Quantum Hall Effect

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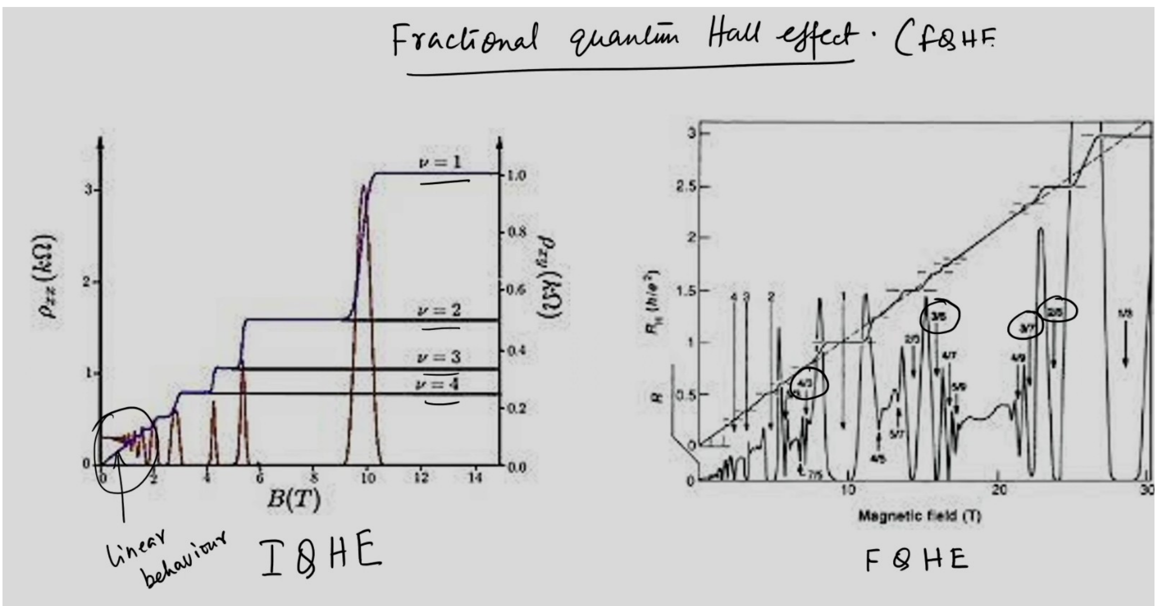
Week-09

Lec 19: Landau gauge in fractional quantum Hall effect

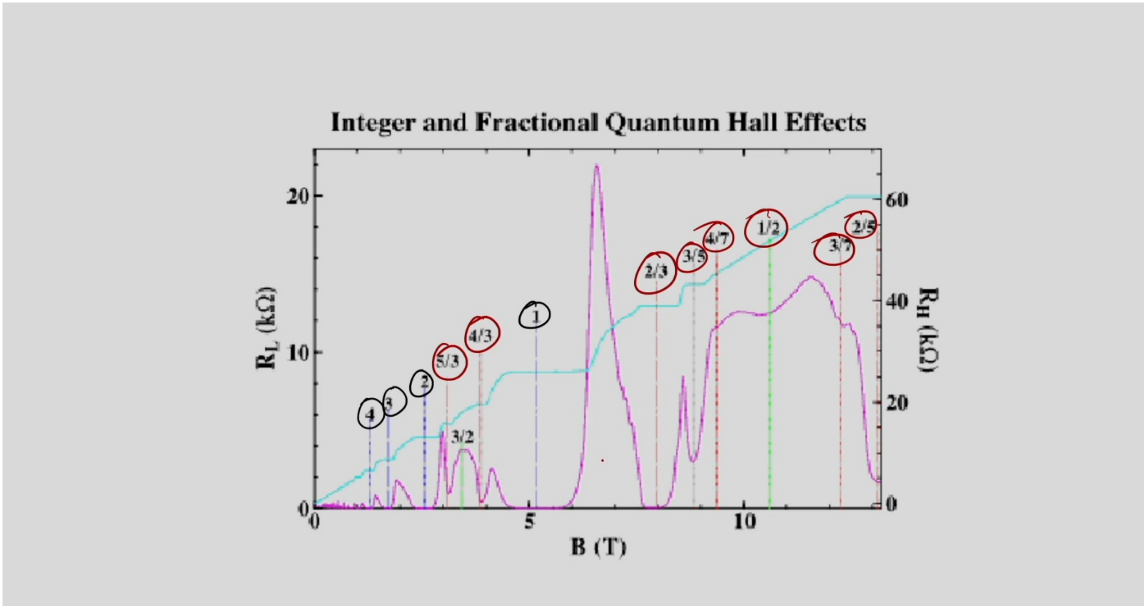
So we have looked at integer quantum Hall effect in great details almost everything that are relevant we have done that. And if you remember that we had shown this picture that is there on your screen. So this is the integer quantum Hall effect. So this is let us abbreviate it as IQHE where you see all those plateaus. So the left hand side is for the magnetoresistance and the right hand side is for the Hall resistance. So even though it is not clear you know what it means.

So this is a nu equal to 1 plateau and then there are nu equal to 2 plateaus nu equal to 3 plateaus and so on. So these are the resistivity plots Hall resistivity plots and along with that there are these peaks in the magnetoresistance behavior as a function of B and you see that the B is pretty large here. I mean it has been taken up to something like 14 or 15 Tesla and we are very well aware that why do these plateaus arise in spite of the system being a dirty two dimensional electron gas with no time reversal invariance or translational invariance. In fact the time reversal invariance is the reason or the loss of time reversal invariance is the reason for these plateaus to be quantized and they are related to the Chern number which is what we have seen and we have also seen that it's a situation in which whenever these Hall plateaus jump from one new value to another new value like jumps up or jumps down depending upon how you want to view it the magnetoresistance that shows a peak there and so on.

So that's there for all the way to low values of magnetic field excepting that at very low values you see that this region you see that there is a linear behavior of the Hall resistance. So linear behavior and this linear behavior is related to the classical Hall effect that we have all seen either at the school level or young undergraduate level where the Hall resistivity or the Hall coefficient is proportional to B linearly in B and the slope gives you information about the density of the carriers and the type of carriers that is whether it's a n-type semiconductor or a p-type semiconductor. Almost soon afterwards people have found this FQHE and in which the fractional plateaus are seen I completely agree that these are not clear fractions but one can actually see a fraction as 3.5 there is one at 3.7 there is one at 2.3 and there is one at 4.3 and so on. So these are plateaus being seen at the fractional values and that's why it's known as FQHE or the fractional quantum Hall effect and this forms the discussion for us for the next few lectures before we wind up the course. So what are the reasons for these fractional plateaus does it mean that there are fractional feelings or electron is divisible where we know that these in terms of the Corbinodic picture there are these the Hall effect or the Hall phenomena is perceived as Hall the quantum pump in which there is an electron that's been transferred from the inner edge of the disk to the outer range and the two electrons being transferred three electrons being transferred and that forms these plateau labels of the plateaus that we have indicated here as 1, 2, 3 and 4 and so on. So if they are fractions that is these labels are fractions would it mean that the electrons are divisible and certainly not in fact that is what we are going to talk about that the electrons are not divisible they are a fundamental you know the electronic charge is a fundamental quantity there is nothing called a third of an electronic charge and so on. But still we see plateaus at fractional values so we start this discussion on the fractional quantum Hall effect and which we abbreviate as FQHE okay.



So to show you one more picture these are probably a little more clear than the earlier one. So you see that these are integer plateaus here and let's show the fractional plateaus by a color so these are these fractional values 5 by 3, 4 by 3, 2 by 3, 3 by 5, 4 by 7, half and so on so forth 2 by 5, 3 by 7 the mostly rational fraction and the proper fractions which means that the denominator is larger than the numerator. But there are also improper fractions that are seen in experiments as well as they are different kind of fractions that are noted in experiments and overall about more than 80 such fractions have been noted overall in experiment. So this is a single figure which shows both integer and fractional quantum Hall effects okay.



So in order to understand we have to again understand that the energies of these electrons in a magnetic field which we already know but does anything change in this fractional quantum Hall effect. It turns out that nothing changes excepting that these wave functions are found to have a radial symmetry and in fact these are called as Laughlin states which we will see and or these Landau levels rather have radial symmetry and these have to be worked out. So what I am going to show is I am going to take a gauge that captures this radial symmetry and this is called as a symmetric gauge and in this gauge we shall do calculations and we will find out the degeneracy of the Landau levels and the wave functions that are relevant for the discussion okay. So let us start with the gauge. So we start with the symmetric gauge and let me mention a priori that there is a lot of mathematics that is going to follow.

They are mostly algebra and simple algebra but they need to be done in order to understand what is the nature of the wave function and how we can write it in terms of you know particularly in terms of complex numbers like Z which is equal to X plus iY and so on and before that I urge the readers to have a look at the simple harmonic oscillator in spherical polar coordinates okay. So you know that one dimensional oscillators have energy which is N plus half H cross omega there is no degeneracy but in 2D in Cartesian coordinates it has NX plus half plus NY plus half H cross omega, omega is still given by this root over K by M and now there is degeneracy there. Now if you go to three dimensional spherical polar coordinate if you solve the problem in three dimensional spherical polar coordinates you will see the importance of the angular momentum operator and the quantum number corresponding to the angular momentum quantum number is important to find out the degeneracy. You can look at any good quantum mechanics book in particular you can look at Cohen Kanoji which gives you a very nice introduction about the this problem that is a simple harmonic oscillator in 3D spherical polar coordinates okay. And this actually would help you a lot in understanding this part because we are still going to get these harmonic oscillator these energies as well as a wave function.

So we start with the symmetric gauge what I mean by symmetric gauge is that for constant magnetic field which we know that in this particular case it is a constant so there is a two dimensional electron gas which is pierced by a uniform field in the Z direction we write down the gauge as half r cross B okay. See if you remember that we have taken this a to be like you know minus y b x cap or we have taken it as x b y cap and so on okay which are same and this called as a Landau gauge. And in this particular gauge what came out is that after we have solved the problem the energy levels came out as n plus half h cross omega where n can take values as 0 1 2 etc. Now depending on the gate chosen of these two that is it is minus y b x cap or x b y cap there is either k x or k y they continue to be good quantum numbers and any value of k x and its quantization would be you know acceptable for any given value of n. And that gives rise to an enormous degeneracy into the problem that is there are very large number of quantum states that are available for the electrons to occupy which means that each Landau level is heavily occupied by electrons or they are heavily degenerate.

However, we are foregoing that constraint that is neither k x nor k y continue to be good quantum numbers, but instead we will see that the L z or the J z that is the z component of the angular momentum quantum number. It is probably safer to call talk about J z because where L z and J z would mean the same thing because there is no spin involved, but we will talk about J z you have to keep in mind that if you look at another book it might actually talk about L z, but they would mean the same thing all right. So, this is equal to half of minus b y x cap plus a b x y cap okay. So, that is the form of the vector potential and this choice of the gauge actually breaks the translational symmetry in both the x and the y directions which means that k x and k y cease to be good or conserved quantities so to say okay. And this will be very helpful for us to understand the fractional quantum Hall effect.

So, we will find out the momentum and the momentum let us write it as the momentum which is P plus E a this momentum let us call it as some momentum called as pi and this is important to understand that this momentum is gauge invariant, but it is not a canonical momentum. What I mean by a canonical momentum is that x and x component of pi do not have these commutation relations which are ih cross. So, if they have then they are called as canonical, but these can be shown as they do not obey the canonical relations, but nevertheless these are gauge invariant momentum. So, we use this just like what we have done it for the first course of quantum mechanics to study linear harmonic oscillators. We have used P and X combinations linear combinations of P and X in order to write a and a dagger here we will do a different thing, but something similar that is we will write down these creation and annihilation operators namely a and a dagger to be so a dagger is creation and a is annihilation operator.

So, these will be written in terms of the x component of pi and the y component of pi. So, what I mean by that is your a is equal to so it is a pi divided by h e B this is equal to pi x minus i pi y and a dagger equal to pi divided by h e B and pi x plus i pi y. So, I hope the discussion that we had done so far is clear and the rational or the motivation is clear the motivation is that we are going to talk about the fractional quantum Hall effect, but however doing a full calculation leading to the conductivity is something that will not do rather because we have done that using the Kubo formula and here there is a lot of numerical calculations involved in order to you know verify various things such as you know the plateaus or the fractionally quantized plateaus and so on. So, we will not do that rather we will concentrate more on the nature of the wave function and that to the lowest Landau level and various things various properties related to the quantum Hall fluid. So, in order to do that we have noted that the most suitable this gauge here is the symmetric gauge and not the Landau gauge that we have seen earlier and in this symmetric gauge we have do this problem that is the problem of electrons in a magnetic field and find out the energy eigenvalues and in order to do that we have written down some annihilation and creation operators in terms of these canonical momentum sorry non-canonical momentum, but gauge invariant momentum which are pi, pi x and pi y the components of that ok.

So, this is of course your a a dagger commutation relation you can easily find out there is something that even though I expect you to do let me do that because we are introducing these thing for the first time. So, this is both get multiplied the pre factors get multiplied and we have a pi x minus i pi y and a pi x minus plus i pi y ok. So, that is your a dagger and so on and so there is and there is a pi x plus i pi y and pi x minus i pi y alright. So, this is something that one has to calculate and let us do that quickly it is h e B and so this is a pi x square plus i pi x pi y plus pi y square plus minus pi x square plus i pi x pi y and minus i pi y pi x and a pi y square and so on. So, if you simplify that a little you get this a a dagger as 2 pi i by eBh pi x pi y minus a pi y pi x now these are simple things you should be able to do and this is nothing but this is i by eBh cross and this is equal to pi x pi x pi y commutator.

Symmetric gauge.

$$\vec{A} = \frac{1}{2}\left(\vec{r}\times\vec{B}\right).$$

$$= \frac{1}{2}\left(-By\hat{x} + Bx\hat{y}\right).$$

$$\left.\begin{aligned}\vec{A} &= \left(-yB\hat{x}\right)\\ &= \left(xB\hat{y}\right)\end{aligned}\right\}\ \text{Landau gauge.}$$

$$\vec{p} + e\vec{A} = \vec{\pi}$$

$$a = \sqrt{\frac{\pi}{heB}}\left(\pi_x - i\pi_y\right).$$

$$a^{+} = \sqrt{\frac{\pi}{heB}}\left(\pi_x + i\pi_y\right).$$

$$\left[a, a^{+}\right] = aa^{+} - a^{+}a = \left(\frac{\pi}{heB}\right)\Big[\left(\pi_x - i\pi_y\right)\left(\pi_x + i\pi_y\right) - \left(\pi_x + i\pi_y\right)\left(\pi_x - i\pi_y\right)\Big].$$

$$= \left(\frac{\pi}{heB}\right)\Big[\pi_x^2 + i\pi_x\pi_y + \pi_y^2 \quad \pi_x^2 + i\pi_x\pi_y - i\pi_y\pi_x - \pi_y^2\Big]$$

So, a and b they do not commute of course which we know and the commutation is expressed in terms of pi x pi y. Now if you remember that if these were canonical momentum then p x p y should be equal to 0 because the x component of p has got nothing to do with the y component but they are not canonical as we have said earlier. So, we just say that they are not canonical because x i pi j this is not equal to delta ij what I mean is I mean ih cross delta ij if you like ih cross delta ij or just let me write it as delta ij and so on okay. Okay you can write that and then of course you have a pi i pi j is not equal to delta ij as well okay so that is why they are not canonical alright. So, these are some of the features of this of course as I said that they are gauge invariant and so on.

So, one can actually prove the commutation relation as this pi i pi pi j plus e Ai so this and p j plus e Aj this is equal to minus e and there is a del a del xi minus del Ai this Aj and this is xj and this is equal to minus e epsilon ijk Bk. I am writing in the tensor notation but you do not have to but you work it out remember that this pi and pj these are canonical momenta and then work out various things you will get this okay. So, if you put in there then a a dagger becomes equal to so i over e B h cross multiplied by minus i e h cross B and these will cancel and will give you equal to 1 okay. So, a a dagger commutation is equal to 1 and these are important okay. Now in order to understand the degeneracy of the Landau levels we are simply now playing with the Hamiltonian various terms of the Hamiltonian and writing them in terms of this pi x and pi y operators and these are momentum operators as I said they are gauge invariant but non-canonical okay.

$$[a, a^\dagger] = \left(\frac{2\pi i}{eBh}\right)\left[\pi_x \pi_y - \pi_y \pi_x\right].$$

$$= \left(\frac{i}{eB\hbar}\right)\left[\pi_x, \pi_y\right]$$

They are <u>not canonical</u> because. $[x_i, \pi_j] \neq i\hbar\delta_{ij}$

$[\pi_i, \pi_j] \neq \delta_{ij}$

$$\left[(p_i + eA_i), (p_j + eA_j)\right] = -e\left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}\right) = -e\,\epsilon_{ijk} B_k$$

$$[a, a^\dagger] = \left(\frac{i}{eB\hbar}\right)\left(-ie\hbar B\right) = 1$$

Introduce, $\boxed{\tilde{\vec{\pi}} = \vec{\pi} - e\vec{A}}$ $\tilde{\vec{\pi}}$ is not gauge invariant. but it is canonical.

And we can introduce another variable so introduce a pi tilde this is of course a vector, vector pi tilde this is equal to a pi minus e A ok. The advantage with this you will see in a moment it has the other property that pi tilde is not gauge invariant but canonical okay. So, we introduce this pi tilde operators and write them in terms of the pi operators all right. And also it has to be kept in mind that the vector potential enjoys a gauge freedom which we can write it as a minus grad Chi where Chi some scalar quantities and the reason that such a thing exists because your curl of A has to give you a B. So, a curl of A prime should also give you a B which is curl of A minus grad chi, Chi being a scalar.

So, let us write that so an arbitrary scalar it can be any scalar okay. So, this is equal to curl of a and minus curl of grad chi and this is equal to 0 simply for the reason that grad Chi is a direction it is a vector which has a particular direction if you try to take a curl of it that is whether it rotates or not and of course does not rotate and that is why it is equal to 0. So, A curl of A prime will give you a b the magnetic field that you want and as well as curl of a. So, that is why you can write down a to be ambiguous by the amount a minus grad Chi, Chi being a scalar okay. And so this will enable us to write a pi tilde prime okay which is equal to A p prime minus e A prime minus grad Chi okay that is what is there if you lift the prime then you have you can lift prime all throughout and then you have to drop the grad Chi as well.

So, the important thing is that for these these commutation relations of these quantities okay let me not write it with a vector anymore we write it with just a tilde. So, I well x say for example and pi y tilde prime is equal to i e h cross by B of course this sort of differ from the commutation relations of pi x and pi y by just a sign, but there is an additional advantage because each of these pi i tilde and pi j prime tilde this is equal to 0 which means that they are canonical. So, this primed momenta they are canonical whereas the unprimed momenta even though they are gauge invariant they are not canonical okay. All right so this is our discussion so far and so this is it is good that it is equal to 0. So, if you want to write down the Hamiltonian in terms of these a and a dagger operator then one can write it as it is a dagger a plus half that is the form of the Hamiltonian you can check that with the definitions of a and a dagger if you write them and then you will get this.

So, your Hamiltonian will become equal to let me write it here so your Hamiltonian will be equal to you know so this p x or rather p plus e A p plus e A whole square by 2 m okay so this is the p changes to p plus e A and so on and then of course you write it in terms of pi and then when you do that then convert it into the a and a dagger and this certainly gives you the number density or the number operator corresponding to this problem. So, this is equal to nothing but equal to h cross omega b n plus half and n is equal to 0 1 2 which denote the indices of the Landau level okay. So, that is our Hamiltonian and written now in terms of the operator so it is n cross I mean n operator which is a number operator and it is written in the Fock states. The problem as we have already figured out that if you do it in the symmetric gauge and when none of k x and k y are conserved quantities it is difficult to see that where the degeneracy is coming from because that is where in the Landau gauge we have gotten this degeneracy from the conserved quantities such as k x or k y depending on the gate chosen. But here both are non conserved quantities and finding the degeneracy is a bit challenging but then we know that what the solution is the solution would be introducing the angular momentum operator.

$$\vec{A}' = \vec{A} - \vec{\nabla}\chi$$

$$\vec{\nabla}\times\vec{A} = \vec{B}$$

$$\vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\left(\vec{A} - \vec{\nabla}\chi\right)$$

$$= \vec{\nabla}\times\vec{A} - \underbrace{\vec{\nabla}\times\vec{\nabla}\chi}_{0}$$

χ : an arbitrary scalar.

$$\tilde{\vec{\pi}}' = \vec{p}' - e\left(\vec{A}' - \vec{\nabla}\chi\right).$$

$$\left[\tilde{\pi}_x', \tilde{\pi}_y'\right] = ie\hbar B.$$

$$\left[\tilde{\pi}_i', \tilde{\pi}_j'\right] = 0. \Rightarrow \text{Canonical.}$$

$$H = \frac{\left(\vec{p} + e\vec{A}\right)^2}{2m}.$$

$$H = \hbar\omega_B\left(a^\dagger a + \frac{1}{2}\right).$$

$$\boxed{H = \hbar\omega_B\left(n + \frac{1}{2}\right)}$$

$n = 0, 1, 2 \ldots\ldots$ indices of the Landau levels.

And in order to do that let me introduce new set of operators okay and the new set of operators are nothing but so this b which is equal to 1 divided by 2 E h cross b and pi x tilde plus i pi y tilde that is your b and the b dagger is of course the minus sign that goes inside the bracket. So, it is 2 E h cross b and pi x tilde minus pi y tilde okay. So, these are the b and the b dagger operators and you can check that b b dagger is equal to 1 okay. And these b and the b dagger will give degeneracy of the levels okay.

So, that we will see how. So, now we of course have one quantum number. So, what are the quantum numbers that we have? So, we have of course n which is which appears in the Hamiltonian which is the eigenvalue of the n operator the number operator which acts on the Fock space giving us a number of oscillators present in the problem and the other one is m which comes from. So, this is included in the energy and this is included in the angular momentum operator. And what I mean by angular momentum operator such as if it is a Z component of the angular momentum then Z is equal to you know it is like r cross p and the Z component of that assuming that we are not talking about spin otherwise this would just be L Z okay. So, our quantum numbers become there are 2 quantum numbers which are n and m and this should label uniquely the Landau levels or the wave functions corresponding to the Landau level okay.

Introduce new set of operators.

$$b = \frac{1}{\sqrt{2e\hbar B}} \left(\tilde{\Pi}_x + i\tilde{\Pi}_y\right).$$

$$b^{\dagger} = \frac{1}{\sqrt{2e\hbar B}} \left(\tilde{\Pi}_x - i\tilde{\Pi}_y\right).$$

Check : $[b, b^{\dagger}] = 1.$

$b, b^{\dagger}$ will give degeneracy of the Landau levels.

What are the quantum numbers? $\qquad J_z = (\vec{r} \times \vec{p})_z .$

n, m → Angular momentum.

↓

Energy

So, let us write these things as a ket comprising of these numbers. So, this is n and m. So, this is equal to a dagger n b dagger m and this is divided by n factorial m factorial. So, this acting on the vacuum 0 0 will give us a Landau level starting from the from the vacuum okay. And of course, we know that that a 0 0 is nothing, but equal to 0 and as well b acting on 0 0 gives nothing, but 0 basically that you cannot lower these n and m indices any further that is why the a acting on 0 0 or the b acting on 0 0 will give you 0.

$$|n, m\rangle = \frac{(a^{\dagger})^n (b^{\dagger})^m}{\sqrt{n!\, m!}} \, |0, 0\rangle.$$

$$a\,|0,0\rangle = 0, \qquad b\,|0,0\rangle = 0.$$

$$H = \frac{1}{2m}\, \vec{\Pi} \cdot \vec{\Pi} = \frac{1}{2m} \left(\vec{p} + e\vec{A}\right)^2.$$

Like a and b in their respective ways they are the annihilation operator you cannot annihilate anything below 0 that is why these things give you 0. And what is the Hamiltonian for this? So, Hamiltonian is 1 over 2 m and we have a pi dot pi that is the momentum and this is nothing, but equal to 1 over 2 m p plus e A whole square okay.

Now we bring in all those non canonical momentum, but that are gauge invariant in order to write H you can also write it in terms of pi tilde it really does not matter okay. We will have to construct the Landau level wave function. So, if you look at one slide before two slides rather you see these energy expression that you see I mean or the Hamiltonian is simply that of a harmonic oscillator it is a simple harmonic oscillator.

And even you can think about it in just one dimension where one n is just the quantum number one single quantum number that can take values n is equal to 0 1 and 2 ok. So, now we have to find the degeneracy and find the wave function. And in order to do that let us you know write down again this a operators which are 1 divided by in terms of this pi over h e B this can also be written as using h cross equal to h over 2 pi. So, we can write this down equal to 2 e h cross B pi x minus i pi y and this is equal to 1 divided by root over 2 e h cross b and this is equal to a P x minus i P y and a plus e A x minus i A y okay. So, if you write down this pi x in terms of P x and A x and pi y in terms of P y and a y.

$$a = \frac{1}{\sqrt{2e\hbar B}} \left(\pi_x - i\pi_y\right).$$

$$= \frac{1}{\sqrt{2e\hbar B}} \left[\left(p_x - ip_y\right) + e\left(A_x - iA_y\right)\right]$$

$$= \frac{1}{\sqrt{2e\hbar B}} \left[-i\hbar\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) + \frac{eB}{2}\left(-y - ix\right)\right]$$

So, this is the thing. Now what I will do is that I will introduce this P x equal to minus i H cross del del x ok. And then from there we will do something else let us see what we do from there. So, it is 2 E H cross b and we have minus i H cross del del x minus i del del y plus e now I will write A x and A y also in terms of e. So, it is e B by 2 and a minus y minus i x ok. This is quite important because you know your a that is the vector potential the A x and the A y they are position dependent quantities okay.

In this case it is very clear that they grow linearly with x and y and because of this really the momentum becomes a non-canonical because of these position dependent terms and that is why it refuses to give rise to those canonical commutation relations that we are very familiar in quantum mechanics okay. So, this is the form of this a and we can also write down a dagger and so on. So, we will use this quantities here that you see and you see here we will use complex numbers for these quantities. So, we will use a z is equal to x plus or x minus i y and we use a z tilde as x plus i y. You might wonder that why I am using the opposite definition that is z is usually taken as x plus i y and z tilde is taken as or z star rather is taken as x minus i y but here we have taken.

So, z tilde is nothing but z star okay. Now we are doing it for a purpose that is z is taken in this particular fashion for a given reason and let us also define a shorthand notation that is half of del del x plus i del del y which appear here as you see which appear here that is del del x minus i del del y or del del x plus i del del y this we write it as delta okay. So, this is delta just a shorthand notation and similarly we will use a half of del del x minus i del del y this is equal to so this is not 0 just to make sure this is like this and this is like this okay. So, this del or del tilde is what these things are and this is important to understand that del z is equal to del z tilde this is or del tilde z tilde is equal to 1 because you see that if you take this del z this is equal to half of del del x plus i del del y and this is going to act on the z which is x minus i y. And this will be half del del x of x will be equal to 1 and del del x of iy will be 0 and i del del y will be of x is will be 0 and del del y of that will be equal to 1 but then there is a minus i square so that will give you a 1 and this will give you a 1 okay then that is why you can check for the other one as well. So, this del z so there is an operator which is of the form that we have shown here.

So, this acting on the z will give you 1 and so on okay. So, in fact now these a and a dagger so we will write a and a dagger in terms of the coordinates z. I mean I am using this inside quotes because x and y are definitely coordinates so z is a coordinate which is a combination of x and y but now in the complex plane because it is x plus iy or x minus iy okay. So, a is equal to a minus i root 2 l B that is a magnetic length that we have studied plus z for l B there is some specific way of writing in order to make sort of the notations simple and this is l B tilde minus z tilde this is four l B this will not be a tilde okay alright. So, this is fine now l B is equal to h cross over e B root over h cross over e B which is what we have studied earlier it is basically the magnetic length and it is related to the guiding center which we have discussed these motion of the electrons in a magnetic field okay. So, these are some of the notations some algebra. Algebra is very simple just finding out commutation relations etc you can easily do that.

Use $z = x - iy.$

$\tilde{z} = x + iy.$

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \partial$$

$$\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) = \tilde{\partial}$$

$$\partial z = \tilde{\partial}\tilde{z} = 1.$$

$$\partial z = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(x - iy)$$

$$= \frac{1}{2}(1+1) = 1.$$

Write a and $a^{\dagger}$ in terms of the 'coordinates' z.

$$a = -i\sqrt{2}\left(l_B\, \partial + \frac{z}{4 l_B}\right)$$

$$a^{\dagger} = -i\sqrt{2}\left(l_B\, \tilde{\partial} - \frac{\tilde{z}}{4 l_B}\right)$$

$$l_B = \sqrt{\frac{\hbar}{eB}}$$

So, this tells you that what we do is a acting on 0 m that has to be equal to 0 okay. So, we will start with the state which is starting with a wave function n m okay. Just making sure that you understand what these n m are n is equal to the quantum number corresponding to the energy and m is still what we are yet to find it is the angular momentum quantum number corresponding to J z or L z okay. This is true because if n is equal to 0 then applying this annihilation operator would yield you 0 this is the usual property of the Fock state that the state where there is no oscillator you cannot reduce it any farther. So, this tells you now I will put the form of I and that tells you that this a is equal to a is equal to lb delta plus a z and a 4 l B and a 0 m that should be equal to 0 okay.

Now that tells you okay so this make sure that this is delta. So, this operator acting on 0 m ket would give you 0 because there is the n quantum number is equal to 0. So, you cannot reduce it any farther and so on. open this up and do a little bit of calculation. So, these 0 m is let us call it as the L psi L L L and is a function of z and z tilde L L L means lowest Landau level okay.

So, if we take the lowest Landau level and then put m equal to 0 as well then this has a form unnormalized form is minus z mod square divided by 4 l B square okay. And this is nothing but it reminds you of the ground state wave function for the harmonic oscillator. So, this is the ground state corresponding to m equal to 0 there is no m here and in terms of the coordinates this is written as exponential minus mod z square by 4 l B square l B is what we said earlier l B is equal to this and this is just the Gaussian like this in the complex plane. So, this is you know so this is psi of z so this is a function of z and z tilde or z or z star and this is that z equal to 0. So, this is of the form of a Gaussian this is well known that the lowest wave function for the harmonic oscillator is just a Gaussian as you go to you know larger and larger or rather excited states you will have a polynomial coming in as a coefficient and this polynomial has properties such as this correspond to even and it has it is an even polynomial and corresponding to odd and we have an odd polynomial.

$$\boxed{\text{Starting with a wavefunction } |n, m\rangle}$$

$$a\,|0, m\rangle = 0$$

$$\underbrace{-i\sqrt{2}\left(l_B \partial + \frac{z}{4 l_B}\right)}\,|0, m\rangle = 0.$$

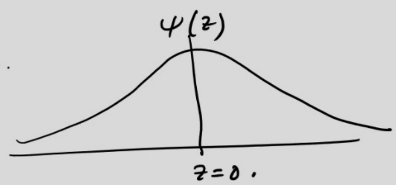
LLL: Lowest Landau level.

$$|0, m\rangle \text{ is } \psi_{LLL}(z, \tilde{z})$$

$$\psi_{LLL, m=0}(z, \tilde{z}) \simeq e^{-|z|^2/4 l_B^2}$$

$$\downarrow$$

Gaussian.



$$\psi_{LLL, m\neq 0}(z, \tilde{z}) = \left(\frac{z}{l_B}\right)^m e^{-|z|^2/4 l_B^2}$$

Thankfully we will be talking mostly about the lowest Landau level here and then we should be happy with this Gaussian form that you see here okay. If you want to have an ansatz that what is what is it for LLL m not equal to 0 and is a function of z and z tilde then it should be some it is like a z over lb to the power m that is a polynomial and then z square divided by 4 l B square okay. So, this is the Gaussian multiplied by a polynomial and so on. Now we will introduce the and just for our notation we will write it as Jz instead of Lz, but as I said that you can see Lz written in some books and notes.

So, we just use a total quantum number notation J. So, this Jz is nothing but equal to h cross which is of course the scale of the problem and z del n minus z tilde del tilde okay. So, just to remind you that these are the definitions of delta and delta tilde okay and this z is of course x minus iy and z tilde is x plus iy. I told you that this is not in the conventional form, but this is what suits us the best in terms of these the definitions that you know of z and z star or z tilde okay. So, this is the z that you have and then of course just to remind you that z is equal to x minus iy, z tilde equal to x plus iy and delta is equal to half of del del x plus i del del y half of del del x plus i del del y and delta tilde is equal to half of del del x minus i del del y okay. These are the definitions and this is that and you can check that your Jz acting on these psi LLm okay.

Even for m equal to 0 it gives you m h cross psi LLm which is a known result even in say hydrogen atom problem. So, you put all these z and delta and then take this the form that you have do not put the form of z keep z as z as z and z tilde same and remember that your z mod square is nothing, but equal to z z tilde okay. So, when you take a derivative with respect to z it will be left with a z tilde and when you take a derivative with respect to z tilde it will be left with a z okay.

So, that is how okay. So, this is that relation that you have. So, m are m is the angular momentum quantum number. So, do not make a sort of confusion between m to be equal to mass and m to be equal to the angular momentum quantum number. If you feel that there is a problem then you can write this m as m J okay, but if you feel it is fine we know how to distinguish between them then you can write it as m okay. So, we have taken a gauge which is symmetric and in the symmetric gauge neither of Kx and Ky were constants and we wrote down the Hamiltonian in that symmetric gauge solved it and got a result which is identical to what we have gotten it with the Landau gauge for electron in a in presence of a magnetic field and it should be because the result should not depend upon the gauge because energy is a is a physical observable or it is a measurable quantity that should not depend on the choice of the gauge and which we have seen. And then we have gone ahead and calculated or rather introduced various quantities such as a and b and wrote down h in terms of a and b looked at the commutation relations of a and b in order to understand do they correspond to and then finally use them in order to write down the Hamiltonian and then Jz and then Jz acting on this psi L L L will give me the lowest Landau level and corresponding to any m.

So, m is the degeneracy m equal to 0 correspond to the lowest Landau level m equal to 1 will also correspond to. So, this value of m will get you the degeneracy that we have even in the case of Landau gauge where the conserved quantity such as Kx and Ky gave us a Landau level degeneracy. Now the Landau level degeneracy given by this value of m or m j that you see I will remove this j and just hope that you do not make a mistake between m to be the angular momentum quantum number and m to be the mass. So, you can write down h equal to 1 over 2 m this m to be the mass and not the m above okay, the quantum number alright. This is equal to p plus e A square and so on and then I can write it down in terms of all these things these are customary but they are quite important too.

Introduce the angular momentum J_z

$$J_z = \hbar \left(z\partial - \tilde{z}\tilde{\partial} \right).$$

$$\left.\begin{aligned} z &= x - iy \\ \tilde{z} &= x + iy \end{aligned}\right\} \quad \partial = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \quad \tilde{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

$$\boxed{J_z \psi_{LLL,m} = m\hbar\, \psi_{LLL,m}}$$

$$H = \frac{1}{2m}\left(\vec{p} + e\vec{A}\right)^2$$

m → mass (and not the q.no. m alone).

$$= \frac{1}{2l_B^2}\left[\left(-i\frac{\partial}{\partial x} - \frac{y}{2}\right)^2 + \left(-i\frac{\partial}{\partial y} + \frac{x}{2}\right)^2\right]$$

$$z = x - iy = re^{-i\theta}, \qquad z^* = x + iy = re^{+i\theta}. \qquad \boxed{z^* = \tilde{z}}$$

So, it is 1 over 2 l B square and this is equal to minus i del del x minus y by 2 square plus minus i del del y plus x by 2 square this is the Hamiltonian and if you use variables such as x minus iy which is what we have done and z tilde equal to or we can do a z star here which is equal to x plus iy. So, z plus iy and write down this as r e to the power i theta or minus i theta and write this as r e to the power i theta plus i theta. So, z star and z tilde are identical okay you can write it with whatever notation you want. So, the derivatives can be written as so del del x can be written as a del del z plus del del z star and a del del y can be written as minus i del minus i del del z minus del del z star. So, this is del del x and del del y as in terms of this z and z star the Hamiltonian can be written as 1 by 2 L b square 1 over z 1 over 4 z square minus 4 del 2 del z del z star minus z del del z plus z star del del z star okay.

And it is basically this has some similarities with the harmonic oscillator, but the similarity is not very well defined because of these you know mixed derivatives like z z del del z and z star del del z etcetera etcetera. But these mixed derivatives are the main thing. Now we can use these same set of ladder operators that we had done earlier which are b and b dagger and so this is equal to b equal to 1 by root 2 z star by 2 plus 2 of del del z this is your b and your b dagger is equal to 1 by root 2 z over 2 minus 2 del del z star you can change all your z tilde to z star if you want. And this is a equal to 1 by root 2 z by 2 plus 2 del del z star. Now I am writing both the a and the b operators together and and your a dagger is equal to 1 by root 2 z star by 2 minus 2 del del z.

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial z^*}$$

$$\frac{\partial}{\partial y} = -i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*}\right).$$

$$H = \frac{1}{2 l_B^2}\left[\frac{1}{4}|z|^2 - 4\frac{\partial^2}{\partial z \partial z^*} - z\frac{\partial}{\partial z} + z^*\frac{\partial}{\partial z^*}\right].$$

$$b = \frac{1}{\sqrt{2}}\left(\frac{z^*}{2} + 2\frac{\partial}{\partial z}\right) \; ; \quad b^\dagger = \frac{1}{\sqrt{2}}\left(\frac{z}{2} - 2\frac{\partial}{\partial z^*}\right).$$

$$a = \frac{1}{\sqrt{2}}\left(\frac{z}{2} + 2\frac{\partial}{\partial z^*}\right) \; ; \quad a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{z^*}{2} - 2\frac{\partial}{\partial z}\right).$$

$$[a, a^\dagger] = [b, b^\dagger] = 1.$$

$$H = a^\dagger a + \frac{1}{2}.$$

So, these are my a and b operators and a and b operators have a similarity that both of them their commutation relations they obey equal to the commutation relations obey equal to 1 and it can be written as so H is equal to a dagger a plus half this is the Hamiltonian of course your h cross Omega etcetera they are absorbed in the definition because of these lb square being there and the angular momentum. So, Jz is written as this is an usual definition that you know in terms of the it is a derivative of only the Phi so this is equal to minus H cross z del del z minus z star del del z star and this is equal to a dagger a minus b dagger b okay. So, that is the form of this z and the eigenvalues of z are z are minus you know they are minus m h cross and m varies from minus n to plus n okay. So, this is quite important that this is the degeneracy of the Landau levels.

Angular Momentum

$$J_z = -i\hbar\frac{\partial}{\partial\phi}.$$

$$= -\hbar\left(z\frac{\partial}{\partial z} - z^*\frac{\partial}{\partial z^*}\right) = a^\dagger a - b^\dagger a.$$

Eigenvalues of z are $-m\hbar$.

$\boxed{m \text{ varies from } -n \text{ to } +n.}$

$b^\dagger$	:	increases m by 1 unit
b	:	decreases m by 1 "
$a^\dagger$	:	increases n by 1 "
a	:	decreases n by 1 "

So, for each n it will vary like this okay. So, b will decrease m by 1 unit and b dagger will increase the m value by 1 unit just like any of the ladder operators whereas a dagger increases n by 1 unit and decreases a decreases n by 1 unit. So, b dagger increases m by 1 unit b decreases m by 1 unit a increases n by 1 unit a dagger a decreases n by 1 unit and so on. So, these are the reasons that we have so far introduced them and we have carefully dealt with them and m goes from minus n to plus n and so on. So, it is like that the harmonic oscillator problem now has 2 indices and these 2 indices are the wave function have 2. So, for a given value of n there are so many values of m possible between minus n to plus n and then you go to another value of n again from minus n to plus n all these values are possible and these n and m can be written as b dagger to the power n plus m divided by root over of n plus m factorial and a dagger to the power n divided by root over n factorial and so this is 0 0 okay.

Just like we create the excited states in the harmonic oscillator by applying these a and a dagger or rather we built excited states by applying a dagger successively once twice thrice and so on in order to get the first excited state, second excited state and third excited state. Here we get the different Landau levels or the wave functions corresponding to the Landau levels by applying both the a and b operators in this particular fashion. Suppose you want to get to 1 1 in which case the n is equal to 1 in that case the m has to become equal to 0 and then you can write down. So, 0 0 what we have written down earlier is a simple Gaussian now we are writing the normalization which is equal to exponential minus z mod square by 4 l B square once again just to remind you that this is equal to a z z star okay. Of course these a 0 0 obeys equal to b 0 0 that obeys equal to 0 so you cannot annihilate any further and if you successively apply b dagger m times on 0 0 you will get a state which is 0 m of course because it creates builds up the m values and this will give us a z to the power m exponential minus mod z square by 4 l B square and this is the normalization is 2 pi 2 to the power m m factorial okay.

So, these are the properties of the Landau levels and as I said that for a given m you will have to have a polynomial which is z to the power m there is a normalization constant that appears there and so on. So, each time you operate a b dagger on the 0 0 state you get a z. So, you operate it twice you get a z square and so on and so forth. Now I will simply write down this n m state which is a general Landau level which is not of too much of relevance to us but still for the sake of completeness because we will be mostly talking about the lowest Landau level.

This is just for an expression that you can may keep it for record. So, it is 2 to the power 2 m plus n and n factorial m plus n factorial that is the normalization and this is equal to a b dagger m plus n and exponential minus z mod square by 4 l B square that is a general Landau level okay. This is the expression for a general Landau level and so on and then you can write down this in terms of these z and z star and so on so forth and let me still write it once. So, that so this is 2 pi 2 to the power 2 m plus n n factorial m plus n factorial and a z star minus 2 del del z star to the power n z minus 2 del del z to the power m e to the power minus z square by 4 l B square. So, these are the general Landau level. So, you want a particular Landau level say n equal to say 3 m equal to 2 and so on so forth you can adjust that power accordingly 3 and 2 and then take a derivative of the Gaussian term that you see on the right and then you know find out the wave function corresponding to that value of n and m.

Wave functions have 2 indices $|n, m\rangle$

$$|n, m\rangle = \frac{(b^{\dagger})^{n+m}}{\sqrt{(n+m)!}} \frac{(a^{\dagger})^{n}}{\sqrt{n!}} |0, 0\rangle$$

$$|0, 0\rangle = \frac{1}{\sqrt{2\pi}} e^{-|z|^2/4l_B^2} \qquad |z|^2 = zz^{*}.$$

$$a|0, 0\rangle = b|0, 0\rangle = 0$$

$$(b^{\dagger})^{m} |0, 0\rangle = |0, m\rangle = \frac{z^{m} e^{-|z|^2/4l_B^2}}{\sqrt{2\pi\, 2^{m}\, m!}}$$

$$|n, m\rangle = \frac{1}{2\pi\, 2^{2(m+n)}\, n!\,(m+n)!} (a^{\dagger})^{n} (b^{\dagger})^{m+n} e^{-|z|^2/4l_B^2}. \quad \text{General LL}$$

$$= \frac{1}{2\pi\, 2^{2(m+n)}\, n!\,(m+n)!} \left(z^{*} - 2\frac{\partial}{\partial z^{*}}\right)^{n} \left(z - 2\frac{\partial}{\partial z}\right)^{m} e^{-|z|^2/4l_B^2}.$$

We will next talk about the properties of these Landau levels and these are in particular called as the Laughlin wave functions which has been written down by Laughlin purely from an intuitive viewpoint and these are the some of the properties we will talk about. Thank you.