Quantum Entanglement: Fundamentals, measures and application

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Week-02

Lec 7: Schmidt Decomposition Method

Hello, welcome to lecture 5 of this course. This is lecture number 2 of module 2. In the last lecture, I have given you a brief technical introduction to quantum entanglement and I started discussing the so-called Schmidt decomposition method, which is a technique to characterize quantum entanglement, particularly in bipartite system. But to understand Schmidt decomposition method, we needed to learn singular value decomposition method, which we have done in the last lecture. In this lecture, I will first revise the singular value decomposition method by illustrating an example, and then we will begin discussing Schmidt decomposition method. So let us begin.

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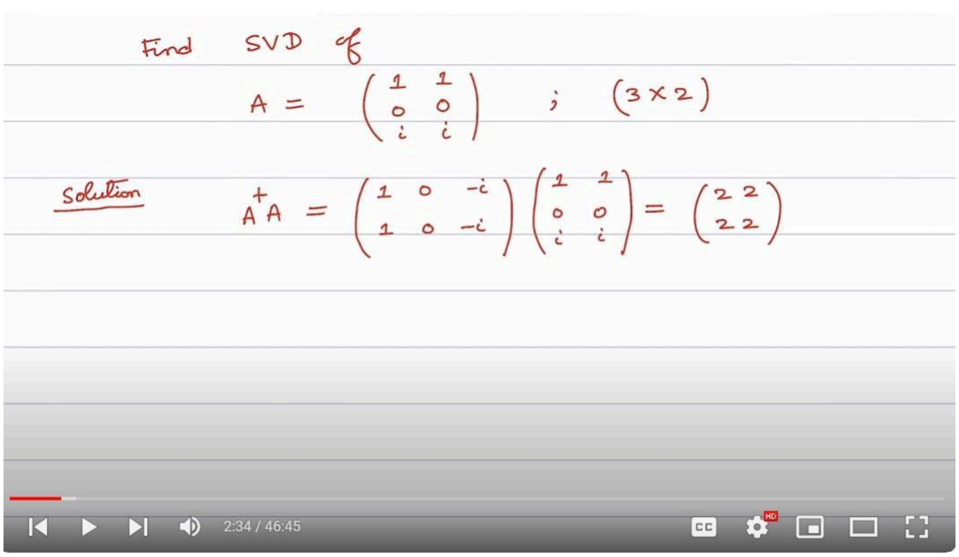

To find singular value decomposition of the matrix, A is equal to 1 0 I 1 0 I As you can see, this is a, it's a 3 by 2 matrix. That means 3 rows and 2 columns. So to find out the

SVD, singular value decomposition of this matrix. So first, let us calculate A dagger A. Now, if A dagger, if you find, then you just have to take the transpose of the matrix A and then take the complex conjugate. So first row would become 1 0 minus I. Second row is also going to be 1 0 minus I. And the matrix A is 1 0 I 1 0 I.

And if you do the multiplication, you will get 2 2 2 2. So as you can see, A dagger A is now a square matrix. You have started with a non-square matrix A and A dagger A is a square matrix of dimension 2 by 2, 2 rows 2 columns.

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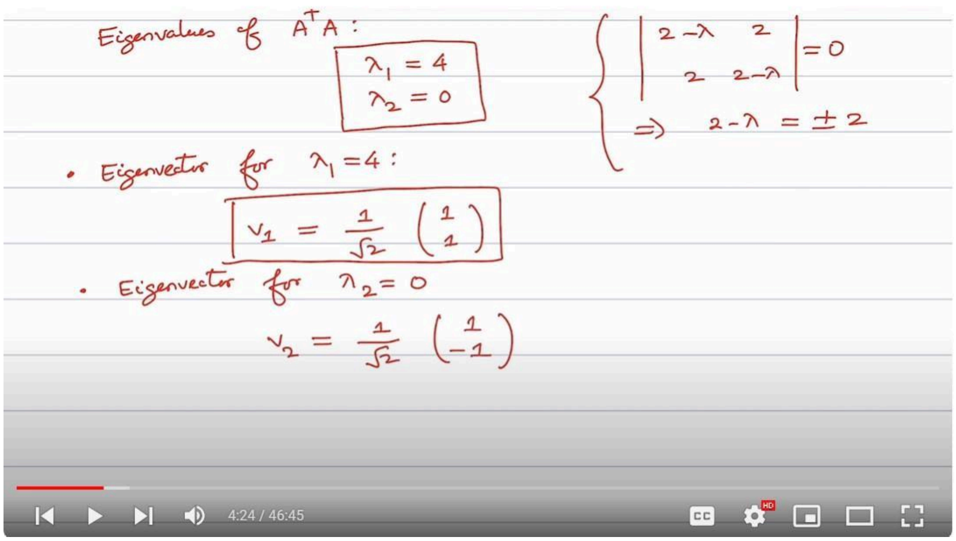

Now, you can find out the eigenvalues of this matrix. Eigenvalues of A dagger A. If you set up the characteristic equation, you will find the eigenvalues as say lambda 1 is equal to 4. And lambda 2, you are going to the second eigenvalue, you will get it to be 0. Okay, I just have solved this equation. 2 minus lambda 2 2 2 minus lambda is equal to 0.

And from this, the characteristic equation you can set up, you will get 2 minus lambda is equal to plus minus 2. So this is going to give you the eigenvalues as 4 and 0. Now let us find out the eigenvectors corresponding to these two eigenvalues. So eigenvectors, eigenvectors vector for say lambda 1 is equal to 4. I think all of you know it. If you do the calculation, you will find the eigenvector for lambda 1 is equal to v as 1 by root 2 1 1.

And similarly, you can show that the eigenvector, eigenvector for lambda 2 is equal to 0.

This is for lambda 1. For lambda 2 is equal to 0, v2, the eigenvector would become 1 by root 2 1 minus 1.

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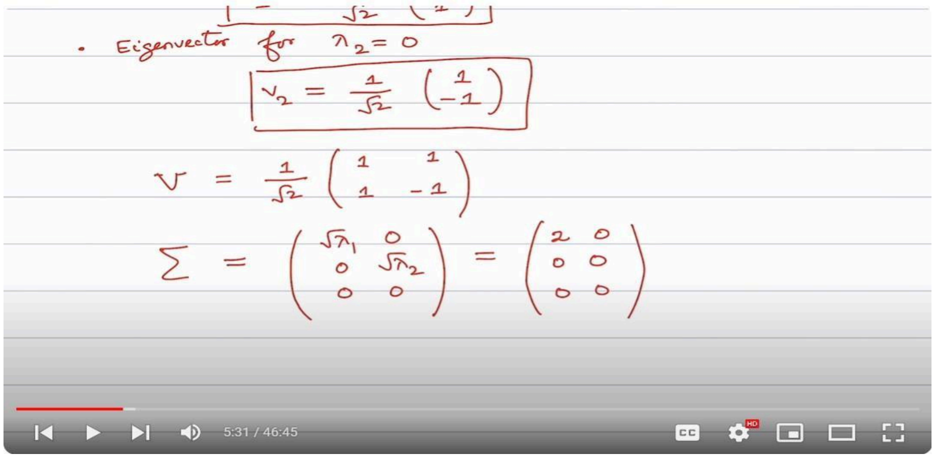

So this can immediately give you the v matrix that we discussed. The capital V matrix would become, you can construct it from these eigenvectors. So first you write v1, v1 is 1 by root 2 1 1 and v2 is 1 by root 2 1 minus 1. So this is going to be your v matrix.

And the sigma matrix is easy to get because already we have got the eigenvalues. So this is going to be a 3 by 2 matrix. It has the same dimension as that of the original matrix A. And it would be square root of lambda 1 0 0 square root of lambda 2 0 0. So therefore, the sigma matrix would become lambda 1 is equal to 4. So square root of lambda 1 is 2 0 0 0 0 0. This is the sigma matrix.

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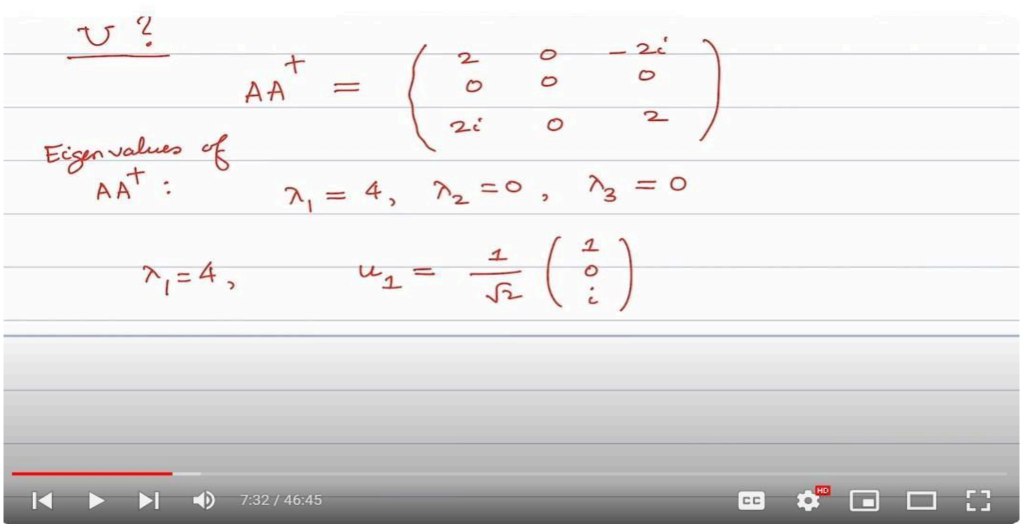

Now, what about the capital U matrix? What about this capital U matrix? So to do that, you have to find out A A dagger and if you do the mathematics, you can show calculation. If you do, you will find that the A A dagger would become 2 0 minus 2 I 0 0 0 2 I 0 2.

And again, you have to find out the corresponding eigenvectors. Again, here the eigenvalues of A dagger A is going to be the same eigenvectors. Sorry, eigenvalues, I should first find out the eigenvalues. Eigenvalues are going to be the same as that of A A dagger. Only thing is that you can have an extra eigenvalue here. So what you are going to get is eigenvalues of A A dagger. Again, you can get it by setting up the characteristic equation. You are going to get the eigenvalues as lambda 1 is equal to 4, lambda 2 is equal to 0. You will get three eigenvalues because it's a three by three matrix and lambda 3 is going to be equal to 0.

And corresponding eigenvectors you can find it out. For lambda 1 is equal to say 0. Ok, so first let me do it for lambda 1 is equal to 4, right? So lambda 1 is equal to 4. You will get eigenvectors as u1 is equal to 1 by root 2 1 0 I.

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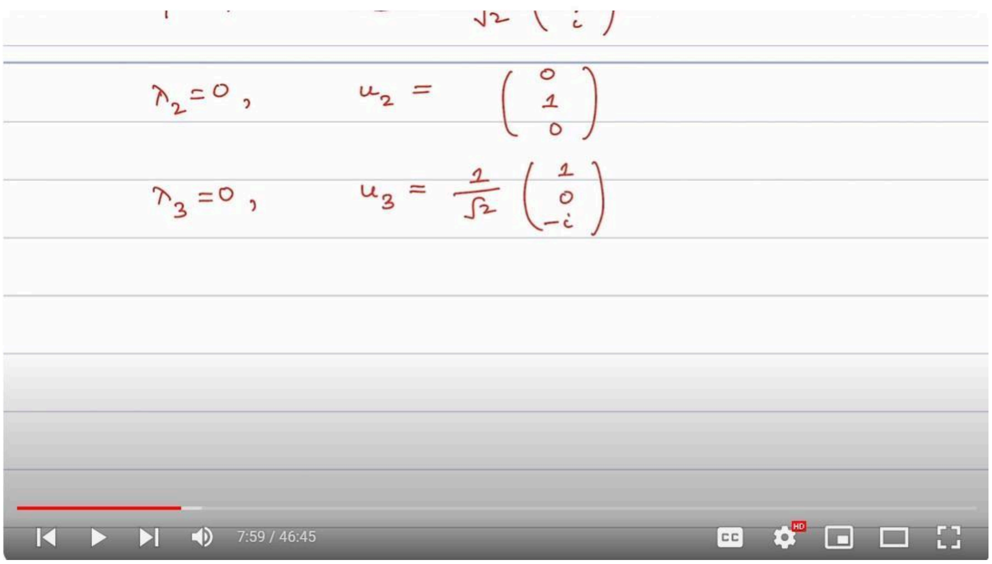

And for lambda 2 is equal to 0, you can get, you can show that the eigenvector is 0 1 0. And for lambda 3 is equal to 0, these eigenvectors are, is going to be u 3 is equal to 1 by root 2 1 0 minus I.

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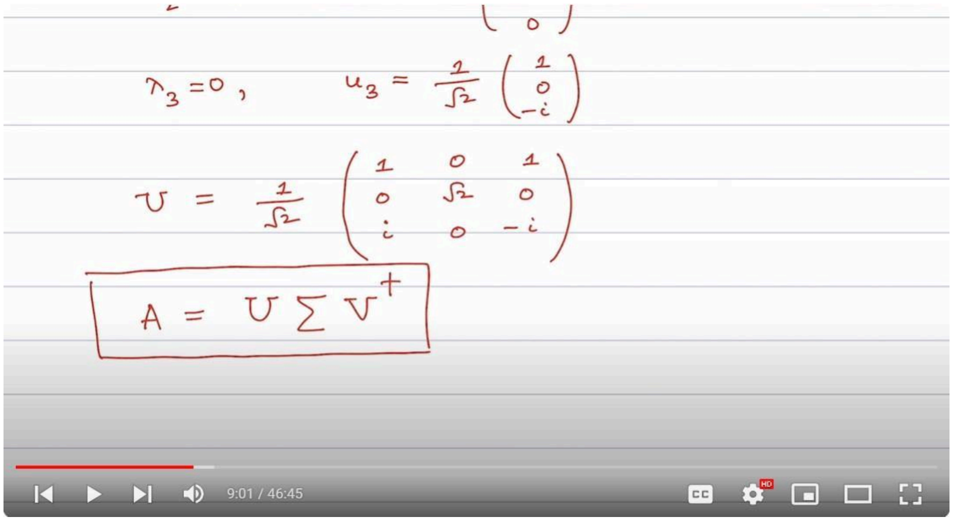

So therefore this capital U matrix, you will get it as 1 by root 2. It is going to be constructed from these three eigenvectors u 1, u 2 and u 3. So just let me write it column wise. It will be 1 0 I 0 root 2 0 1 0 minus I. Ok, so this is your capital U.

So this way you will be able, you are then able to get the singular value decomposition of the matrix A. And that would be capital U sigma v dagger. You have found out this matrix V, V matrix you have found out. So you just have to take the Hermitian conjugate of that. And you can actually verify it taking the matrix products of these three matrices. Then if you work out, you will be able to get the original matrix A.

Now having learned the required mathematical tools, we are ready to discuss the Schmidt decomposition method.

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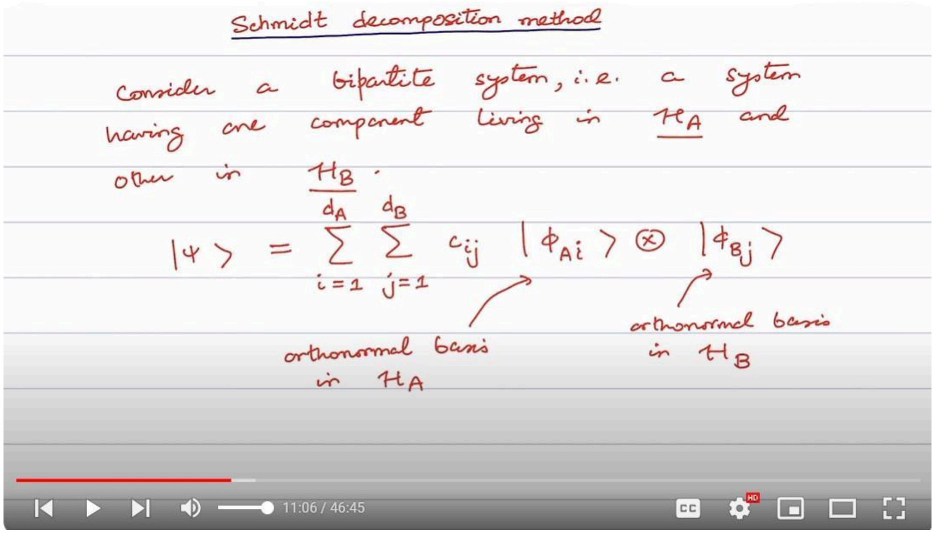

Let us consider a bipartite system and a bipartite system as you know has two components. That is a system having two components. Out of these two components having one component living in, living in the Hilbert space H A. And other living in the Hilbert space H B. So it is a system consisting of two subsystems A and B.

And any arbitrary state ket psi of the system can be expressed as it is a pure state. So I can express it as a direct product state. I am going to explain all the terms one by one. Summation is going from i is equal to one to dA. Where dA is the dimension of the Hilbert space H A.

And this other summation go from j is equal to one to dB. dB is the dimension of the Hilbert space H B. And these coefficients are Cij. And I have this basis ket phi Ai direct product with phi Bj. Where phi Ai are the orthonormal basis.

These are the orthonormal basis in the Hilbert space H A. On the other hand phi Bj are the orthonormal basis in the Hilbert space H B. Okay? And overall the system lives in the Hilbert space. Overall the system lives in the Hilbert space H A tensor product H B.

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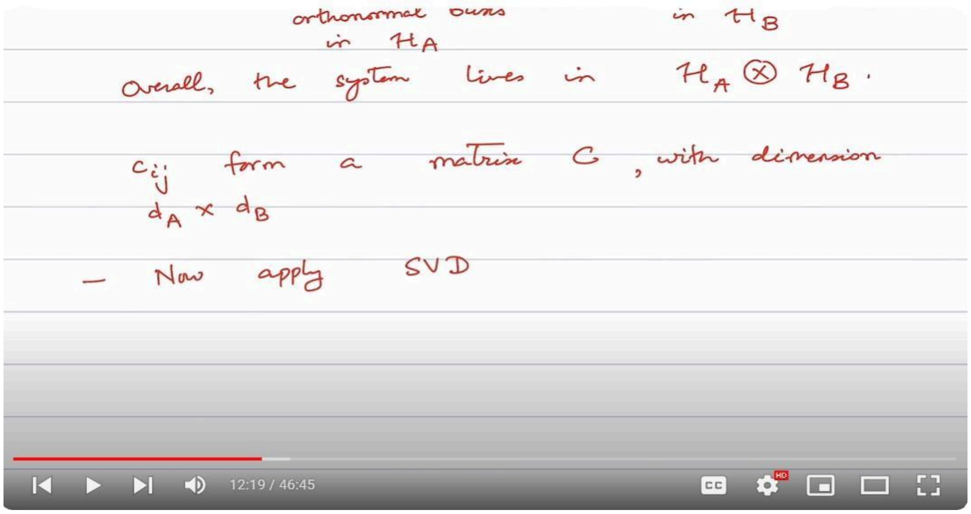

Alright. Now you see these coefficients Cij they form a matrix C with the dimension. Let me write here Cij this coefficient form a matrix say capital C with dimension. That means the size of the matrix is dA cross dB. That means it has dA number of rows and dB number of columns. That's the size of the matrix. As you can see this may be a non-square matrix. And in that case from singular value decomposition one can apply.

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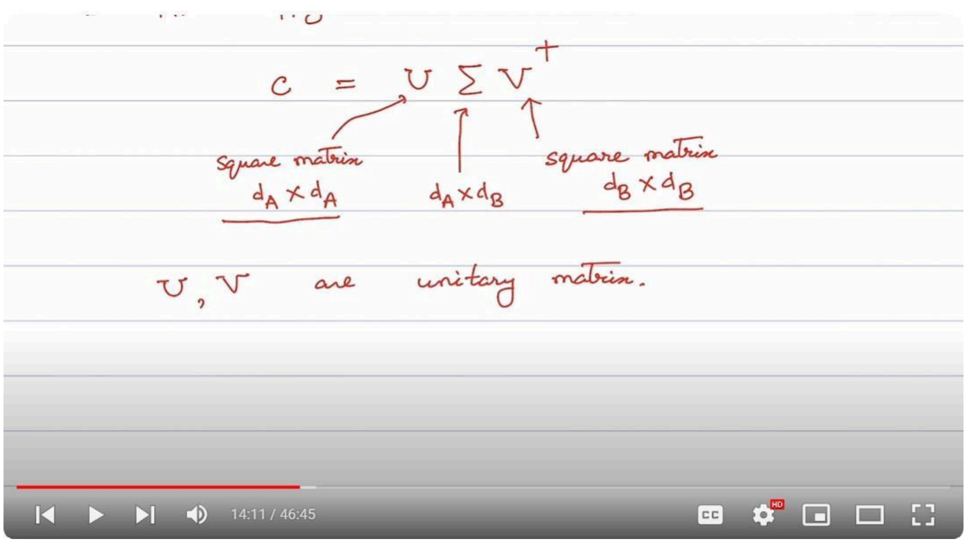

Now apply the singular value decomposition to express this matrix C in the form capital U, capital sigma and. Now here this capital U is a square matrix as you know from SVD method already you have learned. It's a square matrix with size dA cross dA. As you have seen that this capital C is a matrix with size dA cross dB. So capital U is a matrix with size dA cross dB. While V is a square matrix again even its hermitian conjugate is also a square matrix. It's a square matrix with size or dimension dB cross dB.

On the other hand this capital sigma it is a non-square matrix. It may be a non-square matrix because not necessarily that you know A is not equal to B. This SVD method can be applied even to a square matrix as well. But in general this sigma matrix has a size dA cross dB. It's a non-square matrix which has non-zero elements along the main diagonal. And both this capital U, both this capital U and capital V are unitary matrix. And it's very important to remember this U and V are unitary matrix.

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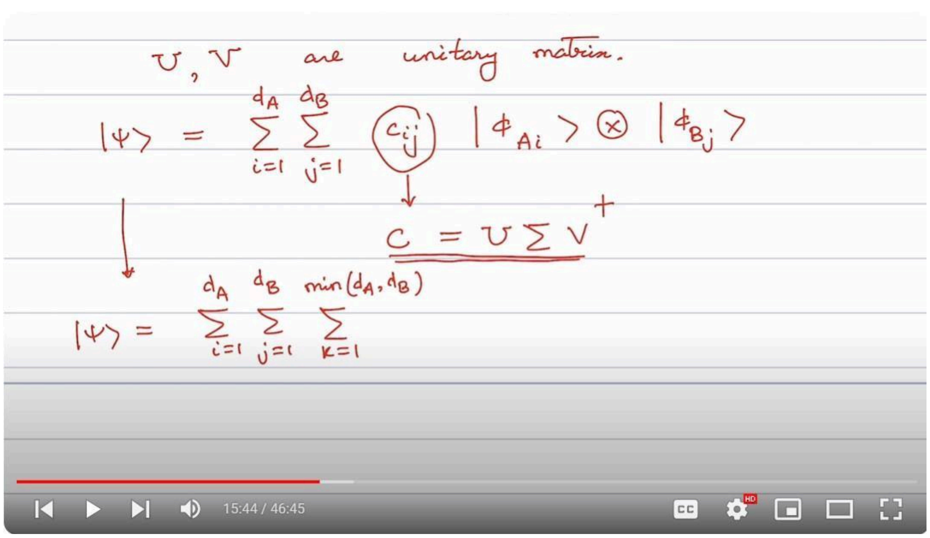

So we can therefore express, let me write this once again. Therefore we can express this ket psi state which we have written as say I goes from 1 to dA, J is equal to 1 to dB. I have Cij phi Ai cross that's a transfer product with phi Bj. Here Cij is a matrix C is I can express it using the SVD method as U sigma V dagger.

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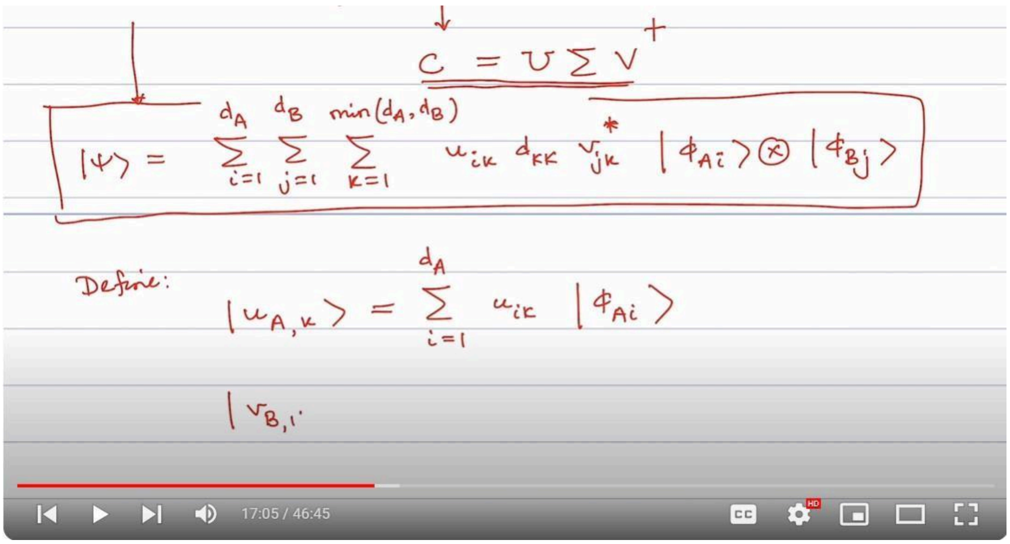

So this ket psi can be expressed as ket psi is equal to summation i is equal to 1 to dA, summation j is equal to 1 to dB and I will now express this matrix capital C in terms of components Cij I am just expressing in terms of matrix. So this can also I express in this form. I am going to write another sum going from say k is equal to 1 to the dimension would be such that it will go from the minimum of dA and dB. All these things will be clear to you once I discuss some example later on. And this coefficient Cij in terms of this matrix C, I can express in terms of capital U, capital sigma and V as follows.

I will write the sum as this Uik dk k because sigma is a diagonal matrix. Its main diagonal only I am going to consider and then I have Vjk star it's a hermitian conjugate and we are having the other parts as phi Ai cross phi Bj. Okay. Now let me define to simplify this big expression I can define let me define quantities. Say ket uak, small uak is equal to summation i is equal to 1 to dA Uik phi Ai and Vak VBk.

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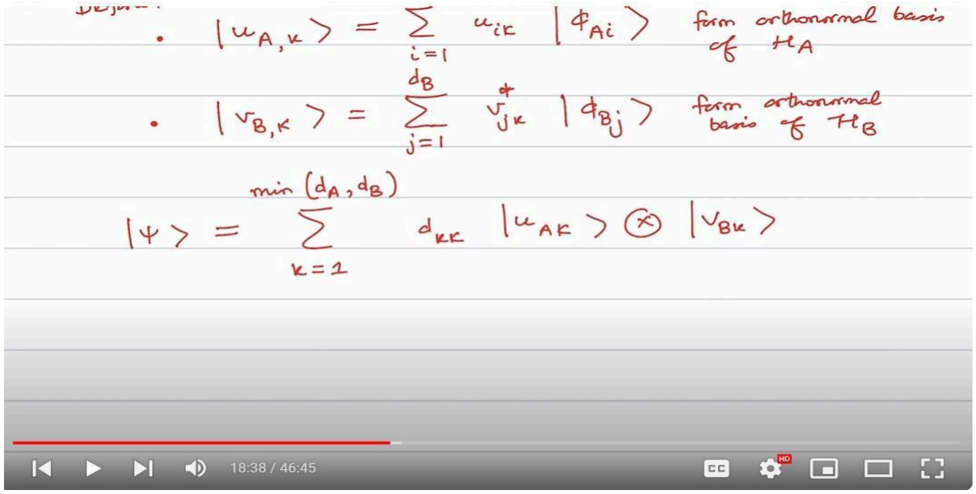

Let me write another ket phi VBk is equal to j is equal to 1 to dB Vjk star phi Bj. Okay. Now due to unitarity of capital U and capital V this ket uak and ubk they form orthonormal basis. Let me write here form orthonormal basis of the Hilbert space HA. And similarly VBk form orthonormal basis of HB. Okay. So using this definition we can obtain this ket psi in this form is going from k is equal to 1 to minimum of dA and dB. Whatever is minimum out of dA and dB then you have this dkk uak tensor product with VBk. Okay.

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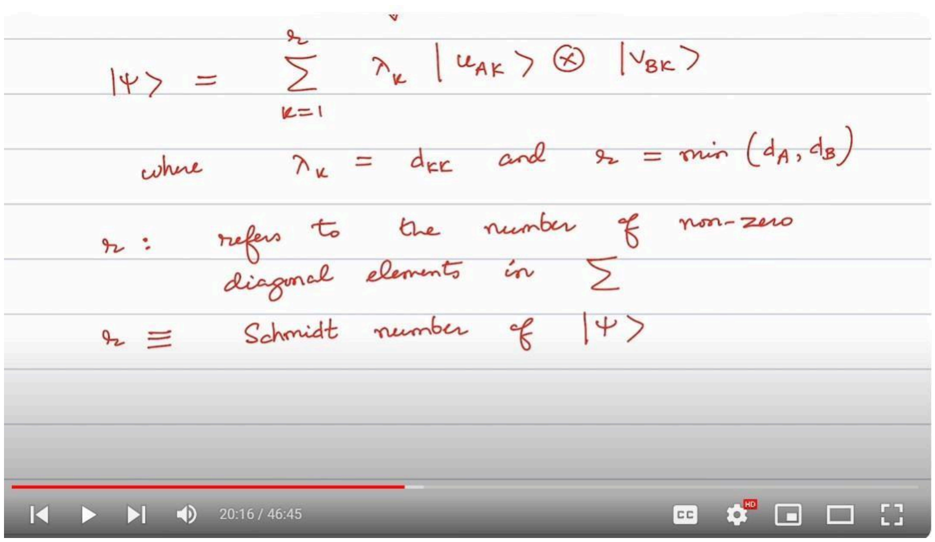

So to write it in a more convenient form I can write this in a more convenient form. Let me write psi is equal to k is equal to 1 to R lambda k uak direct product or tensor product with VBk. Where lambda k as you can see is equal to dkk and R is equal to minimum of dA dB. Actually let me explain this quantity R a little bit. This refer to R refers to the number it refers to the number of non-zero diagonal elements. Non-zero diagonal elements of the or in the capital sigma matrix. And this has a name and this quantity is called R is named as the Schmidt number. It's called it's an important quantity. This is called Schmidt number of the state ket psi.

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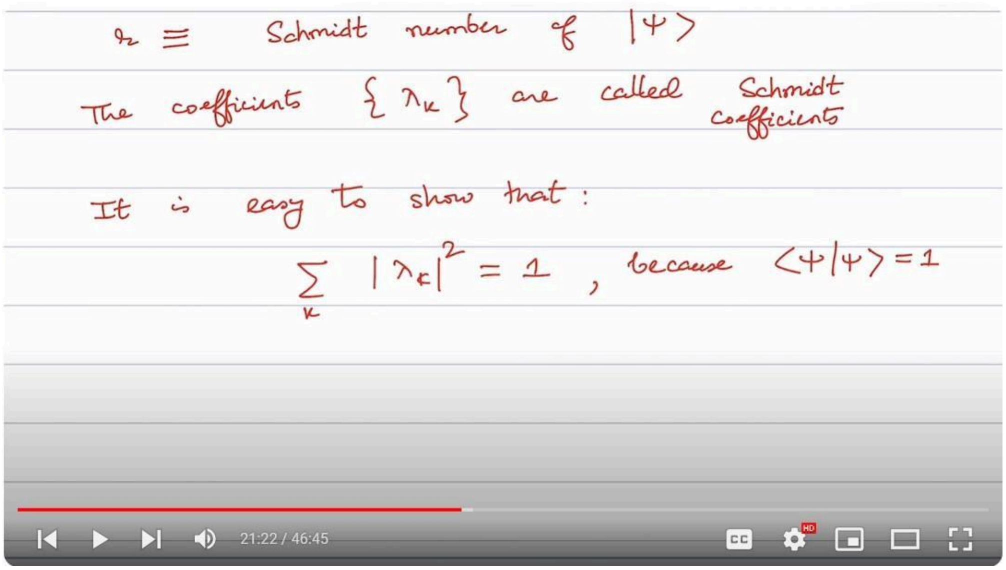

And the coefficients, the coefficients lambdas, the coefficients lambda k has also a name. R called Schmidt coefficients. It is easy to show, easy to show that I encourage you to do it yourself. It's very trivial that summation over k mod lambda k square is equal to 1. And you can show it because of the fact that this state psi ket psi is normalized. So using this fact you can easily show that summation over lambda k square mod lambda k square is equal to 1. Now before I go further just quickly let me also write down the density operator for the bipartite system.

Using this new form where we have written the ket psi in this particular form and this form is known as the Schmidt form. This is what is basically Schmidt decomposition. It may be useful to write the density operator for the bipartite system using this Schmidt decomposition form.

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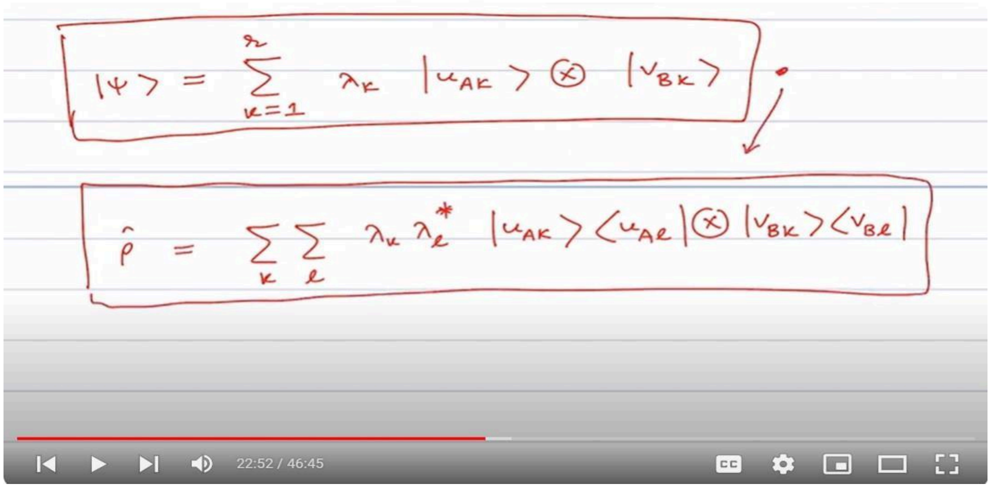

The Schmidt decomposition form that we have is this ket psi is equal to k is equal to 1 to R lambda k U a k tensor product with V b k. This is the so called Schmidt decomposition form. And the corresponding density operator you can easily work out yourself and you can see that it would be summation over k and summation over L lambda k lambda L star U a k the bra U a L direct product with ket V k and bra V b L. I think you can easily follow it. It's easy to follow from the form that I have written from here. Using this you can easily get it.

So we have seen that every bipartite system in a pure state can be rewritten in terms of Schmidt decomposition. Let us now illustrate this method using a quick example.

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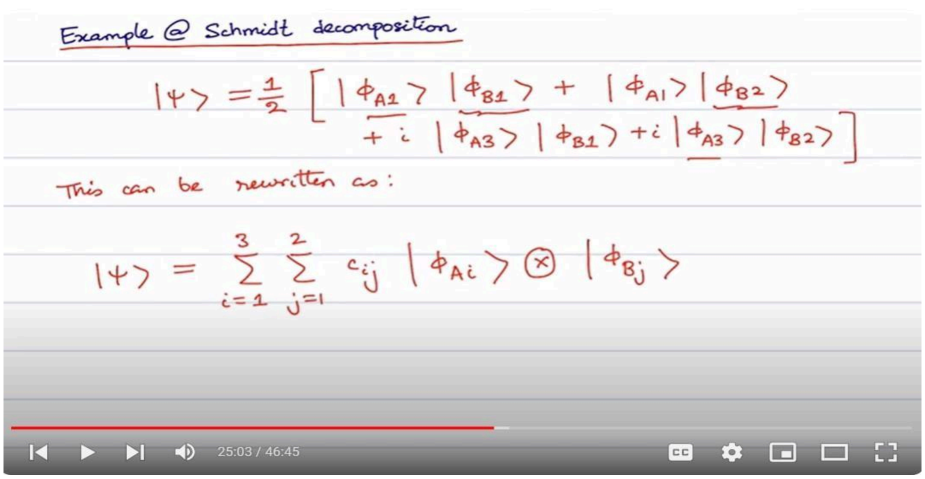

Consider a bipartite state given by this ket. ket psi is equal to half phi a 1 phi b 1 plus phi a 1 phi b 2. All these phi's are orthonormal states phi b 2 plus i phi a 3 phi b 1 plus i phi a 3 phi b 2. Okay this is the ket state given to you. Now this can be rewritten. This particular given state can be rewritten as follows. It can be rewritten as ket psi is equal to summation i is equal to 1 to 3 and summation j is equal to 1 to 2 cij phi ai. As you can see phi ai here i goes from 1 to 3 right as you can see and, in the process, as you can see phi a2 is equal to 0. But anyhow the suffix i goes from 1 to 3 it would be direct product with phi bj and j goes from 1 to 2 as you can see here in this expression. We have phi b1 and phi b2 is there so it goes j goes from that's why from 1 to 2.

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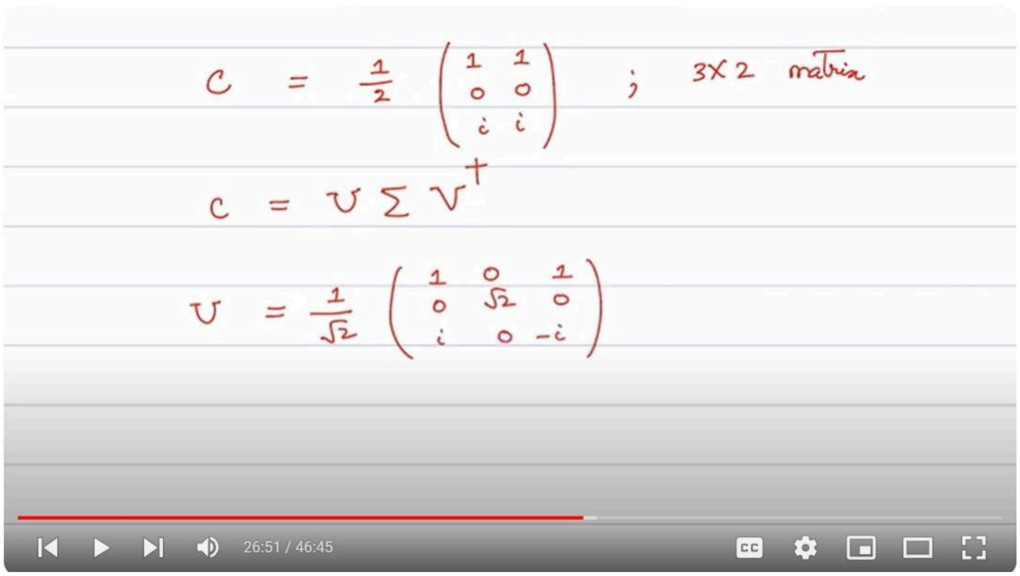

So obviously this is a pure state and this coefficients cij form a matrix. This coefficients cij would be 1 1 0 0 i i. So this is a non-square matrix. This is a 3 by 2 matrix 3 rows it's a 3 by 2 matrix 3 rows and 2 columns.

Okay. And this matrix we can express in the form of SVD. In fact we have actually already worked out the singular value decomposition of this matrix a few minutes back. And we can use those results now from the example that we have done in the context of singular value decomposition. We have exactly used this particular matrix c. Let me write here instead of cij let me write capital C that's the matrix I have formed from this coefficients cij. And now this matrix can be written in the singular value decomposition form where we already worked out.

It would be U capital sigma V Hermitian conjugate of the V matrix. We have already worked out this capital U matrix in the example that we have done a little while back. It is 1 by root 2 1 0 1 0 root 2 0 i 0 minus i.

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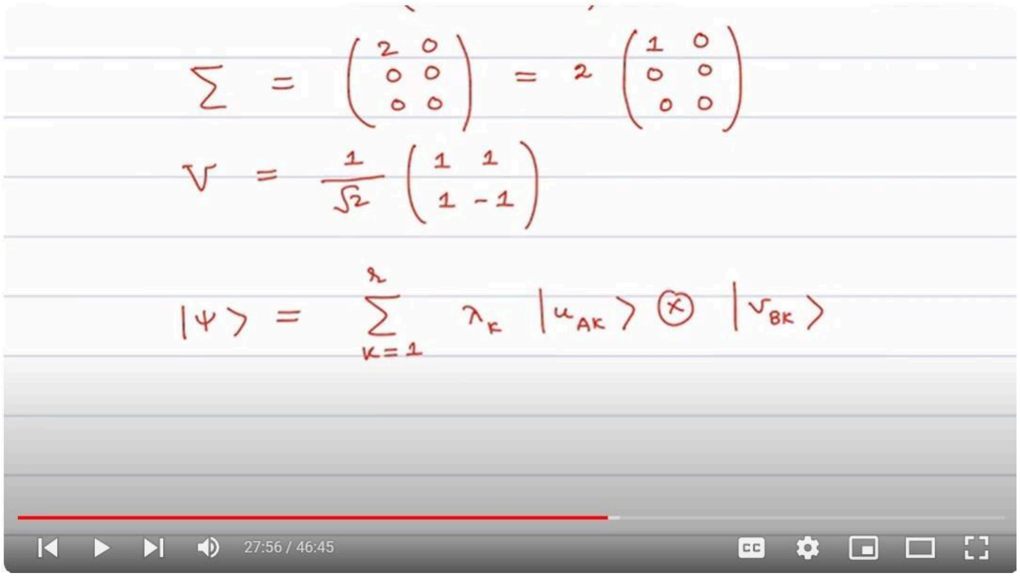

And capital sigma is 2 0 0 0 0 0. So this capital sigma matrix has only one non-zero diagonal element. So or we can write it as 1 0 0 0 0 0. And this matrix V capital V matrix we worked out as 1 by root 2 1 1 1 minus 1. We want to write ket psi in the Schmidt decomposition form as follows. ket psi is equal to summation k is equal to 1 to r lambda k U ak tensor product with V bk small v bk.

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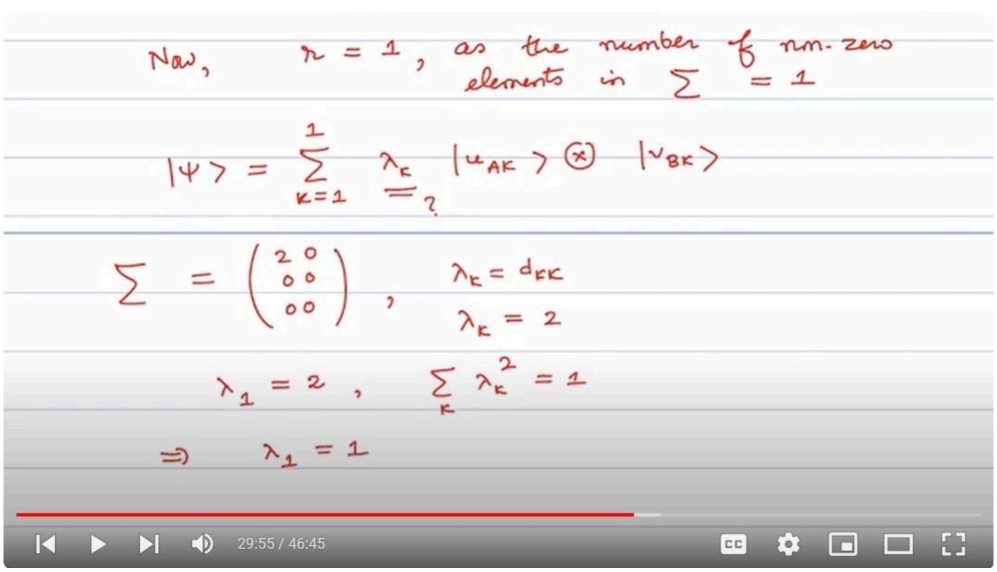

Now r is equal to 1 here. Now r is equal to 1 why because the number of non-zero elements in the sigma matrix is 1. That's why r is equal to 1 as per our definition of r that's called the Schmidt number. r is equal to 1 as the number of non-zero elements in sigma matrix is equal to 1. So ket psi is equal to in the Schmidt decomposition form summation k is equal to 1 to 1 lambda k U ak tensor product with V bk.

Now what about lambda k? What is this? Now from the sigma matrix let me write the sigma matrix we obtain as 2 0 0 0 0 0. And because lambda k is equal to dkk it may occur to that you should take lambda k is equal to 2. But because k runs from 1 to 1 that means we have only one lambda that is lambda 1. So you may write lambda 1 is equal to 2 but you have to be careful here because lambda has to satisfy this equation lambda k square is equal to 1. Right and this means that we have to take the normalized form of lambda and lambda 1 you better take it as 1.

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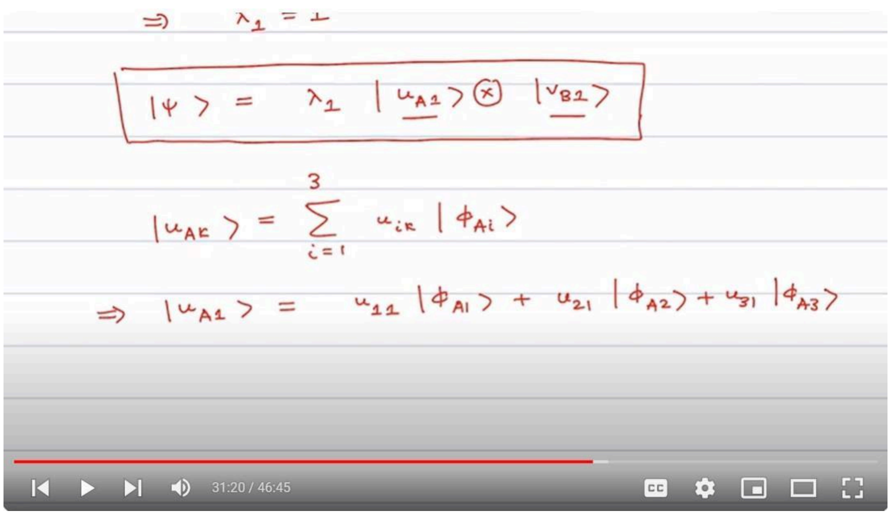

So therefore ket psi we should write as the summation is has no meaning now only we have only one term that is lambda 1 only one coefficient lambda 1. And other terms are U a1 direct product with V b1. So this is the Schmidt decomposition form of the example that we have considered but now let us find out what is U a1 and what is ket V b1.

Now as per the definition of ket U a1 k we had summation i is equal to 1 to 3 small u i k phi a i. And this we can now because k is equal to 1 therefore we can now write U a1 is equal to if I open it up I will have U11 phi a1 plus U21 phi a2 plus U31 phi a3. Right now the fact is that U a2 phi a2 ket phi a2 is equal to 0.

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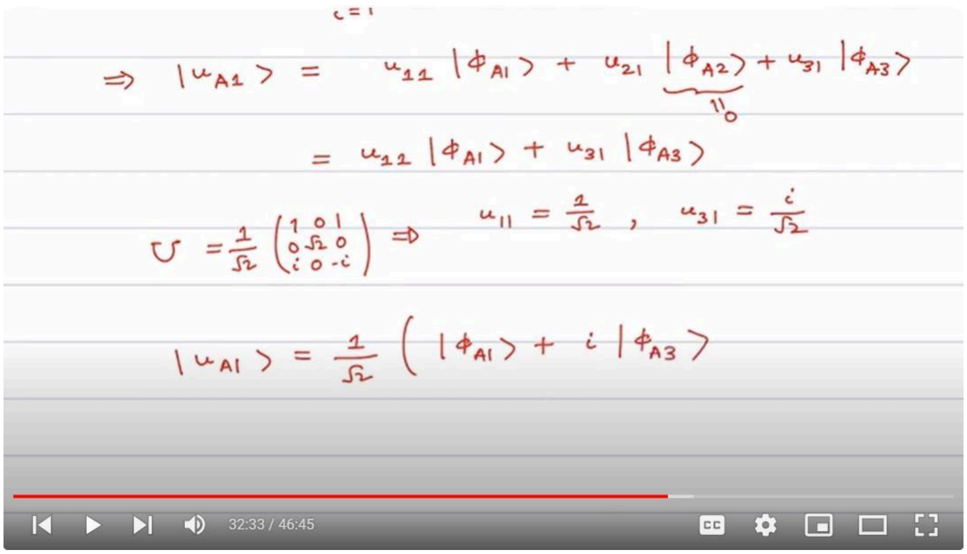

So I will have U11 phi a1 plus U31 phi a3. Now look at the matrix elements of capital U. Capital U matrix if you recall this was 1 by root 2 1 0 1 0 root 2 0 i 0 minus i. So from here you can easily see that U11 is equal to 1 by root 2 and U31 is equal to i by root 2. So therefore you have the ket U a1 is equal to 1 by root 2 phi a1 plus i phi a3.

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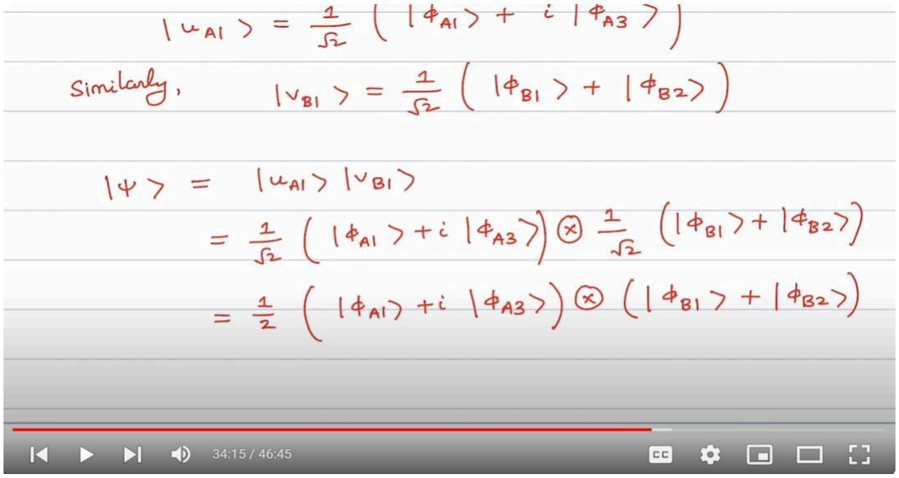

Okay what about V b1? Similarly you can work it out I leave it to you. Similarly you can show that ket V b1 is equal to 1 by root 2 phi b1 plus phi b2. So we can write the given ket psi in the Schmidt form as follows. ket psi is equal to U a1 V b1 and if I write U a1 ket U a1 is 1 by root 2 phi a1 plus i phi a3 tensor product. 1 by root 2 phi b1 plus phi b2 or half ket phi a1 plus i ket phi a3 tensor product with phi b1 ket phi b1 plus ket phi b2.

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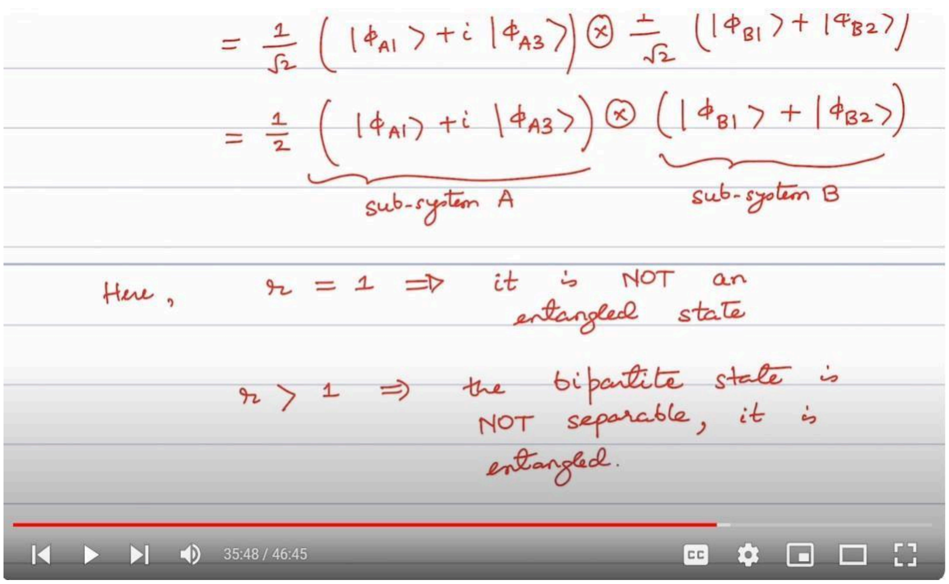

Now you please see that I am able to write the Schmidt form here in such a way that this part refers to the subsystem or one component called A component A or subsystem A on the other hand this one I write for subsystem B. So these states are separable here and in this example here the so called Schmidt number R is equal to 1 and the state is separable so it implies so I can conclude that if R is equal to 1 it is not an entangled state it's a separable state. It's not an entangled state. On the other hand if you find that for a given bipartite state R is greater than 1 that may imply that the bipartite state is not separable. In other words it is entangled.

Ok now some final words on Schmidt decomposition there are various forms of Schmidt decomposition in the literature.

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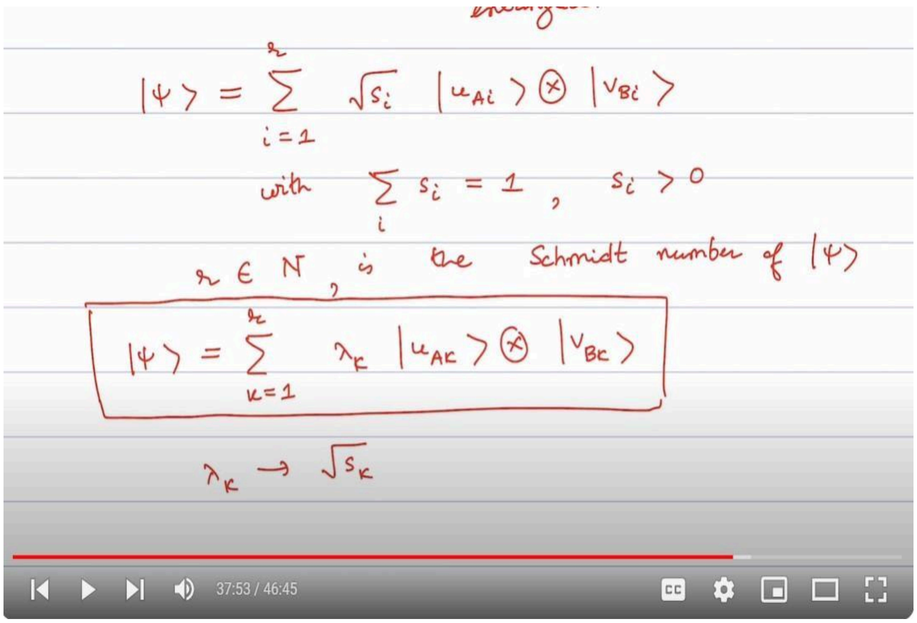

For example sometime you may see this particular form say ket psi is equal to summation i is equal to 1 to R square root of Si uAi tensor product with vBi this is for a bipartite system with the condition that with summation sum over Si is equal to 1 where Si is greater than 0 these are called Schmidt coefficient just like the number lambda i's and R is basically a natural number it's a number and it is called R is the Schmidt number of the state ket psi.

In fact it is basically the same thing we wrote for Schmidt decomposition of ket psi just to remind once again as a final comment we have written and we are going to take this particular form in this course of Schmidt decomposition where k goes from 1 to R lambda k uAk direct product or tensor product with vBk ok so this is the form we are going to use in this course and if you look at the correspondence here lambda k refers to square root of Sk in the previous expression and lambda k is termed as Schmidt coefficient.

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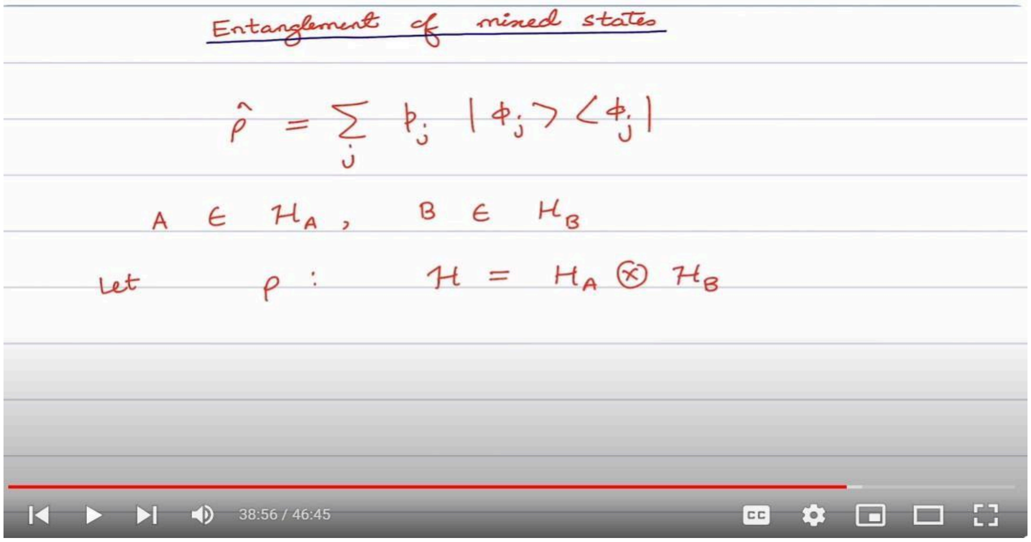

Now before I end this lecture let me now briefly talk about entanglement of mixed states we already know that the mixed state can be represented by a density operator say rho is equal to sum over j Pj ket phi j bra phi j now consider a composite system where it has consisting of say subsystem A and B subsystem A belongs to the Hilbert space HA and the system B belongs to the Hilbert space subsystem B belongs to the Hilbert space HB now let rho be a mixed state over the Hilbert space H is equal to HA tensor product with HB.

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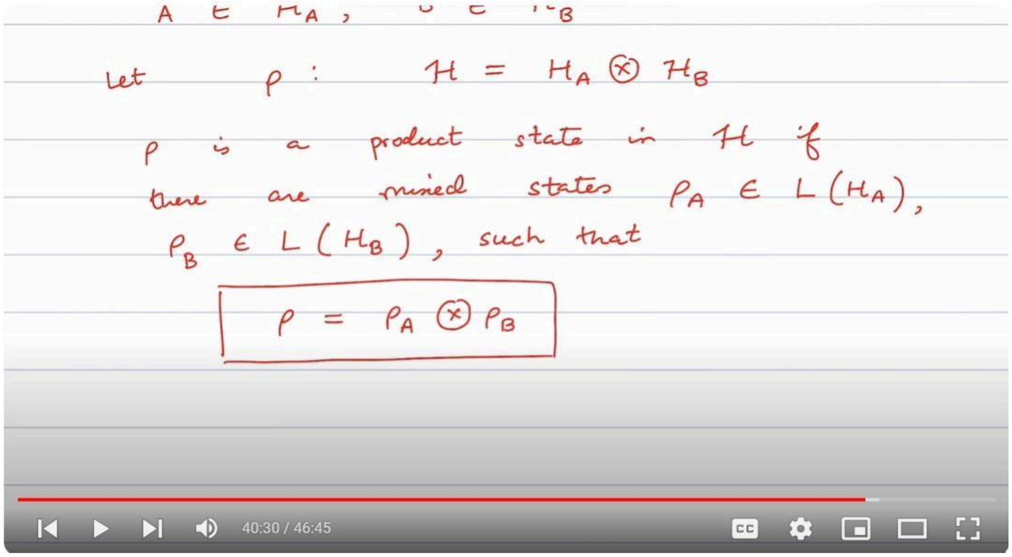

Rho is a product state is a product state in H in the Hilbert space H if there are mixed state there are mixed state rho A belonging to the to a set of linear operators on a Hilbert space HA that means rho A as you know it's a density operator it's an operator so it's a operator on a Hilbert space HA belonging to all set of linear operators that is in the Hilbert space HA similarly say rho B which is the mixed state for corresponding to the subsystem B and it also belongs to the set of linear operators in the Hilbert space HB ok and rho is such that rho is equal to direct product of the density matrices corresponding to the subsystem A and subsystem B and if we can write it in this form obviously rho is a product state.

What about the case where we have a n number of such product states forming a mixed states how can we characterize if they are separable or not there is a concept called convexity which may be useful in this context let us discuss it.

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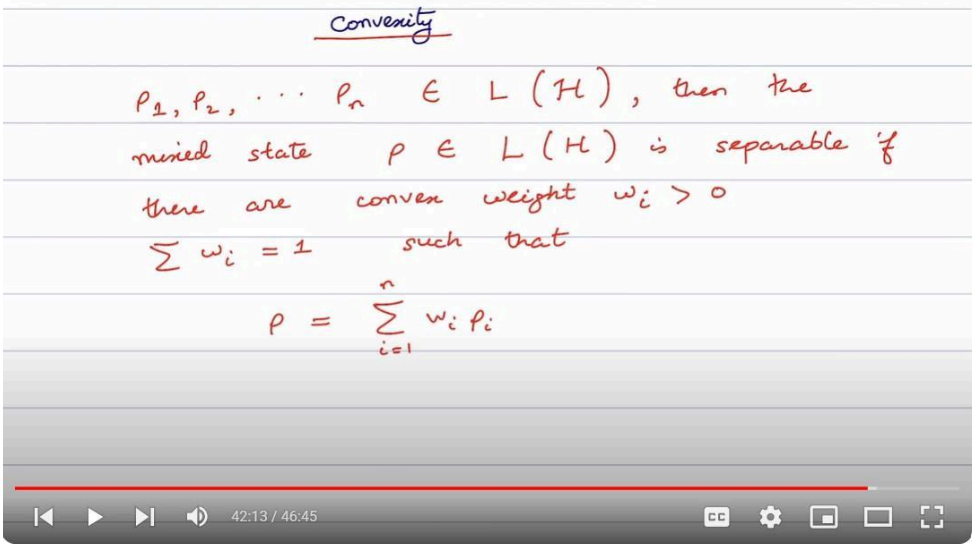

Say we have rho1 rho2 rho n n number of product states are there that belong to the set of product of linear operator in the hilbert space H and In that case then the mixed state which is formed out of this all this product states belonging to the linear operator in the Hilbert space H is separable if there are convex weights I will explain what it is there are convex weights denoted by say Wi which is greater than zero and also sum of all w's is equal to one such that rho is equal to summation i to n, i is equal to one to n, Wi rho i.

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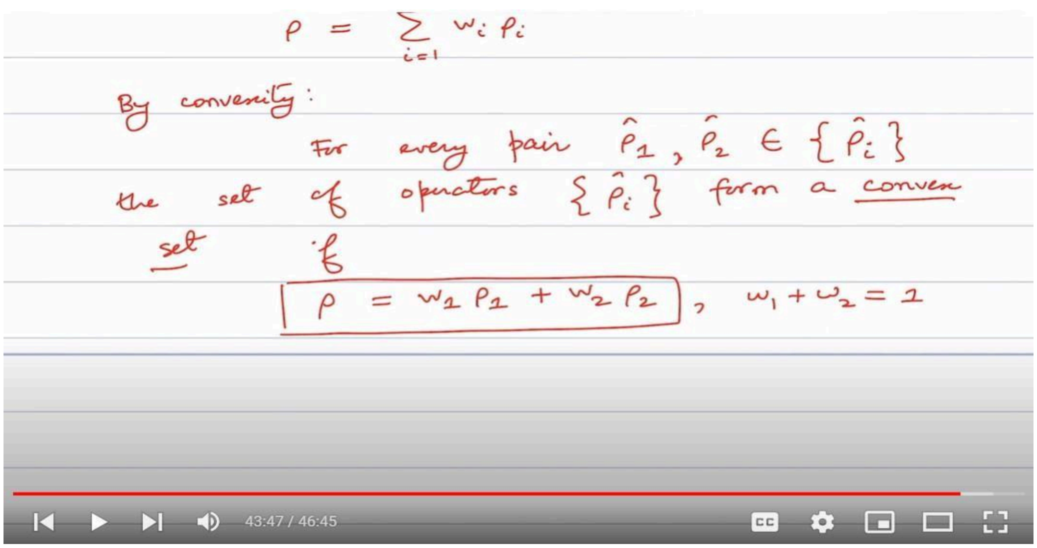

 Now what I mean by convexity is the following by convexity I mean the following I mean that for every pair for every pair rho 1, rho 2 which belongs to one of the rho i's the set of operators the set of operators rho i cap in fact these are also operators rho 1, rho 2, rho i so they form a for every pair rho 1, rho 2 the set of operators rho i form a convex set if rho you can write as w1 rho 1 plus w2 rho 2 so this is valid for every component that is belonging to the set of operators rho 1, rho 2 up to rho n you can write such relations of this type if you can do that with the condition w1 plus w2 is equal to one so this is what we mean by convexity and this is a very simple meaning

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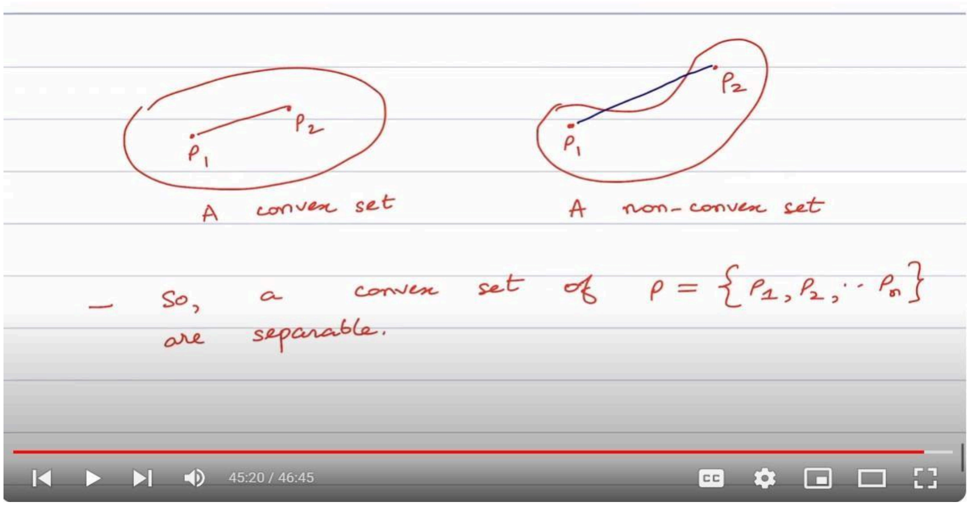

it means that for any two members form a convex set say rho 1, rho 2 they form a convex set if they can be connected by a straight line without leaving the set so this is what I mean by a convex set so as you can see in this convex set rho 1 and rho 2 can be connected to each other without leaving the set on the other hand by a non-convex set I mean the following say I have this member here rho 1 and this is rho 2 and if I want to connect rho 1 and rho 2 by a straight line so as you can see I have to leave the set right as you can see I have to leave the set so it's an example of a non-convex set so a convex set of rho is equal to rho 1, rho 2 up to rho n are separable that means in simple words if the product states rho 1, rho 2 up to rho n form a convex set then they are separable.

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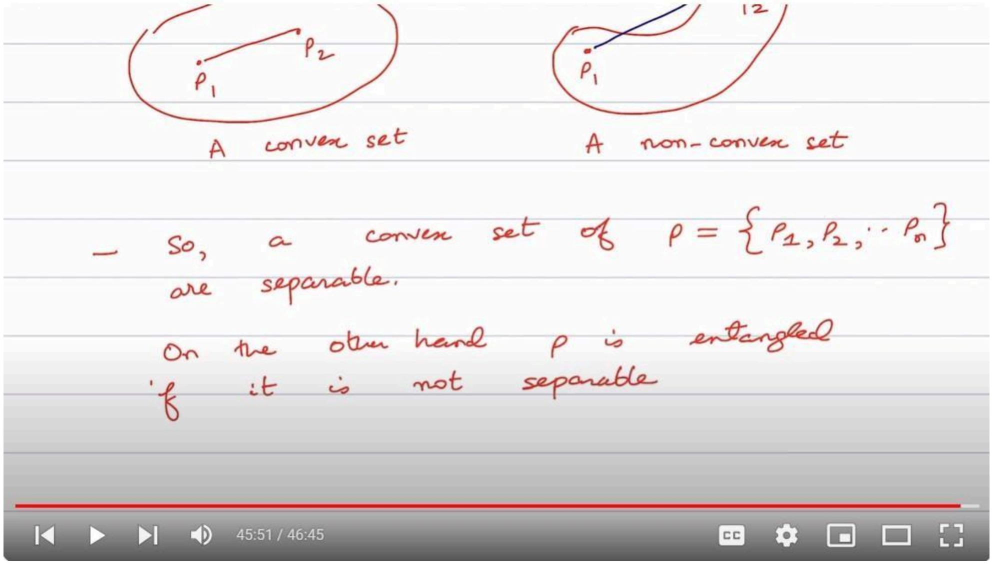

On the other hand rho is entangled if it is not separable. Let me stop now in this lecture we have learnt an important characterizing method of quantum entanglement in terms of the so called Schmidt decomposition method in the next lecture I am going to discuss the EPR paradox and Bayles inequalities and so on this may help you to get an intuitive understanding of quantum entanglement so see you in the next lecture thank you so much you.