of the vector potential vanishes and we know that when curl of a vector vanishes that means it is an irrotational vector which means that it does not rotate it does not curl it is like a vector that is you know either increasing monotonically or decreasing and so on so forth. So nevertheless I mean the whole assumption is or rather the these finding is that the even though the magnetic field is 0 the vector potential still exists and how would we know that the vector potential exists so one can send one electron from one side of the solenoid suppose you are having the solenoid one can one electron can be sent from the right of the solenoid and go and hit a screen at some distance away and another electron can be sent from the left of the solenoid and would again go and incident on the screen at a distance at a certain distance away. Now these two electrons they also can be considered as waves and these two waves when the incident would have a different phase okay and the it will have a constructive interference or a destructive interference depending on the phase relationship between the between the two electrons which goes from the left and the right and this phase difference can be obtained or rather it is finite because the vector potential exists and this is equal to E over h cross and A r dot dl where dl is the length that it travels and A is the vector potential that exists and this is again by using nothing but the stokes law we get it as B dot ds so this is you know that curve C and this is that S it is a B dot ds. So this is the thing and one can actually verify it in experiments which Chambers did in 1960. Now this is a very important thing that if this is a measurable quantity and this is actually verified in experiments which means that the potentials are more important quantities than just the fields and in fact this was earlier not known.

So what is important here is that you know the solenoid actually contains or the singularity in the vector potential is like a vortex as we have said and then you can actually view the solenoid as a hole in the space which is allowed for these vector potential okay. So the quantization arises from the fact that the curves in the space of A the vector potential that enclose the solenoid are non contractible okay so they cannot be contracted and so on and these the winding number thus produced which is a topological invariant which is taking these particle around the solenoid these winding number characterizes you know the distinct homotopy classes etc. So what I wanted to make sure is that as soon as there are these systems with singularities or vortices here for example this there is a vortex which is coming from the solenoid itself and the magnetic monopole is just like a vortex or a singularity at its position. So there when you take a charge or take a particle you know around it, it will give rise to quantized effects and this is the topological quantization that one is talking about. So this will not go away even if you know you can adjust other quantities and this will still remain. So Aronov-Bohm phase is an important it is a sort of indication of topology playing a role in this simple thought experiment okay.

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So let us come to physics and let me you know show you I have already told you this that this 2016 Nobel Prize was awarded to for the theoretical discoveries of topological phase transitions and topological phases of matter. These are by David Thaulus who won half the Nobel Prize and the other half went to Duncan Holden and Michael Costellis. I believe David Thaulus and Michael Costellis they were awarded for the topological phase transitions whereas the topological phases of matter was due to Holden, Duncan Holden which he showed in the mid to late 80s through a number of you know very well cited publications will talk about what is called as a Holden model etc. And this was in 2016 and after that it was these two gentlemen who had popularized again the concept of topology.

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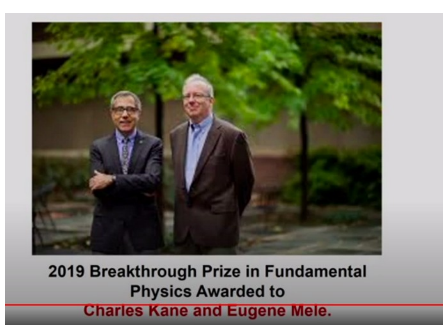

It is not popularization it is more like applicability and into this quantum spin hall phase and they have written down Hamiltonian which does not have a quantum hall effect but it has another topological invariant called as a Z2 invariant and this is non-zero. They won a 2019 breakthrough prize in fundamental physics awarded to Charles Kane and Eugene Milley. Charles Kane is on your right and Eugene Milley on the left.

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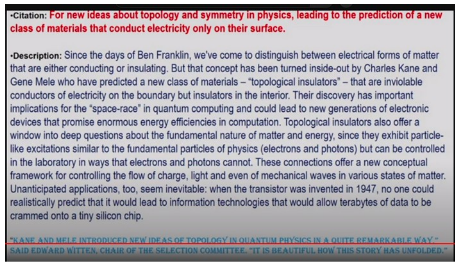

Let me read out the citation is very interesting particularly it is you know it predicts that there would be a lot of applications of these proposals from Kane and Milley on the quantum computation, quantum information etc. and quantum technologies.

So the citation says that for new ideas about topology and symmetry in physics leading to the prediction of a new class of materials which are the quantum spin hall materials that conduct electricity only on the surface and this electricity is actually the spin polarized current. So the description is quite interesting let me read it out for you. Since the days of Ben Franklin we have come to distinguish between electrical forms of matter that are either conducting or insulating. But that concept has been turned inside out by Charles Kane and Gene Milley who have predicted a new class of materials the topological insulators that are inviolable conductors of electricity on the boundary but insulating or insulators in the interior. So this very fact that they behave differently at the edges compared to the bulk if you look at this board that you are the screen that you see if I have these screen the edges of the screen that behave differently than the bulk I would not feel comfortable about it.

In fact most of the systems that we know they have no distinction between the behavior of the bulk and the edges and this is what precisely makes these things so interesting. So the discovery has important implications for the space race in quantum computing and could lead to new generations of electronic devices that promise enormous energy efficiencies in computation. Topological insulators also offer a window into deep questions about the fundamental nature of matter and energy. These they exhibit particle like excitations similar to the fundamental particles of physics electrons and photons such as them but can be controlled in the laboratories in a way that electron and photons cannot be done. These connections offer a new conceptual framework for controlling the flow of charge light and even mechanical waves in various states of matter.

This is what I was saying that topology has proliferated beyond electromagnetic theory or optics or condensed matter physics it has entered into acoustics it has entered into mechanical matters mechanical sort of materials and various other things. I mean unanticipated applications to seem inevitable when the transistor was invented in 1947 and no one could realistically predict that it would lead to information technologies that would allow terabytes of data to be crammed into a tiny silicon chip and that is what all are there in our modern day computers and mobile phones and various other gadgets that we see on everyday life. The last part written in blue is actually by Ed Witten who was the chair of the selection committee and he said that Ken and Millie introduced new ideas of topology in quantum physics in quite remarkable way. He is the chair of the selection committee he says it is beautiful how the story has unfolded.

We will stop here today and we will carry on with more discussions on topology and its relevance to condensed matter physics to be precise and we will see that as the course you know unfolds I am sure you will learn a lot of things about materials and in particular as I said that quantum hall effect is one such material which has been the first topological insulator that means that the bulk of the sample behave differently than the edges and there will be more revelations of different kind of materials you know which are either dirty systems such as 2D electron gases or they are crystal lattices with you know proper symmetries and so on.

We will stop here. Thank you very much.