

Quantum Hall Effect
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Week-09

Lec 12: Composite Fermions, Hierarchy

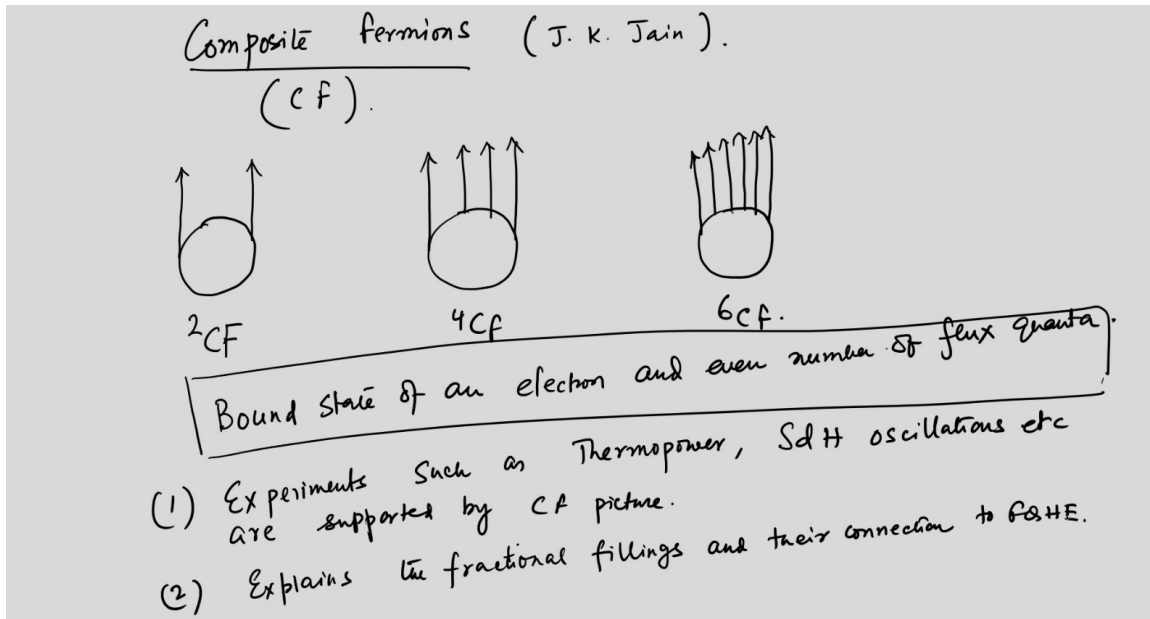
So we will talk about mainly two concepts before we wind up the discussion on the fraction quantum hall effect and as well the course on quantum hall effects. So, this the first one of them is the composite fermions and this was introduced by J.K. Jain, okay. So, this is an alternative approach to understand the fractional quantum Hall effect and particularly to understand the even denominator fractions that are observed experimentally whereas, the Laughlin states are valid for the odd denominators which are of the form of $1/m$ where m being an odd integer. So, particularly you know the fractional filling of $\nu = 1/2$ was to be explained which is one of the simplest fractions, but of course, this cannot be explained with the concept of the Laughlin wave function, the Laughlin states which are for odd denominator, alright.

So, this picture the composite fermion which we will call as CF as abbreviation. So, this is an effective picture it comprises of you know quasi particles which are the electrons they are carrying even number of flux quantum. So, they absorb the flux quantum from the external field and so they sort of are considered as quasi particles and these composite fermions drastically reduces the complexity of the problem and one actually gets starting from a very sort of strongly interacting fermionic system which is the nature of the quantum hall fluid we get a picture of non-interacting fermions. And so this was a very nice idea and not only it is an idea, but it also has the implications of relevance on the experiments such as the Shubnikov-de Haas oscillations and they also verified in thermo power experiments and so on.

So, they are of course, ideas and in reality they are no the electrons do not sort of absorb this even number of flux quantum, but it is a picture that aids us in understanding the fractional quantum hall effect, okay. So, let us try to you know visualize this. So, what happens is that so these are I am drawing an electron to be really large and this is carrying two flux quantum. So, let us call this as 2 CF and here they are 4 flux quantum. So, let us call this as 4 CF and then there are say even larger number of flux quantum which are say the 6 of them.

So, this let us call it as 6 CF, okay. As if it is a bound state of an electron and even number of flux quantum. So, that is the intuitive picture as I told you that it is not a real

picture, it really does not happen that these fermions actually absorb even number of flux quantum, but suppose we assume that there is a situation in sort of for the fractional quantum fluid that these fermions have absorbed or they are really carrying this even number of flux quantum. So, of course, the picture changes intuitively and it gives rise to something that is quite interesting and as I said that it has a relevance or rather it conforms to the experiments that are you know made or rather they are performed and this picture supports those experiments, okay. So, two sort of artifacts happen.



So, it is experiments there are others as well, but we have not I have not written them such as thermo power, SDH oscillations we have discussed this in our earlier lecture, Shubnikov-de Haas oscillations etcetera are supported by the CF picture. Number 2, so it of course, explains the fractional filling of the Landau levels and basically their and their connection to FQHE and third and probably the most important thing which I just said is let me write it in the next page 3 and this is quite important. So, this says that the strongly interacting electrons absorbing most of the flux quantum quanta rather become weakly interacting electrons in a weak magnetic field and this tells you that the latter corresponds to the integer quantum Hall effect which we have seen. So, this picture of FQHE this goes over to IQHE. So, this is for fermions IQHE for composite fermions, okay.

So, this is a big simplification that occurs, okay. So, let us see what happens and how it happens and so on, okay. So, we just said that you know the electrons actually absorb the

flux quanta. So, the effective field is reduced. So, let us write down the effective field which we will call it as B^* which is actually reduced in the CF picture is given by.

So, B^* is equal to B which is an external field and then there are these even number of flux quantum which we write as $2p$ and then each one carrying a flux Φ_0 and the density of the composite fermions. So, B^* of course, that is written. So, let me write B which is the external magnetic field, okay. $2p$ even number of flux quantum Φ_0 of course, you know that that is a flux quantum which is equal to h/e and ρ density of CF. So, this tells you that the magnetic field, the effective magnetic field or the reduced magnetic field is given by this expression which is this B is the external field and now since these fermions have absorbed most of the flux from the external field.

→ (3) The strongly interacting electrons absorbing most of flux quanta become weakly interacting electrons in a weak magnetic field.

Imp. $FQHE \rightarrow IQHE$
for fermions for CF.

Effective field (B^*) in CF picture

$$B^* = B - 2p \Phi_0 \rho$$

B : External magnetic field.

$2p$: Even number of flux quanta.

Φ_0 : flux quantum $= \frac{h}{e}$.

ρ : density of CF.

So, the field now remains as this and let us now see that what this means in terms of the electron filling. So, the filling fraction for the CF let us call that as ν^* in keeping with the B^* which is equal to $\rho \Phi_0$ divided by B^* , okay. And this is of course, understood that the B is pointing in the z direction. So, we should also mention that here. So, B is in this direction, the external field is in this direction and let us call this as equation 1 and this as say equation 2.

That is the definition of a filling fraction which is ρ that is the density of the fermions a composite fermions multiplied by Φ_0 divided by the B^* . So, a putting 1 in 2 what I mean is put this B^* $B - 2p \Phi_0 \rho$ in 2, one gets a ν^* to be equal to $\rho \Phi_0$ divided by $B - 2p \Phi_0 \rho$, okay. So, I write Φ_0 as h/e and this

is E and this $B - 2P\phi_0\rho$. Now what I do is I take this ρ also down and then sort of $\rho H E$ to be down. So, this is like a 1 divided by E over ρH and a $B - 2P\phi_0\rho$ and then little bit of algebra.

So, I take this in so it is a $E B$ by ρH and a minus $2P\phi_0\rho$ into E by ρH . So, ρ will cancel and E by H is 1 over ϕ_0 . So, that will cancel as well and we are left with $E B$ over ρH which is nothing but the filling fraction of the original fermions, okay. So, this is the inverse of it rather. So, it is 1 over ν minus $2P$ and so on.

Filling fraction for $\bar{\nu}$ C.P.

$$\nu^* = \frac{\rho\phi_0}{B^*} \quad (2)$$

Putting (1) in (2)

$$\begin{aligned} \nu^* &= \frac{\rho\phi_0}{B - 2P\phi_0\rho} = \frac{\rho h}{e(B - 2P\phi_0\rho)} \\ &= \frac{1}{\frac{e}{\rho h}(B - 2P\phi_0\rho)} = \frac{1}{\frac{eB}{\rho h} - 2P\cancel{\phi_0\rho} \frac{e}{\rho h}} \\ &= \frac{1}{\frac{1}{\nu} - 2P} \quad (3) \end{aligned}$$

So, this is the thing and so let us call this as equation 3 and then we can write down the ν^* in terms of this ν to be equal to $\nu - 1 - 2P\nu$ and let us call this as equation 4. So, this is the expression for the filling of the composite fermions and its relation to the filling of the actual fermions which are there in the system and this P is equal to 1, 2, 3 and $2P$ denotes the number of vortices or the number of flux quantum which are actually vortices I will just come to that, okay. So, it can be actually inverted. So, 1 over ν^* this is equal to a plus minus. So, this is equal to 1 over $\nu - 2P$ and so this is the you know the filling fraction relationship between the filling fraction and just take a note that we have written both plus and minus signs and they would you know give rise to a new different fractions.

So, to say both signs are actually allowed. So, plus and minus signs are allowed depending on whether B^* is in the direction of B or positive, okay. So, this is let us call it as 5 is it is same 4 and 5 are same is just written in a different language and this can again be written in another language in which you can see that. So, if you write

down the ν in terms of ν^* then it becomes equal to a $2P\nu^*$ plus 1 and this is the equivalent expression where ν is the filling of the original problem of the Hall conductance or Hall resistance. And so, for ν less than 1 which is what we are talking about fractional filling ν^* is greater than 1, okay.

$$\nu^* = \frac{\nu}{1 - 2P\nu} \quad (4)$$

$$\boxed{\frac{1}{\nu^*} = \pm \left(\frac{1}{\nu} - 2P \right)} \quad (5)$$

\pm signs are allowed depending on whether B^* is in the direction of B or opposite to it.

$$\boxed{\nu = \frac{\nu^*}{2P\nu^* + 1}} \quad (6)$$

for $\nu < 1$ (fractional filling), $\nu^* > 1$.

And when the ν^* becomes large it corresponds to the or rather it is an integer, okay. It is equal to 1 or greater than 1 and becomes an integer then it corresponds to the non-degenerate state and the non-degenerate state of course, would correspond to weak interaction or no interaction at all which means that we go to the case of IQHE, okay. So, let me make this a little more clear because. So, the filling factor is defined in the following way. It is also called as filling fraction.

So, this is equal to a ρ divided by a degeneracy, okay. So, now, if ρ that is a density of electrons fermions that becomes equal to the degeneracy which means that the integer number of Landau levels are filled and if it becomes 2 say the ρ is twice of the degeneracy which means 2 Landau levels are completely filled and this situation would correspond to exactly what we have studied in the context of IQHE that is integer quantum Hall effect and the system corresponds to non-degenerate states and non-degenerate scenario and that sort of should explain the non-interacting electrons or very weakly interacting electrons which we have studied in the context of IQHE, ok. So, this is the picture that emerges out of this composite fermion picture and let us try to understand what is its relation to the Laughlin state. This is quite a nice explanation. So, follow me carefully.

So, remember the Jastrow factor and the Jastrow factor actually says that, so there is a Z_i minus Z_j , I might have written Z_j and Z_k , these are just indices which you can take according to your convenience. So, there is this M and then of course it is multiplied by the Gaussian. Let us look at the Jastrow factor only. You would sort of recognize that for each electron, so Z s are the coordinates of the Z_i denote the coordinate of these electrons and this basically enforces the anti-symmetric property by when M is you know an odd integer and it also says that if Z_i becomes equal to Z_j , this is not only 0, but it is 0 M times because of the product that is involved here. I hope this notation is clear.

We write a summation by this and we write a product by this. So, this is a sum and this is a product, okay. So, this product of all i , i is less than j and then it is a 0 of m th order and that is what comes out from the Jastrow factor and that is what is embedded in the Laughlin state. Now, of course we need one 0 of those m zeros and one 0 we need because of we want to enforce the exclusion principle. What happens to the m minus one zeros? So, these m minus one zeros account for what are known as vortices.

So, let me write as one 0 ensures Pauli's exclusion principle. The rest m minus one zeros sort of account for the vortices and what are the vortices or what is a vortex? A vortex is actually a sort of a topological excitation. So, suppose a complex number, a complex number z equal to $r e^{i\theta}$ all of you are familiar with this polar representation. It has a vortex at the origin, okay. So, which means that if a particle sort of encircles this origin it picks up a phase 2π .

$$\text{filling factor } \nu = \frac{p}{\text{degeneracy}}.$$

\sum : sum
 \prod : product

Relation to the Laughlin state

$$\text{Jastrow factor } \prod_{i < j} (z_i - z_j)^m$$

'1' zero ensures Pauli's Exclusion principle.

The rest $(m-1)$ zeros account for the vortices.

Vortex A complex number $z = r e^{i\theta}$ has a vortex at the origin.

So, that theta actually changes from theta to theta plus 2π and the complex number remains unchanged in the sense that because you have a $2\pi i$ exponential $2\pi i$ is equal to 1, okay. So, that is the meaning and suppose these the FQHE wave function contains a factor $z - z_0$ to the power $2p$, okay. We are writing M as a $2p$ in the sense that these are just to make p to be any integer not only odd integer. So, $2p$ will be.

So, a p could be any integer. So, that $2p$ is even, $2p$ is an even integer. So, you see here you have these $m-1$ is an even integer because m is an odd integer. So, $m-1$ is an even integer and so we write that as $2p$. So, this means that a fermion 1 sees fermion 2 to carry $2p$ vortices or $M-1$ vortices, okay. And so basically this means that even number of vortices which is what the this composite fermion picture it sort of you know proposes.

So, let me tabulate a few p and ν and ν with ν^* to be an integer and I told you that this corresponds to the IQHE when ν^* becomes an integer. So, this is a p and a ν which is equal to a ν^* and which is equal to a $2p\nu^* + 1$ the formula that we have written earlier and let me just show it for p equal to 1. So, this gives rise to one third, two fifth and three seventh and so on, okay. For ν^* to be an integer and for p equal to this p equal to 1 here let us say p equal to minus 1 we get other integers which are two third, three fifth and five seventh and so on. And so 2 will give rise to 1 by 5 and 2 by 9 and minus 2 will give rise to 2 by 7 and 3 by 11.

So, you see all these fractions have been seen in experiments they have been realized in experiments and that is why just having ν^* to be you know a integer values with p to

be plus minus 1 plus minus 2 you can extend it to plus minus 3 etcetera. We are getting all kinds of fractions which were not realized in the Laughlin wave function I mean with the Laughlin wave function which is only valid for $1/m$. So, $1/3$ is in that form, but none of these other $1/5$ is also of that form, but none other are included in the Laughlin wave function and these are called the non Laughlin states. So, let me show you a picture an intuitive picture of this whole thing.

Suppose the FQHE wavefunction contains a factor $(z - z_0)^{2p}$. $2p$: even integer.

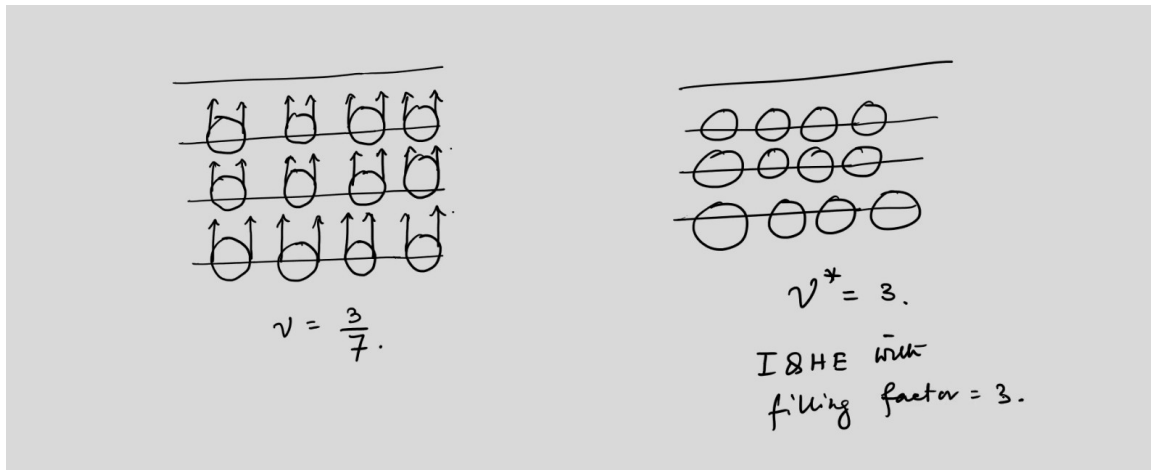
Fermion '1' sees Fermion '2' to carry $(m-1)$ vortices.

Table of p and ν with $\nu^* = \text{an integer}$.

p	$\nu = \frac{\nu^*}{2p\nu^* + 1}$
1	$\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \dots$
-1	$\frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \dots$
2	$\frac{1}{5}, \frac{2}{9}$
-2	$\frac{2}{7}, \frac{3}{11}$

Laughlin wave function yield fraction $\frac{1}{m}$.

So, we will draw so this 1, 2, 3, 4. So, we will have this usual fermions and then the composite fermion picture and sort of where filling fractions will be compared. So, these are these fermions which are carrying these vortices of the flux quantum and this corresponds to the original picture of fermions. This is like this 4 of them in each of the 3 Landau levels and this corresponds to a filling fraction to be $\nu = 3/7$ okay. So, this is a fractional quantum Hall effect and when we go to the non interacting picture or the picture of so this okay so this corresponds to $\nu^* = 3$ okay. So, this is the as a schematic plot this shows i Q H E with filling factor equal to 3 and this corresponds to the f Q H E with filling factor equal to $3/7$ okay.



So, this is the problem of f Q H E becomes a problem of i Q H E of the composite fermions okay. And let us do the last thing in this regard we still are missing a number of fractions okay even though we have been able to go beyond the Laughlin picture and get a lot of other fractions from this composite fermions, but we still are not have not got all the fractions that are experimentally observed okay. And there is a hierarchy picture which again is very subtle the idea is very subtle though you know getting a lot of other fractions are not too difficult. So, this sort of you know relies on the fact that there are other filling fractions that are observed in experiments starting from some parent states one can get Laughlin like daughter states okay. So, from a given parent state so these are Laughlin like daughter states okay.

So, these Laughlin like daughter states and a parent states which are very interesting here and these fractions at any level another important thing the fractions at any level in this hierarchy scheme are denoted as continued fractions. You might have seen this continued fractions even in school these the fractions carry on and here we will have these different at a given level of the parent state we will have a multiple daughter state emerging out and we will give a large number of these fractions coming out and so on. It is not that all the fractions are seen in experiments which are predicted from this hierarchy scheme, but quite a few of them have been observed okay. And it is not that you know at a given sort of Laughlin the daughter states at a given sort of hierarchy of the daughter states there could be many fractions and the probability to observe these fractions in actual experiments it seems to be the same whereas some of them are just not observed at all some of them are of course observed that tells you that this of course doesn't distinguish between one fraction at a given level from another fraction. Now physical picture is necessary and the physical picture really relies upon not the ground state which we have been talking about so far, but it relies on the excited state.

And so you have a problem if you say a quantum mechanical problem and you have a Hamiltonian and you may be able to solve that Hamiltonian exactly or maybe there are you know mathematical difficulties or there are computational difficulties and so on, but it is very easy to understand that these quantum mechanical Hamiltonian will have a large number of states possible which are energy states that could be degeneracies and so on, but forgetting the degeneracies for the moment there are ground state there are first excited state and there are second excited states and so on rarely ever one needs to go very far away from the ground state in condensed matter physics because we always talk about low lying excitations which are which are the most important thing. I mean something that's far away from the ground state may not be taking part in either the thermodynamic or the transport properties at all. So it's quite important that and we are sort of you know understanding that the Fermi level lies very close to the ground state okay and that's why the ground state is the most important thing. However in certain scenarios you may have to go beyond the ground state and take into account excitations and these excitations in this fractional quantum Hall fluid are called as quasi particles and quasi holes depending upon the shortage of density or the excess of density. So if you have excess of density of particle like excitations which are like quasi particle excitations and if you have deficit then of course you have hole like or the quasi hole like excitations okay.

So Halperin wrote down so conjectured that these excited states have a form okay which are given by the ψ of Z this is a P of Z K which is our usual Jastrow factor I mean Z K minus Z J product of that and so on so forth. And then there is a Q which is again a function of Z K and the ubiquitous these Gaussian that would of course be there okay. We are familiar with this, this is a Jastrow factor and this accounts for quasi particle and quasi hole excitations.

Hierarchy picture.

- 1) From a given parent state one can get a number of Laughlin-like daughter states.
- 2) The fractions at any level in this hierarchy scheme are denoted as continued fractions.

Halperin conjectured that these excited states have a form,

$$\psi(z) = \underbrace{P(z_k)}_{\text{Jastrow factor}} \underbrace{Q(z_k)}_{\text{Accounts for quasiparticle \& quasihole excitations}} e^{-\sum_j |z_j|^2 / 4l_B^2}$$

And what did he proposed for these Q Z K? So the Q Z K he proposed to have a form which is equal to it is again a product of J less than K and then it just goes all the way up to M and it is a Z J minus a Z K whole to the power plus minus 1 over M okay. We will not go into the details that was the ansatz that was made by him and of course the P Z K that is the Jastrow factor of course has this form J less than K to the power M it is a Z J minus Z K to the power 2 P okay.

So if you combine them then psi of Z so this is the excitation above the Laughlin state and the excitation has been included by these this Q Z K term which of course you know talks about either particle or hole the excitations of the system. So you know at the middle of the plateau a quantum hall plateau the density is uniform okay. And so there are number of states that are pinned to a defect or an impurity. Now as the magnetic field is ramped up it is increased beyond a certain value of the magnetic field these fields or these states actually break away from the impurity and they would cause a density imbalance. So if the density becomes larger we have a quasi particle excitation and if the density falls below then it becomes a it becomes a deficit it becomes a quasi hole excitation okay.

So this is the idea of this so total psi of Z becomes the product it is J less than K and of course it goes to m and then it is a Z J minus a Z K whole to the power 2 P plus minus m 1 over m and then of course that Gaussian thing will of course be there. So it is Z J square divided by 4 l B square okay. So that is a form of the wave function the excited state wave function m is still an odd integer which is what Laughlin had said. Now of course you can understand that this gives you instead of just m this gives you the total J

max. So J max the magnitude of that it becomes equal to $2P \pm 1/m$ and I will have to multiply it by N because there are N electrons okay and I am just talking about the magnitude.

So this is N electrons and $2P \pm 1/m$. So this filling fraction corresponding to this scenario can be obtained if we note that the area of the quantum Hall fluid is given by a equal to so N then it is $2P \pm 1/m$ which is that J and then we multiply it by the π into a $2m$ and l_B square okay. So that is the area and the number of states within an area A is given by N curly N okay. So N is equal to a $B A$ over ϕ_0 where A is given by this which I just wrote so this is equal to ϕ over ϕ_0 this is equal to $2P \pm 1/m$ and $m^2 N$ okay.

$$\begin{aligned} \phi(z_k) &= \prod_{j < k}^N (z_j - z_k)^{\pm 1/m} \\ \rho(z_k) &= \prod_{j < k}^N (z_j - z_k)^{2P} \\ \psi(z) &= \prod_{j < k}^N (z_j - z_k)^{2P \pm \frac{1}{m}} e^{-\sum_j z_j^2 / 4l_B^2} \\ |J_{\max}| &= N(2P \pm \frac{1}{m}) \\ \text{Area of the QH fluid } A &= N(2P \pm \frac{1}{m})(\pi 2m l_B^2) \\ \text{No. of states within an area } A \text{ is given by,} \\ \nu &= \frac{BA}{\phi_0} = \frac{\Phi}{\Phi_0} = (2P \pm \frac{1}{m})m^2 N \end{aligned}$$

So that is the number of states that you obtain and when we get the number of states the filling fraction is given by this. So ν is equal to 1 by $2P \pm 1/m$ square plus minus m okay that is the filling fraction and this filling fraction can be expressed in terms of a continued fraction but importantly it needs to be noted that there is a term which is this which is in addition to the m term that is there in the denominator.

So the Laughlin state only had this m term because Laughlin states only you know remained in the ground state so it is a ground state wave function. So the hierarchy so the continued fraction or rather ν can be written. In terms of the continued fraction as okay so ν is equal to 1 divided by so this is for you know for different values of P and different values of m okay odd integer m and different values of P . So this is equal to 1

plus m plus minus 1 divided by $2P_1$ plus minus 1 divided by $2P_2$ plus minus and so on so forth okay. So this is the continued fraction that we talked about and so this is like a P is having all these values 1, 2, 3 etc.

Now the filling fraction is given by,

$$\sqrt{v} = \frac{1}{2pm^2 \pm m}$$

\sqrt{v} can be written in terms of the continued fraction as,

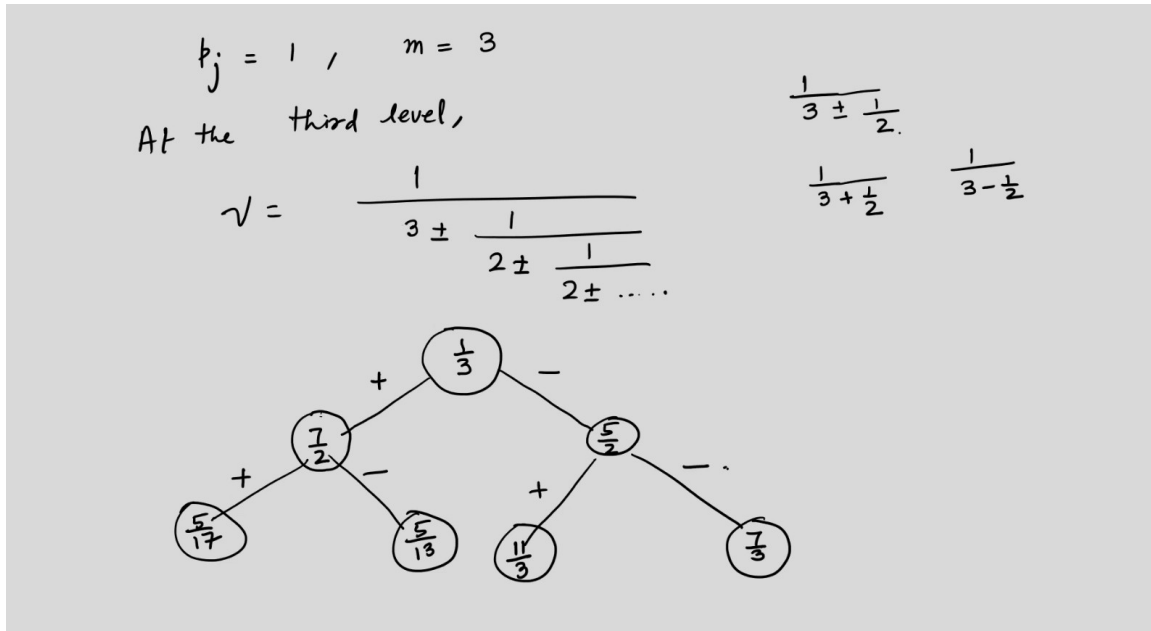
$$\sqrt{v} = \frac{1}{m \pm \frac{1}{2p_1 \pm \frac{1}{2p_2 \pm \dots}}}$$

And m has values 3, 5, 7 etc. and then one can actually form these continued fraction. So let me show you one example of this continued fraction. So let us just write for all the P 's let us call it a P_j to be equal to 1 just to have a simple form. So and m let us say equal to 3 that is also simple. So at the third level one has you know so ν becomes equal to 1 divided by 3 plus minus 1 divided by 2 plus minus 1 divided by 2 plus minus and so on okay.

So at the third level for corresponding to this thing okay and I mean till the third level then you can just drop everything out here and then calculate what fraction is coming out from this continued fraction. So let me show you the picture for m equal to 3 and this P_j equal to 1. So the picture is like this so we have this so at the third level so this there is one so this is the parent state is equal to 1 over 3 okay. So m equal to 1 over 3 and there are two branches that come out from the plus and the minus sign. So let us write the plus branch here on the left and the minus branch on the right.

So at the P_1 level so this is P_j equal to 1 so P_1 level at the all the P_j is equal to 1 but you can have other fractions where P_1 equal to 1 and P_2 equal to say 2 and so on so forth okay. So this is equal to 7 by 2 that comes out from the positive branch okay that is like 1 over 3 and plus so this is how it is coming is that 1 over 3 and then you have a plus and a minus and 1 over 2 okay. So this becomes equal to so there are 1 by 3 plus 1 half that is giving you the 7 by 2 branch and then you know so that is the plus branch and the

minus branch is coming out as 3 minus half and that is that branch gives you 5 by 2. So these are the fractions that are there and let me go to another level so this plus branch here that gives you a 5 over 17 and of course there is a 5 over 13 from the minus branch. Now this one will have a plus branch and the minus branch as well and this is coming out as 11 by 3 do it carefully and 7 by 3 coming from the plus branch and the minus branch.



So at the third level so starting from one Laughlin fraction we get so many non Laughlin fractions coming out of this you know the hierarchy picture okay and it gives you of course a large number of fractions not all of them are seen and at the same level it sort of even though it predicts that the probability of getting this fraction should be same but that does not happen.

Okay I wish to close the discussion of this course here so if you wish me to you know go back and talk about what has been done so far and so we have started with you know mesoscopic systems. So this two-dimensional electron gas is a mesoscopic system and then we have talked about Landauer formula then we have talked about in the related context we have talked about S matrix and we have talked about the discovery of the Hall effect the classical Hall effect as well as the quantum Hall effect and how Hall effect has a very important thing as metrology in determining the unit of the resistance okay and of course we have talked about classical Hall effect and this trajectory of the charged particles etc and so on so these are all part of these classical Hall effect. Then we came to this quantum Hall effect it's of course discovery Landau levels filling

fraction and when I say QHE I really mean the integer quantum Hall effect filling fractions etc inclusion of you know spin electric field magnetic field into the problem that is Zeeman electric and magnetic fields and we have talked about the incompressibility of the quantum Hall fluid, is a very important thing in fact these plateaus actually at the plateaus the μ doesn't even if you change the number of particles μ remains constant and so on and then of course we have talked about the chiral properties of this QHE so these are chiral edge modes edge states and so on we have talked about SDH oscillation which is an oscillation of the magnetization and so on in presence of the magnetic field and then we have talked about so SDH oscillations, Kubo formula.

Then we had gone to this you know QHE in a crystal lattice. So first we have talked about square lattice and it's quite important to understand so all these things are in 2D electron gas and then first we have talked about square lattice and we have learnt how Peierl's coupling sort of modifies the hopping term there in the across the one from one site one lattice site to another then we have talked about the you know the topological invariant which is the Chern number in this particular case and then very importantly we have talked about graphene in reasonable amount of details what happens to the Landau levels in graphene and how they are different from the 2D electron gas and what's the quantization like for the Hall plateaus in graphene and they are significantly different than that of the 2D electron gas. We have also done Hofstadter butterfly these are fractal structures that are shown by the energy as a function of the field and then we have talked about importantly we have talked about spin Hall effect and so graphene is thought to be a candidate however because of extremely small Rashba spin-orbit coupling that is quite difficult to observe so this is a non-dissipative spin transport and its applications to spintronics this has been touched in brief.

Summary.

- 1) Mesoscopic systems, Landauer formula.
S-Matrix, Hall effect, Metrology
- 2) IQHE : discovery, Landau levels, filling fractions,
in 2DEG. Spin, Electric & magnetic fields, incompressibility of in QH.
fluid, Chiral edge states, SdH oscillations,
Kubo formula.
- 3) QHE in a crystal lattice : Square lattice, Peierls coupling,
Topological invariant, graphene,
Hofstadter butterfly
- 4) Spin Hall effect : Graphene, non-dissipative spin transport
Spintronics

- 5) Fractional Quantum Hall effect : Symmetric gauge, Landau level
Laughlin states, properties,
Plasma (analogy), Composite
fermions and hierarchy picture.

And finally we have done fractional quantum Hall effect. And there we have started with you know the symmetric gauge now this is by no means complete this particularly this part of fractional quantum Hall effect because of the lack of time I had to skip many important things such as the fractional statistics the great statistics and then you know the many of these other things such as the topological quantum computation etc. those are definitely very interesting however, quite advanced topic as well so this is symmetric gauge we have done then we have of course here I think we should also say that we have done the BHZ model in the context of a quantum spin Hall insulator the symmetric gauge we have done and then we have done the Landau levels then we have done the Laughlin states properties plasma analogy with plasma rather plasma analogy composite fermions and the hierarchy picture. So with this I end the course hope you have enjoyed as much as I did these some of these are slightly advanced topics and you need to supplement it with a literature a number of literature has they have been cited along with but there are more now finding out a relevant literature is not a problem with this internet being so fast and you know so resourceful I hope that you learn more going beyond the course and take this course as a sort of the first level course in understanding the quantum Hall effect of course there are other things such as anomalous quantum Hall

effect etc. we did not do in this course which will be done in another course related to topology and condensed matter physics.