

Quantum Hall Effect
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Week-02
Lec 4: Quantum Hall Effect

So, let me connect with the discussion that we were having last time, the previous discussion about the quantization of the Hall plateaus. So, this is the effect that we have seen that this is the quantum hall effect or the quantized plateaus which it is famous for. So, we have to understand these origin of this plateaus and how they are so robust to disorder and impurities as they are and also we have to understand these green plots which represent the ρ_{xx} or the magneto-resistance or the longitudinal resistance. So, let me just write that ρ_{xy} would be called as Hall resistivity and ρ_{xx} will be called as one can call it as a longitudinal resistance resistivity and or it is also called as a magneto resistivity. Basically the resistivity that arises because of the magnetic field and so these are as you see the green plots actually vanish most of the time excepting when the red curves show a jump from one plateau to another then it sort of shoots up and then again goes to 0 and it remains 0 till the plateaus exist in the Hall conductivity. This is a quantum phenomena as we have said earlier and we will be talking about this.

So, the quantum mechanical nature of the electrons in presence of a magnetic field will have to be worked out and we have to understand how these plateaus arise and why they are so robust so much so that they actually give the benchmark of resistance the plateaus actually h over e square sets the scale of resistance which is 25.813 kilo ohm and we have to understand all that. But before that for one last time let me do the classical analysis in order to understand the behavior of charged particle in a electromagnetic field or a magnetic field say and we have of course done this problem at equilibrium and we wanted to understand that what happens or how the Hall voltage develops in the transverse ages of the sample when there is a longitude current being sent from one end to another and that was a analysis that was done and at equilibrium when the force vanishes. But we have not talked about what is a trajectory of the particle in presence of an electromagnetic field and this is important because the trajectory being actually a path that is followed by a particle a charged particle should not really depend upon whether we are talking about classical mechanics or quantum mechanics.

Of course in quantum mechanics we rarely discuss or worry about the trajectory of the particle for the reason that the trajectory is it is there is an uncertainty principle which is underlying that and which governs the trajectory. So let us look at the classical trajectory of a charged particle and this is done with an intention that will be dealing with a

quantum mechanical system soon. So let us talk about two cases the case 1 being where there is no electric field and there is only a magnetic field which is constant which is not equal to 0 it is finite and it is uniform. So this is the situation 1 that is case 1 say and the energy is constant so please do not confuse between the electric field E and the energy, energy is a scalar so energy is a constant and energy is constant because of the reason that the power delivered is equal to 0 which is nothing but the Lorentz force is $\mathbf{V} \times \mathbf{B}$ and that is why this is equal to 0. So we can write this as $m \mathbf{V} \cdot \frac{d\mathbf{V}}{dt}$ which is equal to $\frac{d}{dt} \left(\frac{1}{2} m \mathbf{V} \cdot \mathbf{V} \right)$ and this is equal to $Q \int \mathbf{V} \cdot (\mathbf{V} \times \mathbf{B})$ which is equal to 0.

So this is your $\mathbf{F} \cdot \mathbf{V}$ because your \mathbf{F} is the Lorentz force when there is no electric field it is only the magnetic field will create a Lorentz force. So $\mathbf{V} \cdot (\mathbf{V} \times \mathbf{B})$ equal to 0 the reason is simple this $\mathbf{V} \times \mathbf{B}$ is a vector which is perpendicular to both \mathbf{V} and \mathbf{B} and that is why if you take a dot product with \mathbf{V} it will give you 0 and that is why $\frac{d}{dt} \left(\frac{1}{2} m \mathbf{V} \cdot \mathbf{V} \right)$ equal to 0 which means $\frac{1}{2} m \mathbf{V} \cdot \mathbf{V}$ equal to constant and this is what we mean by the energy being a constant that is the only energy that is left in the problem. So this you know results in separate equations or separate dynamics for the longitudinal and the transverse direction. So if you write your \mathbf{V} to be having a parallel component and a perpendicular component and these dynamics are uncoupled or decoupled and the motion takes place independent of each other in absence of an electric field. So let us talk about the perpendicular motion and what we mean by the perpendicular motion.

So we have $m \frac{dV_x}{dt}$ this is equal to $q V_y B$ so this is say equation number 1 and the equation number 2 will be $m \frac{dV_y}{dt}$ equal to minus $q V_x B$. We are continuing with the same situation where E equal to 0 and B is uniform and not equal to 0 and we are just discussing or rather deliberating upon the nature of the motion and this is quite instructive. If you take a double derivative of this so differentiate with respect to time t and then of course use 2. So then $m \frac{d^2 V_x}{dt^2}$ so this is equal to $q \frac{dV_y}{dt} B$, I mean into B and then I can take this M to be in the denominator this is in the denominator. So this now I will use $\frac{dV_y}{dt}$ to be here so this is equal to q into B by M and then this is equal to minus q into $V_x B$ divided by m .

Classical trajectory of a charge particle.

Case-I $\vec{E} = 0$, $\vec{B} \neq 0$ uniform.

Energy is constant $\vec{F} \cdot \vec{v} = 0$

$$m \vec{v} \cdot \dot{\vec{v}} = \frac{d}{dt} \left(\frac{1}{2} m v^2 \right) = \underbrace{q \vec{v} \cdot (\vec{v} \times \vec{B})}_{=0} = 0.$$

$$\frac{1}{2} m v^2 = \text{Constant.}$$

$$\vec{v} = (\vec{v}_{||}, \vec{v}_{\perp})$$

(i) Perpendicular Motion

$$m \dot{v}_x = q v_y B \quad (1) \rightarrow \text{differentiate w.r.t } t.$$

$$m \dot{v}_y = -q v_x B \quad (2) \text{ use (2)}$$

$$\ddot{v}_x = \frac{q \dot{v}_y B}{m} = \frac{q B}{m} \cdot \left(-\frac{q v_x B}{m} \right) = - \left(\frac{q B}{m} \right)^2 v_x = -\omega_B^2 v_x$$

$\omega_B = \frac{q B}{m}$: cyclotron frequency.

So this becomes equal to minus qB by m square and V_X and this is equal to nothing but minus which we have written it earlier that $\omega_B^2 V_X$ or ω_B is nothing but equal to qB over m which we have called as the cyclotron frequency even earlier. So just a quick recap we are talking about no electric field and a charged particle is moving with a velocity V which has components V_X and V_Y and there is a B there that is a magnetic field because of magnetic field there is a Lorentz force and in this particular case we know that the energy is constant because $\vec{F} \cdot \vec{v} = 0$ and now we are analyzing the motion by writing down the equations of motion. So $m \dot{V}_X$ is equal to $q V_Y B$ because \vec{F} is equal to $q \vec{v} \times \vec{B}$ and that is why the X component or the derivative of the X component of the velocity is related to the Y component of the velocity and similarly for the Y component is X component but with a negative sign and now I did some little manipulation here in order to write this V_X double dot. Now the end result is that the V_X double dot becomes equal to minus $\omega_B^2 V_X$ and all of you know that this corresponds to a harmonic motion where V_X will harmonically depend on time and so this is what the solution is and the solution is that V_X is equal to V_{\perp} cosine of $\omega_B t$ and similarly if you do the same calculation for V_Y then V_Y turns out to be m divided by qB into V_X dot which is equal to a minus q divided by m and a V and a sine $\omega_B t$. So, this is the trajectory for the perpendicular motion.

So, this is the trajectory in the X direction and this is the trajectory in the Y direction

and why we are talking about the perpendicular because we are considering a perpendicular to the magnetic field. So, if you integrate and want to know that how the trajectory looks like then you will get so integrating these equations we have already written a 1 and 2 so this could be 3 and 4. So, integrating 3 and 4 and using you know some initial conditions so X is equal to X_0 plus a V_{\perp} divided by ωB and a $\sin \omega B T$ because I integrate it with respect to T . So, this will give me the X motion and the Y motion is almost similar harmonic, but out of phase by an amount $\pi/2$ this is a $\cos \omega B T$. So, this is the X motion and the Y motion and it is clear that if you superpose this motion you get a circular motion and let me show that circular motion let us see how neatly I can draw a circle not very good, but this is that circular motion that we are talking about this point $X_0 Y_0$ it may look slightly you know shifted towards the positive side, but that is not what is intended and this things are so this is the radius of that.

So, this is where the positive q motion of the particle that is clockwise and this is the motion when q is negative and if you want to know the you know the axis because this is very important. So, this is the X and this is the Y and this is Z and this is the direction of B . So, B is in the Z direction and this is the planar motion. So, this is the orbit of a charged particle about a guiding center $X_0 Y_0$ okay. So, this is called as a guiding center this point about which the motion takes place and this is pretty much the motion of a charged particle only in a magnetic field.

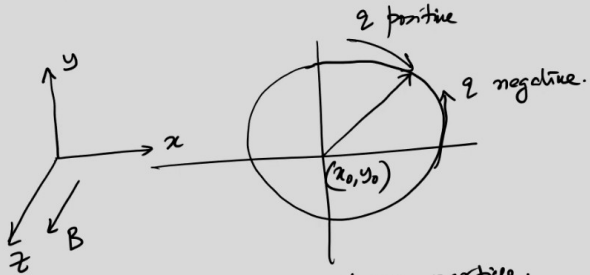
So, let us look at case 2 now when we will talk about $E \neq 0$ $B \neq 0$ okay, but E is perpendicular to B okay. So, it is basically a generalized condition of the one that we just saw and we can write down the equation of motion which is F which is equal to $m \dot{v}$ which is equal to $q(E + v \times B)$ okay. Now the total force has a component or rather a contribution coming from the electric field and earlier of course, for $E = 0$ which is the case 1 which we saw your v_{\parallel} was a constant. So, the parallel component of v , but now because of this the parallel component actually has an acceleration which is given by the $q E_{\parallel}$ and the parallel component of E divided by m . So, what happens is that there is no fixed guiding center that you are seeing here the guiding center remains fixed and the charged particle either in the clockwise direction or anticlockwise direction depending on its sign would just move around this circle, but now this guiding center keeps getting drifted because of the electric field let us see that okay.

$$v_x = v_{\perp} \cos \omega_B t \quad (3)$$

$$v_y = \frac{m}{qB} \dot{v}_x = -\frac{|q|}{2} v_{\perp} \sin \omega_B t \quad (4)$$

Integrating (3) & (4)

$$x = x_0 + \frac{v_{\perp}}{\omega_B} \sin \omega_B t$$

$$y = y_0 + \frac{q}{|q|} \frac{v_{\perp}}{\omega_B} \cos \omega_B t$$


Orbit of a charge particle.
about a guiding center (x_0, y_0) .

Case II $E \neq 0, B \neq 0$
 $E \perp B$

$$\vec{F} = m \dot{\vec{v}} = q (\vec{E} + \vec{v} \times \vec{B})$$

for $E = 0, v_{||}$ constant

$$\dot{v}_{||} = \frac{q E_{||}}{m}$$

So, the guiding center which was earlier we have denoted it by x_0, y_0 . So, this guiding center x_0, y_0 it moves perpendicular to both E and B okay. So, this guiding center moves in that direction. So, the charged particle drifts the charged particle drifts with a velocity V_d . The reason that I am writing this is that there is another component of velocity or rather another contribution to the velocity which you also will have to worry about.

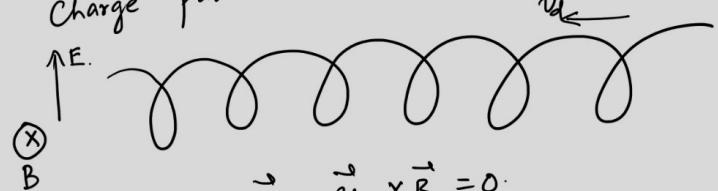
So, that is because of the guiding center that you know moves in a direction which is perpendicular to both E and B the charged particle has to drift and this drifting actually is represented by like this and so on okay. So, the drift actually happens in this direction and in particular suppose you have B to be pointing inside the screen. So, that you know this is usually represented because you see an arrow the back of the arrow if you see it looks like a cross and that is why when you write it with a cross a direction which means it is going into the plane of the paper in this case it is a plane of the board or this whiteboard that you are seeing and we have electric field in this direction. So, one is going piercing inside the board and then the other the electric field is in this direction and then the drift happens in a direction which is perpendicular to both which in this particular case it happens in this direction that is from left to right okay. So, how do we explain this drift velocity or what is the corresponding equation that it satisfies? The drift velocity satisfies this equation which is $V_d \times B = 0$.

So, this is the net force is 0 and this is the equation that it would satisfy and defines this V_d is defined by this equation okay. And so, how do we get this? So, we can actually

get you can take a cross product with \vec{B} cross product with \vec{B} that gives you 0 equal to \vec{E} cross \vec{B} and \vec{V}_d cross \vec{B} cross \vec{B} . I am sure you can have seen these kind of a cross product triple cross product \vec{A} cross \vec{B} cross \vec{C} and there is a particular formula that one uses which you can look up and this is equal to nothing, but this is like \vec{E} cross \vec{B} and a plus \vec{V}_d dot \vec{B} and \vec{B} here and minus a \vec{B} square and a \vec{V}_d okay. So, this is the equation for this or rather this after you simplify it becomes this part and then your \vec{V}_d becomes equal to \vec{E} cross \vec{B} divided by \vec{B} square okay. So, \vec{B} square you can write it with a square and a mod, but this is also fine.

So, this is the drift velocity of the charged particle in presence of an electromagnetic field where the electric and the magnetic fields are crossed with respect to each other that is their perpendicular with respect to each other all right. So, there is also another contribution to this velocity. In fact, this velocity is coming from even if there is no \vec{E} term that is the electric field term there will be a velocity due to the Lorentz force which is what we have seen earlier and so, this let us call it as \vec{V}_L and this L stands for Lorentz force. So, let us call this as a Lorentz velocity and this Lorentz velocity can be found out by taking a d/dt of these orbit that we have talked about and this is like exponential $i\omega_B t$ minus some initial t naught okay. So, this is purely because of the Lorentz force which is nothing, but so, \vec{F}_L is nothing, but $q \vec{V}$ cross \vec{B} okay.

The guiding center (x_0, y_0) moves perpendicular to both \vec{E} & \vec{B} .
 Charge particle drifts with a velocity \vec{v}_d .



Cross product with \vec{B} ,
 $\vec{E} + \vec{v}_d \times \vec{B} = 0$.

$0 = \vec{E} \times \vec{B} + (\vec{v}_d \times \vec{B}) \times \vec{B} = \vec{E} \times \vec{B} + (\vec{v}_d \cdot \vec{B}) \vec{B} - |\vec{B}|^2 \vec{v}_d$

$\vec{F}_L = q(\vec{v} \times \vec{B})$

Lorentz velocity $\vec{v}_d = \frac{\vec{E} \times \vec{B}}{B^2}$

$\vec{v}_L = \frac{d}{dt} (\vec{r}_L e^{i\omega_B(t-t_0)})$

$\vec{v} = \vec{v}_L + \vec{v}_d + \vec{v}_{||}$

So, this is a Lorentz velocity this will get added to the drift velocity and that net velocity

will become equal to so, it is a V_L plus V_d okay. And of course, because of this this electric field there is also a velocity which will be there which is the parallel component of the velocity and this is the total velocity of the particle and this total velocity can be integrated in order to find r as a function of t and this is done in all books classical electrodynamics books and especially one can look at Griffiths' book on electrodynamics and it is nicely worked out there that it is actually a helix a moving helix which you know sort of moves in a direction which is perpendicular to both E and B and if you have non uniform B that creates another complication and in which maybe you will have you know the radius of the moving orbit changes and so on. The reason that we have discussed is that we needed to understand the trajectory of the charged particle and we are really dealing with electrons in an electromagnetic field or in a magnetic field. So, and these electrons are governed by laws of quantum mechanics and we will have to do a quantum mechanical treatment of this problem and the finally, the problem would lead to giving us this quantum Hall effect or the integer quantum Hall effect which I showed you even at the beginning of the discussion today.

Alright. So, let me sort of do a quantum mechanics. Now this is of course, a large number of electrons involved but they are mostly semiconductors or metals that we are talking about and they have electrons which the large number of electrons being there, but still the inter-particle interaction can still be ignored. So, you can talk about them as free electrons and the Drude theory would explain their the conductivities and the resistivities as we have talked about. Let us write down a quantum mechanics of a charged particle and first let us do it only for the magnetic field and we will see that the physics is not radically altered when you include the electric field there are some ramifications of that which we will talk about. And how do we do this problem? Since we are talking about single charge particle because the electronic interactions can be neglected here they are weakly interacting or they are not interacting at all we can solve a Schrodinger equation in presence of a magnetic field.

So, how does one solve a Schrodinger equation in presence of a magnetic field? So, let us write down the Schrodinger equation in its simplest form it is minus \hbar^2 by $2m$ and ∇^2 and a ψ of say r and this r is well there is a ψ here which is a function of r and there is a function of r and this if you write it the time independent Schrodinger equation then this is a formula or rather this is the equation that you need to solve and you have solved it in a variety of situations in which you had say the particle in a box say for example, in which ψ is equal to 0 inside the box and it is equal to infinity at the walls or rather outside the box it is infinity which means that the particle cannot escape or you have done a problem in which these are finite steps or they are finite wells or they are finite barrier or they are infinite barriers such as a delta function barrier and so on so

forth which are all part of the quantum mechanics course first course on quantum mechanics that you have learned. Now we are going to solve the same problem and we are going to put this equal to 0 because we claim rather assert that there is no interaction that needed to be taken into account even if it is there it is very small and can be ignored and now you have to somehow include the presence of the magnetic field. And if you remember that the magnetic field for a particle charged particle that enters through the momentum. So, the momentum which is written as p which was without any magnetic field or any other thing is equal to m into v which is called as a mechanical momentum this thing changes into in presence of a magnetic field this is p minus $e A$. So, if this changes from p to p minus $q A$ where q is the charge and A is the vector potential and this vector potential is an important quantity and we know that the B actually is given by curl of A .

So, this fixes the gauge now why I say a gauge because you know when b is perpendicular that is in the z direction then A can be either in the x direction or can be in the y direction or can be in any of x and y direction I mean it is in a direction which is perpendicular to the z direction. So, say you have asserted that b is in the z direction. So, the understanding is that we have a sort of planar charges. So, there are charges here and which is you know a material like this say for example, a thin material and there is a magnetic field which is perpendicular and acting in the z direction. So, this is the z direction all right.

So, so B is equal to $B_z \hat{z}$ in that case a can be equal to you know minus B_y that is the x component 0 0 it can also be 0 b_x 0 and it can also be you know half equal to half r cross B and this is for a constant magnetic field. So, this is equal to like B_y by 2 minus B_x by 2 and 0. So, these are B into y and B into x . So, all three forms of the vector potential would create the same B which is in the which is constant and in the z direction you just have to convince yourself by taking the curl of this and so these are called gauge freedom you know I mean in the sense that it the final result would not depend upon which A you use okay. And this A is also you can add or subtract say for example, you can add a gradient of a scalar quantity and see this scalar quantity does not matter because when you take a curl of this a curl of A gradient would always be 0 and you can understand it in this particular fashion that a gradient has a particular direction in space in which you ask this question that if λ is a scalar quantity say it is like a heat density or something or there is a sound wave that are or sound field that are produced because of some loudspeakers and so on and they are placed in four corners of the room and you may want to ask this question that in which direction the sound intensity changes the fastest and in which case you have to find the gradient in which it changes the fastest and that gradient is actually a direction and because it is a direction if you take

a curl of a particular direction it has to give you 0 that is why there are these choices of λ along with this minus gradient of λ that these are called the gauge choices.

Finally, the results that you get are physical results physical observables so they will not depend upon which choice of λ you have taken or whether what choice of λ you have taken and it there is a final result and it will not depend upon that all right. So, let us take this gauge as let us fix the gauge to be equal to A_x equal to that is the x component of A is minus B_y . So, A is equal to so the first choice that we have talked about let us take that, but it really does not matter and this is called as a Landau gauge. So, we do this and then put this in the Schrodinger equation and now make sure that your q is nothing but minus e because we are talking about electrons specifically. So, we will have this equation which is 1 over $2m$ because it is a p square over $2m$. So, we have a p_x minus $e B_y$ by square and plus a p_y square over $2m$ and a plus p_z square over $2m$ you can actually put the $2m$ outside maybe let us do that. So, that I do not have to write these and this the only thing and then this ψ is equal to $E \psi$ is the equation.

Quantum mechanics of a charge particle in Magnetic field

$$\frac{-\hbar^2}{2m} \nabla^2 \psi(\vec{r}) + \underbrace{V(\vec{r})}_{=0} \psi(\vec{r}) = E \psi(\vec{r})$$

$$\vec{p} = m \vec{v}$$

In presence of a magnetic field $\vec{p} \rightarrow \vec{p} - q\vec{A}$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{B} = B_z \hat{z}$$

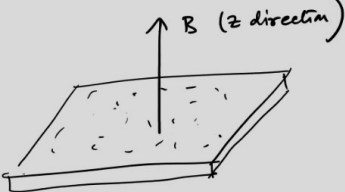
$$\vec{A} = (-By, 0, 0)$$

$$= (0, Bx, 0)$$

$$= \frac{1}{2} \vec{r} \times \vec{B} = (\frac{By}{2}, -\frac{Bx}{2}, 0)$$

def's fix $\vec{A} = (-By, 0, 0) \xrightarrow{\vec{A} \rightarrow \vec{A} - \vec{\nabla}\lambda}$ Landau gauge.

$$\frac{1}{2m} \left[(p_x - eBy)^2 + p_y^2 + p_z^2 \right] \psi = E \psi \rightarrow \text{Schrodinger Equation}$$



So, this is a Schrodinger equation that you have to solve. Now, it is very clear that the motion in the z direction is exactly like a free particle it is just like a particle in a box and that direction is quantized and in fact, it really does not matter if you are talking about perfectly 2 dimensional electron the z component cannot be there even if it is there it is quantized. So, the electron cannot escape in the z direction because we were talking about a 2 dimensional problem.

So, that problem then let us eliminate the z part of the kinetic energy or this p_z^2 square over $2m$ and let us simply write it as like a $\frac{1}{2m}$ and then the p_x^2 minus $e B y$ whole square plus a p_y^2 square and so on ψ is equal to. Now, this ψ is a function of x and y only and this is equal to let us talk about an energy let us let us say that is denoted by $\epsilon \psi$ of x, y . I hope you understood that why p_x^2 minus $e B y$ is written because the vector potential is assumed to be only present in the x direction. So, it is p_x^2 minus $e A$ or p_x^2 plus $e A$ rather because p_x^2 minus $q A$. So, p_x^2 plus $e A$ that A has only x component.

So, only the p_x changes and nothing else the p_y remains as it is. Now, this is of course, a 2 dimensional Hamiltonian you can call it some $H(x, y)$. So, if you write it symbolically it is like $H(x, y) \psi(x, y) = \epsilon \psi(x, y)$ and this is what you have to solve for. So, you have to solve for the eigen solutions and the eigen solutions are nothing, but the $\psi(x, y)$ and the energies are ϵ are the 2 things then one has to solve for. There is something interesting that happens here if you want to ask that which component of the momentum is conserved then it is easy to see that this p_x and $H(x, y)$ is conserved and the reason is that there is no x variable here had there been an x variable then there would be no commutation between p_x and h , but because we have decided to take the vector potential in the x direction and with a $B y$ that is why it this p_x commutes with the Hamiltonian.

And from elementary quantum mechanics as you know that if an operator commutes with the Hamiltonian then the corresponding quantum number remains conserved or there are quantities that remain conserved here of course, the p_x and that is the x component of the momentum remains conserved and the quantum number or the sort of quantity that remains conserved is called as a k_x . So, k_x is $\hbar k_x$. So, p_x is equal to constant which tells you that $\hbar k_x$ is equal to constant okay. So, $\hbar k_x$ is constant and now this k_x because we are talking about a quantum mechanical problem this k_x is quantized as $2\pi n_x$ over l_x you know n_x divided by l_x okay. It is just that just the way we have learned that particle in a box the k vectors are quantized the k vectors are quantized because of the presence of boundaries there and that is we have seen that k equal to you know $n\pi$ over l here because of the periodic boundary condition you get a factor of 2.

So, this k_x is equal to $2\pi n_x$ by l_x and n_x are $0, 1, 2, 3$ and so on okay. So, these are the values that are sorry the 0 can be neglected because 0 means that the particle is not there. So, it is actually $1, 2, 3$ and so on okay. So, any integer the rest of the relation can be written as. So, if p_x is a constant I can absorb it as a constant in the Hamiltonian and can write down the resultant Schrodinger equation as follows okay.

So, this is equal to p_y^2 over $2m$. p_y is not conserved for the reason there is a y in the Hamiltonian and p_y and y do not commute this is the uncertainty principle that you have learnt and particularly y and p_y is equal to $i\hbar$ cross okay. This reason that p_y is not constant and I can write down then this $1/2$ that is $\frac{1}{2} m$ and E_B over m^2 and y minus y_0 square and now my wave function actually becomes function of only one variable that is y because you see there is no x anywhere in the Hamiltonian. So, I can write this as simply f of y and this is equal to ϵ and f of y okay. And what is the y_0 ? y_0 is the constant thing which is p_x over eB and this p_x is the x component of the momentum which is argued that y it is constant and of course, for a given problem e and B are constants B is the uniform field. So, this y_0 is actually p_x by eB and this is it has a name called as a magnetic length and written as L_B because it depends on B okay.

Now, this tells you that if this is a constant there is a p_y^2 and then there is a half something m into some cyclotron frequency. So, it is a half $m \omega^2$ and then y minus y_0 square and then you have this wave function which is only a function of y and that is why from $\psi(x,y)$ we have written in f of y and then energy into f of y . And this clearly represents that the equation for the Schrodinger equation for a harmonic oscillator oscillating in the y direction about not about 0 , but about this y_0 or the magnetic length which is given by this p_x over eB okay. So, if it is a harmonic oscillator problem then we do not need to solve any further we can get this the solution to be exactly of the form which is ϵ equal to n plus half \hbar cross ω_B where ω_B is equal to of course, ω_B is equal to eB over m which we have said several times that this is equal to the cyclotron frequency, okay. So, a problem which was simple enough to begin with we find the solution at least till now we have found the energy. So, the energy is nothing but it is n plus half \hbar cross ω_B just following the solution for the energy for a harmonic oscillator n can have $0, 1, 2, 3$ etcetera all the integers and here it can be equal to 0 because the harmonic oscillator allows for a solution with a quantum number n equal to 0 .

$$\begin{aligned}
 & \frac{1}{2m} \left[\underbrace{(p_x - eBy)^2 + p_y^2}_{H(x,y)} \right] \psi(x,y) = E \psi(x,y). \\
 & [p_x, H(x,y)] = 0. \\
 & p_x = \text{Constant} = \hbar k_x \quad k_x = \frac{2\pi n_x}{L_x} \quad n_x = 1, 2, 3, \dots \\
 & \left[\frac{p_y^2}{2m} + \frac{1}{2} m \left(\frac{eB}{m} \right)^2 (y - y_0)^2 \right] f(y) = E f(y). \quad [p_y, y] = i\hbar. \\
 & y_0 = \frac{p_x}{eB} = l_B \quad (\text{magnetic length}). \\
 & \boxed{E = \left(n + \frac{1}{2} \right) \hbar \omega_B} \Rightarrow \omega_B = \frac{eB}{m} : \text{cyclotron frequency}. \\
 & n = 0, 1, 2, 3, \dots
 \end{aligned}$$

So, this is the energy or the eigenvalues of the problem, let us see the eigenfunction. So, the eigenfunction now let us call it because we can solve for of course, f of y that is what we will do, but then we know that the total wave function which is equal to f of or ψ of x y ψ of x y is actually a free particle in the x direction multiplied by this f of y okay. This is the total wave function for the planar motion of the electron that is in the x direction if you take this as the x direction it propagates like a free particle because p_x is constant or k_x is constant. So, it will propagate like a free wave there and will have a harmonic motion in the y direction.

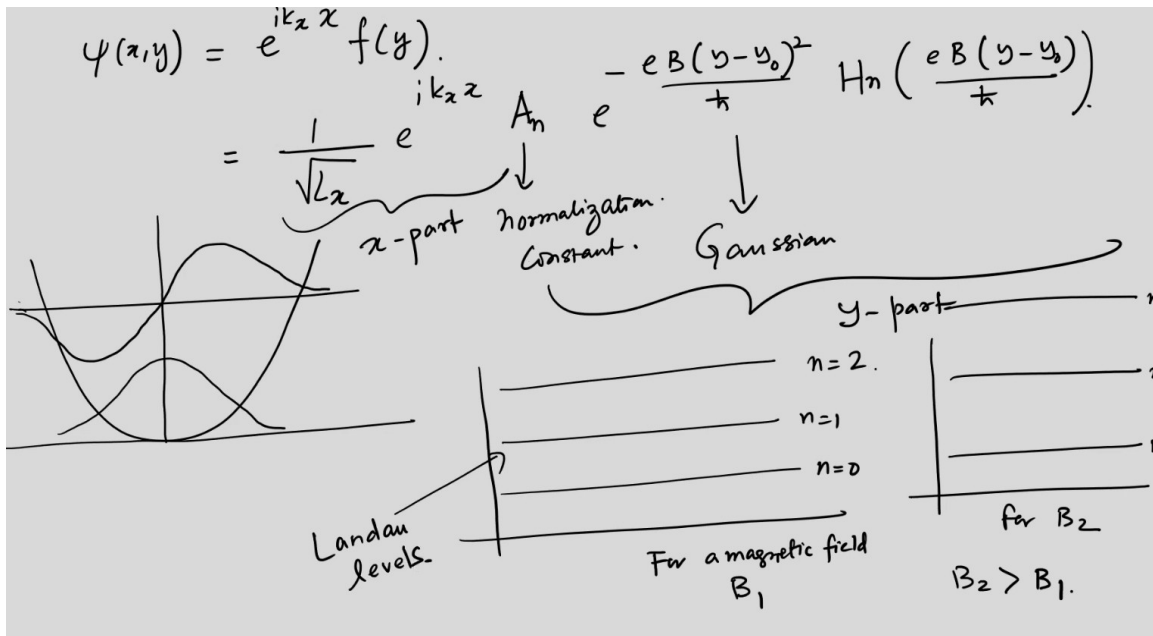
So, in the y direction it will have a harmonic motion will make a execute a harmonic motion about a point which is given by y_0 and this y_0 inversely depend upon b okay. So, this is nothing, but I use a normalization for the x direction box normalization this is equal to exponential $i k_x x$ and now I do not try to normalize it is not you know required here I use this as a normalization constant and use the this okay. So, let me write the normalization not with A capital n because you have a small n this thing there. So, let us write it with a A_n . So, A_n is a normalization constant and the wave functions are for a harmonic oscillator are known to be a convolution of a Gaussian which is exponential minus αx^2 or αy^2 multiplied by a polynomial function which is called as a Hermite polynomial and this polynomial has a property that when n is even the polynomial is even that is it contains terms such as x to the power 0 $x x^2$ x to the power 4 and so on.

When n is odd it contains terms which are $x x^3$ x to the power 5 and so on. So, this

is that form that we have to write down. So, this is equal to $e^{-B y^2/2}$ whole square divided by \sqrt{h} cross which is the Gaussian part just reminding you that this is the harmonic oscillator and you have a Gaussian like this for the ground state and this is for the first excited state it is like this and so on okay. So, this is an odd function. So, this is a ground state and this is the first excited state and so on.

So, and then there is a Hermite polynomial and this Hermite polynomial is written as $H_n(B y/\sqrt{h})$ divided by \sqrt{h} cross etcetera. The exact form is not important for us at this moment, but we need to know that at least there is a Hermite polynomial and then there is a Gaussian term which comes with a normalization constant and this part is purely because of the x part and this is because of the y. And as I said that the z component even if you consider it is like a free particle in the z direction. In particular if you confine electrons in two dimension then that term does not arise. Now if you go to this last slide you have $n + \frac{1}{2} \hbar \omega$ and this n equal to 0 1 2 3 etcetera, these are called as the Landau levels.

So, the energy levels have a name which are called as a Landau levels and Landau is a Russian physicist and the wave function is actually a freely propagating part in the x direction multiplied by a harmonic part which is executing a harmonic oscillation which has a Gaussian term and a Hermite polynomial. When n is even the polynomial is even when n is odd the polynomial is odd. So, these Landau levels so if we really plot it how does it look like the Landau levels will simply look like this. They are equidistant for a given value of B. So, this is say n equal to 0 n equal to 1 n equal to 2 and so on, but if you draw for another magnetic field it may look like this and so on. So, then n equal to 1 n equal to 2 n equal to 3 and so on.



And the reason is that ω_c which is the cyclotron frequency it is directly proportional to B . So, as you increase B let us say that this is ω_c for a magnetic field B_1 and this is for B_2 and of course, B_2 is greater than B_1 and these are called as a Landau levels. So, now what are the properties of the Landau levels and we would eventually get to that that these Landau levels are the most important things in understanding the structure of the plateaus and these are not really very sharp levels because of disorder this get broaden we will come to that. This one very important things that these Landau levels are infinitely degenerate or the degeneracy is very high.

So, degeneracy of the Landau levels. So, there are many many states corresponding to each one of the n values. So, n equal to 0 has very large number of states which means that these states can be occupied by electrons. So, each Landau level is occupied by many many electrons. So, let us understand the how that degeneracy arises and where does it come from. It comes from the value that k_x is a constant and this result that you have got is independent of k_x .

So, k_x does not play a role here which means any value of k_x would give rise to this level and this k_x has a n_x associated with it which denote quantum number of the states which means that corresponding to any value of n here a given value of n there could be a very large number of values of n_x that are possible. So, this in principle this is infinite but the degeneracy is actually limited by some factors and let us you know try to understand that what factors do they depend upon. But I hope that it is clear that these levels are heavily degenerate and the degeneracy is because they are coming from the

fact that any value of k_x would satisfy this equation or any value of n_x would satisfy these energy levels. So, each of the energy levels do not depend upon n_x and any value would be then acceptable solution.

So, if you want to know the degeneracy let us look at this. So, we have written this down earlier that the y_0 is equal to. So, we have y_0 equal to p_x over eB we have written this earlier and which we called as l_B which is a magnetic length and this is nothing but \hbar cross k_x divided by eB because p_x is \hbar cross k_x and this is equal to \hbar cross k_x is $2\pi n_x$ by l_B okay. So, this is your y_0 and we want to find that what is the maximum degeneracy. So, we want to find what is n_x max okay. And n_x max all others are constant the y_0 is the is basically a length scale in the y direction okay.

Along the y direction there is some value of y which is determined by of course, a magnetic field. Now this y_0 can at the most be l_y that is the length of the sample in the y direction that y_0 cannot be outside the sample. So, the maximum value of y_0 will be l_y and so, maximum value of n_x max can be obtained if you substitute y_0 as l_y . So, this is equal to $e b l_x l_y$ because I am putting y_0 as l_y and this is equal to $2\pi \hbar$ cross.

So, that is the maximum degeneracy. So, this is equal to eB into A , A is the area of the sample do not confuse it with anything else it is the a is the area of the sample. And divided by this \hbar because $2\pi \hbar$ cross is nothing, but \hbar I can write this as \hbar over e and so, I can I can take that B this thing I mean I can leave that B there and put this electronic charge below the in the denominator because \hbar over e is a very important quantity which is called as a flux quantum which is denoted by some ϕ_0 . So, this is equal to a flux quantum which has a particular value in Weber some 10 to the power minus 14 10 to the power minus 15 value. So, you see that let us call this as g the degeneracy. So, g is equal to it depends on the B the magnetic field the area of the sample and a ϕ_0 and this tells you that this degeneracy can be infinite because B if you go back to the first slide that we had and you see here the magnetic field goes all the way up to 14 15 Tesla.

So, this is about 15 Tesla. So, there is a 15 Tesla magnetic field if you want if you have facilities you can increase that magnetic field even more. So, and the sample size is up to you okay. So, this is really a very large degeneracy that is coming out of each one of the Landau levels are degenerate. In fact, this has important repercussions or implications on the fractional quantum Hall effect okay.

So, two things that we notice here. So, you note that the degeneracy is of course, what I told you that degeneracy is proportional to B and A I told you B is a magnetic field and A is the area and the second is that you know the trajectory can be the trajectory is centered about some y_0 is SHM in y direction and centered about y_0 . So, these are two very important you know outcome of this exercise that we have done that is we have solved a charged particle a single charged particle in presence of a perpendicular or a transverse magnetic field. We have taken a particular gauge and suppose we have taken the gauge in which the magnetic potential is in the y direction in that case you will have just the x and y being interchange that the system the particle will propagate like a free particle in the y direction and will execute a simple harmonic oscillation in the x direction. So, x and y they have no specific significance and the third gauge which we will talk about later it is called as a spherical gauge or symmetric gauge and in that case you have both x and y component and these Landau levels are actually found out to be circular. So, now there are some incidental similarities which I will just point out and we will discuss them as we go along and the similarities are that the Hall resistivity it shows plateaus in multiples of h over e^2 these plateaus are h over e^2 .

Degeneracy of the Landau levels.

$$y_0 = \frac{p_x}{eB} = \frac{\hbar k_x}{eB} = \frac{\hbar 2\pi n_x}{eB L_x}$$

$$g = (n_x)_{\max} = \frac{eB L_x L_y}{2\pi\hbar} = \frac{eB A}{h} = \frac{AB}{\frac{h}{e}}$$

A : Area of the Sample
 $\frac{h}{e}$: flux quantum $= \Phi_0$.

$$g = \frac{BA}{\Phi_0}$$

Note! (i) Degeneracy is proportional to B and A
(ii) Trajectory is SHM in y -direction centered about y_0 .

So, h over e^2 say n e^2 or something where n is an integer. So, consider n_0 that is the density of carriers. So, define a quantity called as ν or let us call this as just what we called here as ν let me call this as ν where ν takes values 1 2 3 etcetera we just change this because the too many n 's n is used for density as well. So, consider charge density of the carriers to be n_0 . So, ν is defined as n_0 divided by g by A now this g by A is that the degeneracy per unit area. So, let us not have the geometry of the sample coming in because you can have any geometry, but if you divide it by A then it

becomes a fundamental quantity which only depends on the value of the magnetic field and we are now talking about a given value of the magnetic field and this is equal to nothing, but $n_0 h$ divided by $e B$ and this is equal to.

Consider charge density n_0 of the carriers.

$$\nu = \frac{n_0}{\phi/A} = \frac{n_0 h}{e B} = \frac{n_0}{(\frac{h}{e}) B}$$

$$= \frac{\text{number of electrons}}{\text{number of flux quanta.}}$$

So, this is equal to a number of electrons and I can write this as n_0 divided by h over e into B . So, that is the flux quantum. So, this is number of electrons and divided by the you know the number of flux quanta and so on. So, if ν assumes a value integer that is when the number of electrons and the number of flux quanta they become commensurate fractions or even fractions in the fractional quantum hall effect then there are plateaus that are seen in the resistivity. So, these depending upon now you tune in the value of the magnetic field go back to this first slide which we had you see the magnetic field is being continuously tuned from a value 0 to 15 Tesla.

So, magnetic field is continuously ramped up and you are doing this experiment and what you see is that when this quantity which we just found out that this quantity becomes an integer then there are these Hall plateaus which are seen which we saw last. So, when this becomes an integer you have a plateau which means that this ratio becomes an integer you see plateaus then you again tune B this goes from being an integer to a fraction then you no longer see plateau and then this thing the resistivity gives shows a jump and then you go to the next integer such that it shows another plateau and that is how this new actually would be you know a sort of it will go from one integer value to another integer value and it will pull along the resistivity of the material to be having plateaus and so on and from one plateau to another and this is what we are going to see more carefully. I will give you a list of references that you should follow some of them are advanced references and you can follow them along with my lectures

here some of them are of the same level often I do not remember them, but next discussion I will provide you with those references. Thank you.