Quantum Hall Effect

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Lec 14: Dirac Hamiltonian, Hofstadter Butterfly

So, we will do a quantum Hall effect in graphene and this will be a derivation of the Landau levels and after that we will just talk about the conductivity without explicitly calculating them, but then you know that a conductivity can be calculated using a Kubo formula. There is one small thing that is important in this context is that see the plateaus come when you know there is a flux that is threaded through the system and that flux has to match with the flux quantum and the flux quantum has a value which we have denoted several times by this, this is has a value some 2 into 10 to the power minus 15 Weber. So, this flux one has to match for an external field to thread through the graphene or the honeycomb lattice. Now, this honeycomb lattice has a this sides of the lattice constant is like 2.46 angstrom and if one does a back of the envelope calculation the area of this unit cell which is just a honeycomb structure is like root over 3 by 2 a square and this comes out to be something like 0.05 nanometer square 0.051 nanometer square. So, if I have to multiply a magnetic field with this area in order to find the flux then that magnetic field has to be few kilo Tesla's or even more and that is a very large magnetic field. So, that is why graphene if you have to see quantum Hall effect in graphene the magnetic field has to be very large larger than what we have seen for the 2D electron gas or the gallium arsenide structures that we have talked about earlier. All right, we ignore this part for the moment and pretend that everything is similar to the quantum Hall effect in a 2D electron gas that is the momentum the mechanical momentum gets you know renormalized by this vector potential and it happens here as well excepting that we now have a lattice structure and not only the lattice structure there is two atoms per basis.

So, it is a matrix equation that we have to solve and we will go ahead and solve that matrix equation and find the energy corresponding to the Schrodinger equation which we will call as a Landau levels. Just to remind you that the Landau levels for a 2D electron gas has a form which was n plus half h cross omega where omega is e B over m and that is the cyclotron frequency which depends on the magnetic field. We will see here how Landau levels that depend on both n the quantum number relevant quantum number and B the magnetic field. All right, so we will use a gauge similar to earlier so we will use a Landau gauge that means your a equal to minus B y x cap we could take the other Landau gauge as well that is B x y cap it will give rise to same result and just to remind you that the Hamiltonian in this particular case is of the form it is a v F and a sigma dot P I have absorbed the h cross here or we could write it as h cross v F into sigma dot k where p is the momentum or k is the wave vector. And this Hamiltonian in presence of a field transforms into it becomes h cross v F sigma dot p minus e A this is the standard prescription which we call as Peierls coupling and this is how the momentum transforms and so on.

So we will write down the explicitly this equation and right now we are doing it for a given K that is a Dirac point but however we generalize it to Dirac points right like writing the Hamiltonian in unified manner for both the Dirac points but that will do later let us carry on with this calculation for the moment. So the matrix equation becomes equal to v F into we will write down the so this is your sigma x and p x minus e A x because this is so we have taken it as a x equal to minus p y and plus so this is your sigma and then we have a sigma y i 0 and p y there is no change in p y because the gauge is purely in the x direction the two component wave equation psi A and psi B so this is equal to psi A and psi B okay. So remember that sigma does not denote the spin degrees of freedom it denotes the sub lattice degrees of freedom which are nothing but the A and B sub lattices. So this 2 by 2 structure is purely because of the sub lattice structure which we call as pseudo spinner and it does not have anything to do with the actual spin of the real spin of the electrons okay. So let us call this as equation 1.

Quantum Hall effect.

$$\Phi_0 = \frac{h}{e} = 2\times 10^{-15}\ \mathrm{Wb}.$$

$a = 2.46\ \mathring{A}$ Area : $\frac{\sqrt{3}}{2}a^2 \simeq 0.05\ nm^2$

$B \sim$ a few Kilo Tesla.

Landau gauge $\vec{A} = -By\hat{x} \Rightarrow A_x = -By$.

$$H = v_F\, \vec{\sigma}\cdot\vec{p}$$

In presence of a field,

$$H = v_F\, \vec{\sigma}\cdot(\vec{p} - e\vec{A})$$

$$v_F\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}(p_x - eA_x) + \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}p_y\right]\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix} = E\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix}.$$

Let us write down the two equations because this is a 2 by 2 equation we can explicitly write them opening up in terms of psi A and psi B. So this is v F and a p x minus I p y plus E b y and this is psi A psi B this is equal to E psi a so the psi B is related to psi A and vice versa. So v F is equal to p x plus i P y plus E B y well we can write only one bracket here so this is equal to psi a equal to E psi B okay. So this is equation number 2 let us call them as you know 2 A and 2 B maybe so this is 2 A and this is 2 B okay. These two equations just we get it from the from equation 1 and if you eliminate one of them in favor of the others we eliminate psi a and solve for psi B then what one gets is that so basically from 2 B or 2 A you calculate psi A and put it back into 2 B that is exactly what is done.

So this is equal to p x minus I p y this is just algebra that I am doing here so it is a e B y and p x plus I p y plus e B y and psi a this is equal to E square psi a so this is just psi b is being eliminated or we could have done the other way around. So here of course we do the we put 2 B into 2 A not 2 A into 2 B so let us write it that eliminating psi B okay. And you keep simplifying it so this becomes equal to p x square plus eB y square plus i p x plus there is a commutator there so e B y and which is a commutator with a p y this is just from the two signs that we get here and then a p y square and this is a commutator bracket. So let me change this bracket to another one so put a big first bracket here and psi a this is equal to e square by v F square psi A just to remind you v F is the Fermi velocity of electrons in graphene. And then simplifying further so it will be p x square plus e B y square minus e h cross B e h cross b plus p y square we simply use the commutation relations p x of course commutes with p y but p y does not commute with y and y p y we use as i h cross.

So once we do that so we get so this is psi A is equal to e square by v F square psi A let us call this as equation 3. So this is the equation which gives you psi A or rather the the Schrodinger equation for psi A. So one has to remember that you know p x does not commute with x or and so on. So the wave function for this or the solution of this would be like a psi a would be some you know f of y which is some function of y. So this is a free particle motion in the x direction because there is no x in this inside the Hamiltonian term that is this term here there is no x but there is A y.

So it is a free particle in the x direction and like a harmonic oscillator in the y direction okay y because p y will not commute with y. So this is simply equal to the wave function is i k x x of course there is a complicated normalization constant which we are not writing but there it is there okay. So if we substitute so this is the solution I hope you understand where the solution comes from this was discussed in details when we did the 2D electron gas problem the same problem in 2D electron gas and then so this is some function of y which we do not know and then there is a propagating part or a plane wave part which is in the x direction okay. So what we do is that we substitute 4 into 3 and what one gets is the following so one gets as p x plus e B y square again a little bit of algebra but these are required in order to arrive at the final result we have solved for the wave function or rather made an ansatz of the wave function unnormalized wave function but we still are far from knowing the eigen value of the problem and both are required. The eigen function will tell you what the electronic wave functions are corresponding to the Landau levels and the Landau level their dependence on the quantum number as well as b have to be found out as well okay. So it is e h cross B plus a p y square so along with me you should verify all these things if I have missed a factor or something you should correct it v f square and this is equal to f of y exponential i k x x.

$$\left.\begin{aligned} v_F\left[\left(p_x - ip_y + eBy\right)\right]\psi_B &= E\psi_A. \quad (2a)\\ v_F\left[\left(p_x + ip_y + eBy\right)\right]\psi_A &= E\psi_B. \quad (2b)\end{aligned}\right\}(2)$$

Eliminating ψ_B.

$$v_F^2\left[\left(p_x - ip_y + eBy\right)\left(p_x + ip_y + eBy\right)\right]\psi_A = E^2\psi_A.$$

$$\left(p_x^2 + (eBy)^2 + i\left[(p_x + eBy), p_y\right] + p_y^2\right)\psi_A = \frac{E_2}{v_F^2}\psi_A. \qquad (3)$$

$$\left(\underbrace{p_x^2 + (eBy)^2 - e\hbar B + p_y^2}\right)\psi_A = \frac{E^2}{v_F^2}\psi_A \qquad (4)$$

The solution,

$$\psi_A = f(y)\, e^{ik_x x}$$

Substituting Eq. (4) into Eq. (3).

$$\left[\left(p_x + eBy\right)^2 - e\hbar B + p_y^2\right] f(y)\, e^{ik_x x} = \frac{E^2}{v_F^2}\cdot f(y)\, e^{ik_x x}.$$

So this is substituting that this replace p x as minus i h cross del del x. Here there is one small point that needs a mention that even though we are doing it on a lattice we are substituting this momentum operator as minus i h cross del del x which is known of course but if you remember your del del x or d dx is nothing but f of x plus h say for example, minus f of x divided by h limit h tends to 0 this is the definition of the derivative the first derivative or you can also write it as you know limit h tends to 0 f of x plus h minus f of x minus h divided by 2 h that is also a central difference formula or the direct formula which is mostly used in mathematics both are acceptable. And how we go from a continuum version to a lattice version is what I am showing here. So if you replace p x by minus i h cross del del x it becomes equal to minus i h cross del del x not del x plus e b y square f of y exponential i k x x minus e b h cross f of y exponential i k x x minus h cross square del 2 f y del y 2 exponential i k x x and this is equal to e square by v F square f of y exponential i k x x. So, this is the Schrodinger equation and we can operate this del del x on these function f of y exponential i k x and that will give me a k x term and so on.

And similarly you know the del 2 of course f y del y 2 there is no dependence. So this del y 2 will be acting only on the f of y. So, if you do a bit of simplification then what comes is that h cross k x plus e B y square and of f of y exponential i k x x minus e h cross b f of y exponential i k x x minus h cross square del 2 f del y 2 this f is a function of y and exponential i k x x and this is equal to E square by v f square f of y exponential i k x x ok. Of course, the exponential term etcetera that cancels out from both sides because they are not equal to 0. So, further simplifying let me see the equation number.

So, let us call this as equation number 5. So, if you simplify then this becomes equal to h cross square and then a k x plus e B y square minus e B over h cross minus del 2 del y 2 and so this f of y equal to e square by v f square f of y okay. So, this is the same as Schrodinger equation that we started with its in a much simplified form here. So, this will be like del 2 f of y del y 2 plus e b over h cross square h cross k x divided by e b plus y square f of y minus e b by h cross f of y equal to h cross e square divided by h cross square v f square f of y okay. So, this is the further simplified form and the finally, we can also write it as minus just changing the sign there del 2 plus e b by h cross square y minus y 0 square f of y minus e b by h cross f of y equal to e square by h cross square v f square f of y okay. Where y 0 is called as a guiding center we have seen this or the magnetic length so to say let us call this as equation 6 all right your y 0. So, let me write down y 0. So, y 0 equal to minus h cross k x over e B okay.

Replace $p_x = -i\hbar\frac{\partial}{\partial x}$

$$\frac{d}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}$$

$$\left(-i\hbar\frac{\partial}{\partial x} + eBy\right)^2 f(y)e^{ik_x x} - eB\hbar\, f(y)\,e^{ik_x x} - \hbar^2\frac{\partial^2 f(y)}{\partial y^2}e^{ik_x x} = \frac{E^2}{v_f^2} f(y)\,e^{ik_x x}$$

$$(\hbar k_x + eBy)^2 f(y)e^{ik_x x} - (e\hbar B)\, f(y)e^{ik_x x} - \hbar^2\frac{\partial^2 f(y)}{\partial y^2}e^{ik_x x} = \frac{E^2}{v_f^2} f(y)e^{ik_x x} \quad (5)$$

Simplifying,

$$\hbar^2\left[(k_x + eBy)^2 - \left(\frac{eB}{\hbar}\right) - \frac{\partial^2}{\partial y^2}\right] f(y) = \frac{E^2}{v_f^2} f(y)$$

$$\frac{\partial^2 f(y)}{\partial y^2} + \left(\frac{eB}{\hbar}\right)^2\left(\frac{\hbar k_x}{eB} + y\right)^2 f(y) - \frac{eB}{\hbar} f(y) = \frac{E^2}{\hbar^2 v_F^2} f(y)$$

$$-\frac{\partial^2 f(y)}{\partial y^2} + \left(\frac{eB}{\hbar}\right)^2 (y - y_0)^2 f(y) - \frac{eB}{\hbar} f(y) = \frac{E^2}{\hbar^2 v_F^2} f(y) \quad (6)$$

$$y_0 = -\frac{\hbar k_x}{eB}$$

So, this is that y 0 and using this we have del 2 f of y del y 2 plus e square divided by h cross square v f square plus e B over h cross this is f of y minus e B over h cross square y tilde square I will tell you what y tilde is and f of y equal to 0 where y tilde is equal to y minus y 0 okay which we have written. So, this is only for simplicity.

So, we introduce this then finally, in terms of this equation the above equation is written as del 2 f of y del y 2. So, we get a second order differential equation linear differential equation in y. So, this is of course, it is not a linear equation because there is a y square involved okay. So, second order differential equation right. So, this is e B over h cross and e square divided by h cross square v F square and h cross over e B just bear with me for some time this is only algebra that I am doing in order to simplify things and so there is a plus 1 minus e B by h cross y tilde square and f of y equal to 0.

So, we get a differential equation f of y and the solution will give us the Eigen functions that we have made an answer about. So, let us take again introduce another variable. So, introduce say alpha square equal to e B over h cross such that this is written as del 2 f of y del y 2 plus alpha square e square by h cross square v F square and h cross over e B plus 1 minus alpha square y tilde square and f of y equal to 0. So, inside the square bracket let us introduce another one another variable which is equal to beta equal to e square divided by h cross square v F square and this is actually divided by alpha square. So, this alpha square should go down which I for to write it there and alpha square and plus 1 let us call that as beta. So, this quantity is equal to beta. So, we have 1 over alpha square del 2 f y del y 2 and plus beta minus alpha square y tilde square and f of y equal to 0. So, what equation number would this be 6. So, let us call it as 7. So, now, we have to solve for f of y.

$$\frac{\partial^2 f(y)}{\partial y^2} + \left[\frac{E^2}{\hbar^2 v_F^2} + \frac{eB}{\hbar}\right] f(y) - \left(\frac{eB}{\hbar}\right)^2 \tilde{y}^2 f(y) = 0.$$

Introducing $\tilde{y} = y - y_0$.

$$\frac{\partial^2 f(y)}{\partial y^2} + \frac{eB}{\hbar}\left[\frac{E^2}{\hbar^2 V_F^2}\left(\frac{\hbar}{eB}\right) + 1 - \frac{eB}{\hbar^2}\tilde{y}^2\right] f(y) = 0.$$

Introduce, $\alpha^2 = \frac{eB}{\hbar}$.

$$\frac{\partial^2 f(y)}{\partial y^2} + \alpha^2\left[\underbrace{\frac{E^2}{\hbar^2 V_F^2 \alpha^2} + 1}- \alpha^2 \tilde{y}^2\right] f(y) = 0.$$

Introduce. $\beta = \frac{E^2}{\hbar^2 V_F^2 \alpha^2} + 1$

$$\frac{1}{\alpha^2}\frac{\partial^2 f(y)}{\partial y^2} + \left(\beta - \alpha^2 \tilde{y}^2\right) f(y) = 0. \qquad (7)$$

If you introduce a new variable which is equal to say q equal to eta y tilde where eta is some scaling then we get this as del 2 f of y del q 2 plus beta minus beta minus q square f of y equal to 0. You have to now refer to this to Bransden and Joachain for the chapter on simple harmonic oscillator there you had got the similar looking equation called as a Weber's equation and from here quantization of the energy levels come and this energy level quantization comes as beta equal to 2 n plus 1. So, see Bransden and Joachain ok. So, what is done is that in order to solve this Weber's equation you have to assume a power series solution and when you put a power series solution inside at all values of y that is at small y and large y you want to have a solution which is a polynomial times a Gaussian type of function its exponential minus say y square by 2 and things like that, but that is not possible until or unless you assume that there is a polynomial would get truncated it is not an infinite series because an infinite polynomial series would become an exponential again and when it becomes an exponential again it sort of defeats the purpose because the exponential along with the Gaussian actually makes it a diverging exponential function which is not allowed, but this is the only solution to the problem where you actually truncate the polynomial up till a given number of terms. So, that it does not blow up and become a or rather it does not become an exponential function and this is given in Bransden and Joachain and see the chapter on simple harmonic oscillator.

So, beta becomes equal to 2N plus 1 that tells you that your E square divided by h cross square v F square eta square becomes equal to plus 1 becomes equal to 2N plus 1. So, the eigenvalue so, 1 will get cancelled from both sides this 1 will come here. So, this E becomes equal to E plus minus becomes equal to plus minus h cross v F or omega B which is e B over m and root over n that is the structure of the Landau level and the unnormalized wave functions we have already talked about, but let me do a little bit more on that. So, instead of the equidistant Landau levels that we have obtained for the 2D electron gas that is N plus half h cross omega where N equal to you know 0 1 2 etcetera. Here the Landau levels do not depend on the index N in a linear fashion rather it depends on it in a quadratic fashion okay.

So, omega B is actually root over 2 e B over h cross okay. We can actually write it as let me not write it as omega B of course, it depends on B, but omega B actually has been used for the cyclotron frequency. So, we will write it as omega tilde ok which is this alright. So, now what is the structure of the wave function? The wave function is f of q which is equal to some a N exponential minus q square by 2 and then there is a Hermite polynomial in q and Hermite polynomial has a property that when N is even it only comprises of even powers of q that is q to the power 0 q square q 4 q 6 etcetera and when is N is odd it is an odd function of q that is it contains odd powers of q like q q q q to the power 5 and so on. So, as you change q to minus q this odd term changes sign while the even term remains same.

So, one can find actually the normalization by using this the total wave function to be. So, this is like minus infinity to plus infinity psi square equal to dy equal to 1 and this will give you there is a little lengthy calculation, but it is done several places. So, this is equal to eta divided by 2 N N factorial root over pi whole to the power half okay. So, that is the normalized wave function because it depends on this N okay.

Introduce a new variable,

$$q = \eta \tilde{y}$$

$$\frac{\partial^2 f(y)}{\partial q^2} + (\beta - q^2) f(y) = 0.$$

See Bransden & Joachain simple Harmonic oscillator.

$$\beta = 2n+1$$

$$\frac{E^2}{\hbar^2 v_F^2 \eta^2} + 1 = 2n+1$$

$$\boxed{E_{\pm} = \pm \hbar v_F . \tilde{\omega} \sqrt{n}.}$$

$$\tilde{\omega} = \sqrt{\frac{2eB}{\hbar}}.$$

$$f(q) = A_n e^{-q^2/2} H_n(q).$$

$$\int_{-\infty}^{\infty} |\psi(q)|^2 dy = 1. \Rightarrow A = \left(\frac{\eta}{2^n n! \sqrt{\pi}}\right)^{1/2}.$$

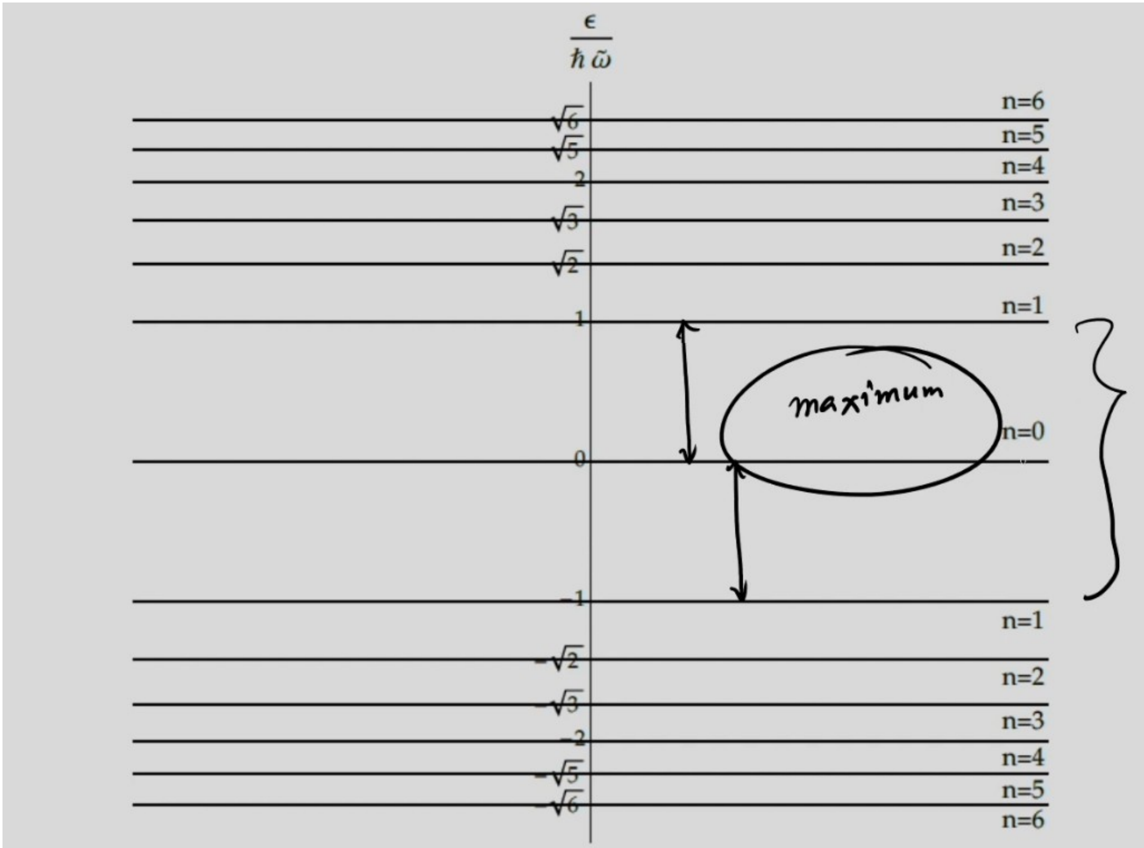
So, this is and then you can put it and then get these are the Landau levels and so on and the Landau level wave function let me write it the full wave function these only the y part or the q part because q and y are related which we have said this is equal to.

So, psi N x y this is given by some e B over h cross whole to the power one-fourth 1 by 2 to the power N N factorial this to the power half and then we of course, have this exponential functions which is freely moving in the x direction and then we have these exponential E b over h cross y minus y 0 square and then the Hermite polynomials which is nothing, but root over e B by h cross and y minus y 0. So, in this gauge the wave function is the particle actually propagates like a free particle in the x direction whereas, it executes a simple harmonic motion in the y direction about some y 0 which is given by the expression that we have talked about earlier. So, this energy of course, does not depend upon the quantum number corresponding to k x and that is why it is degenerate. So, any quantum number k x. So, k x is also like quantized like 2 pi over L x into N where N is the corresponding quantum number let us call it as M.

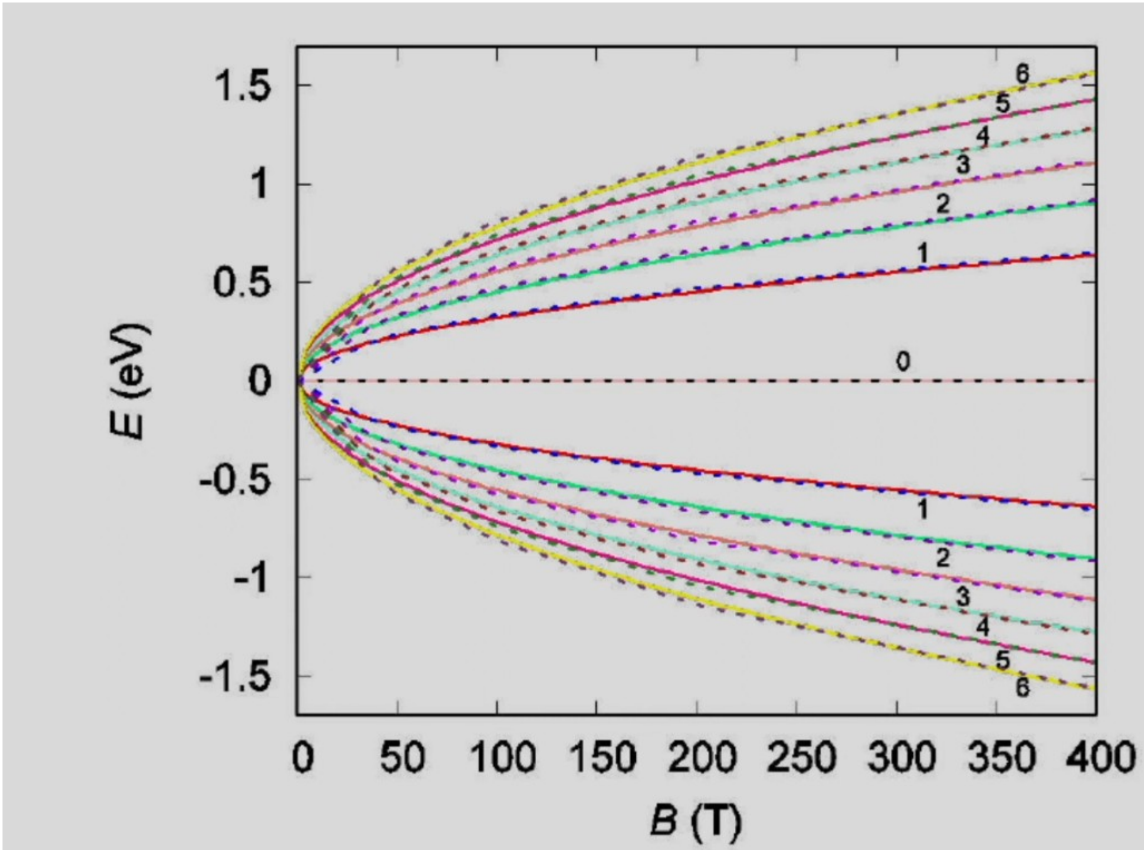
$$\psi_n(x,y) = \left(\frac{eB}{\hbar}\right)^{1/4}\left[\frac{1}{2^n n!}\right]^{1/2} \exp(ik_x x)\,\exp\left[\frac{eB}{\hbar}(y-y_0)^2\right] H_n\left(\sqrt{\frac{eB}{\hbar}}\,(y-y_0)\right)$$

So, this is true for any value of M and that is why it is degenerate and practically the degeneracy could be you know infinitely large okay. So, this is the main ansatz is or rather the inference of this problem is here where we have found the Landau levels to have this particular form okay. Let me show the structure of the Landau levels here. So, this is the structure of the Landau levels this is because it is a square root dependence. Now you see that instead of the Landau levels depending rather they are equidistant they are this case in graphene they are not equidistant they in fact, are as you go larger and larger N values you get lower and lower differences between the Landau levels.

And not only that there are these all the values above N equal to 0 and all the values below N equal to 0 are equally probable that is they can take positive and negative value which was unlike the case of a 2D electron gas okay. So, this is the structure of the Landau levels they are not equidistant the maximum is here okay. So, this is the maximum distance between the Landau level and because of these N equal to 0 and N equal to 1 Landau levels or N equal to minus 1 here both positive and negative integers are allowed. So, the maximum difference comes here and these sort of differences are so large that there is a one can see actually Landau levels in at room temperature or even larger than that okay. Let me show you how the Landau levels depend on B if you see this expression that I have shown you here there is a omega tilde which depends upon root over B.



So, this actually depends on these Landau levels actually depend upon like root over N and B something like that okay. So, a square root of B as well as square root of N I am leaving out other factors at the moment, but it is like root over N and B and which is what you see here you see that E as a function of B has this structure there is a central N equal to 0 level and these are 1, 2, 3, 4, 5, 6 above N equal to 0 that is 0 energy and these are the values below 0 N equal to 0. And this is B in Tesla as I said earlier that you need much larger values of the magnetic field in order to see quantum Hall effect in graphene. So, it is swept till about 400 Tesla okay. So, once we have gotten the Landau levels and the wave functions etcetera one can easily use the Kubo formula that we have talked about that we have described how to calculate conductivity.



So, using that one can calculate the conductivity and finding the conductivity one would know that how does the conductivity depend upon this N okay. It was sigma xy equal to Ne square over h for a 2D gas whereas, is may not be equal to n e square by h. And this difference in the behavior of both the Landau levels between the 2D electron gas and graphene arises from the fact that we are talking about chiral massless fermions relativistic fermions in this case which are of course, as we said several times earlier that they are not really relativistic the scale is set by the velocity scale is set by the Fermi energy instead of the speed of light, but still the dispersion is linear. So, this linear dispersion along with the massless behavior of the chiral fermions what do I mean by chiral fermions is that if the fermion sort of goes around one of the Dirac points in a clockwise direction it will go anticlockwise in the other Dirac point about the other Dirac point at t equal to 0 okay. So, before we go to talk about the quantum hall plateaus or how the quantization looks like it is probably instructive and I will go through it quickly.

However, I sort of suggest that you should look at these derivation quite carefully and maybe sort of get used to it these are matrix equations that we are solving and then we are of course, converting them into a set of equations and this set of equations are being solved you know self consistently. Now, what I will do is that this is the situation at one k point and you need to actually talk about the two k points the two Dirac points or the two valleys together in order to you know do this to understand in a more detailed fashion the structure of the Landau levels and the wave functions contributing to the n equal to 0 Landau level. See here a very importantly because the difference between the n equal to 0 and n equal to 1 is very large as I have shown the mostly the n equal to 0 Landau level is mostly talked about or even in experiments one sees the lowest Landau level. So, I extend this and in a sketchy manner talk about both the k and k prime points. So, we do an analysis with both k and k prime points together unified description.

Then we will write down the Hamiltonian for both k and k prime as so, this is like v F now it will not be a 2 by 2, but it will be a 4 by 4 2 coming from the sub lattice degrees of freedom that is a and b and multiplied by another 2 coming from the two valleys. In addition if you really want to talk about the spin degrees of freedom then you will have two more degrees of or rather this size of the matrix goes up by another factor of 2 which means it becomes 8 by 8, but that is not required as long as the spin orbit coupling can be ignored and in graphene fortunately or unfortunately the spin orbit coupling is quite weak. So, this is 0 and px plus ipy and 0 0 this is a minus px plus ipy 0 0 0 and this is like 0 0 0 px minus ipy and 0 0 px plus ipy 0 and so on. So, this is the wave function or rather this is the Hamiltonian that one has to solve and by solving again one gets this Landau levels which you have already got. So, we have done it for only one k point now I want to demonstrate as I said that I will not do a very detailed calculation like the one that I have done earlier I will only sketch the salient you know steps so that you can fill it in.

$$\psi_n(x,y) = \left(\frac{eB}{\hbar}\right)^{1/4}\left[\frac{1}{2^n n!}\right]^{1/2} \exp(ik_x x)\, \exp\left[\frac{eB}{\hbar}(y-y_0)^2\right] H_n\left(\sqrt{\frac{eB}{\hbar}}\,(y-y_0)\right)$$

Both $\vec{K}$ and $\vec{K}'$ points. (unified description).

$$H_{KK'} = v_F \begin{bmatrix} 0 & (p_x+ip_y) & 0 & 0 \\ -(p_x+ip_y) & 0 & 0 & 0. \\ 0 & 0 & 0 & (p_x-ip_y) \\ 0 & 0 & (p_x+ip_y) & 0 \end{bmatrix}$$

$$\psi = \begin{pmatrix} \psi_A^{K'} \\ \psi_B^{K'} \\ \psi_A^{K} \\ \psi_B^{K} \end{pmatrix}$$

So, we have this psi which will be let us write it as a component like a psi A k prime a psi B k prime and a psi A k and psi B k. So, that is the 4 by 4 structure one because of this k that is the valley degree of freedom and the other would be the sub lattice degrees of freedom. So, if we write down this so this E into psi A k is equal to v F px minus ipy psi B k and E psi b k is equal to v F px plus ipy and a psi A k and so on so forth. So, everything that you know we have done so we again substitute solving these equations. So, E square just exactly similarly that we have done earlier it is k it is equal to h cross square v F square px minus ipy px plus ipy and psi A k.

Now remember this px and py they include the these vector potentials which we have not written here but they would eventually include this just simplifying it. So, E square psi B k is equal to h cross square V f square px plus ipy px minus ipy and a psi B k. So, now we get two equations and these two equations can be solved and you have while you solve there will be so what we do is that we change px to px plus e B y I am just using a different gauge same as Landau gauge but here we are changing the px to px plus eby. So, again of course in the last thing also we have taken the gauge in the x direction. So, that is the Landau gauge and then once you do that and do some bit of simplification what one gets is the following it is h cross square v F square psi B k equal to px plus eby square plus e h cross B plus py square and this is equal to psi B k and so on.

These k and k are the Dirac points I should write it with a vector but I forgot writing it here. So, please put a vector everywhere. So, we arrive at again this E square by h cross square V f square minus e h cross b psi b k equal to px square say let us call it as tilde plus py square tilde plus psi b this is psi b and k and so on. So, where your px tilde tilde square equal to px plus e B y square and so on. And of course your py tilde just to have a symmetry we have written it as py tilde both as with tilde but this is what.

$\vec{k}, \vec{k}'$ Dirac points

$$E\psi_A^{\vec{k}} = v_F (p_x - ip_y)\psi_B^{\vec{k}}$$

$$E\psi_B^{\vec{k}} = v_F (p_x + ip_y)\psi_A^{\vec{k}}.$$

Solving

$$E^2\psi_A^{\vec{k}} = \hbar^2 v_F^2 (p_x - ip_y)(p_x + ip_y)\psi_A^{\vec{k}}$$

$$E^2\psi_B^{\vec{k}} = \hbar^2 v_F^2 (p_x + ip_y)(p_x - ip_y)\psi_B^{\vec{k}}$$

$$p_x \to p_x + eBy.$$

$$\frac{E^2}{\hbar^2 v_F^2}\psi_B^{\vec{k}} = \left[(p_x + eBy)^2 + e\hbar B + p_y^2\right]\psi_B^{\vec{k}}$$

$$\left(\frac{E^2}{\hbar^2 v_F^2} - e\hbar B\right)\psi_B^{\vec{k}} = \left(\tilde{p}_x^2 + \tilde{p}_y^2\right)\psi_B^{\vec{k}}$$

$$\tilde{p}_x^2 = (p_x + eBy)^2, \quad \tilde{p}_y^2 = p_y^2$$

So, I am doing exactly the same calculation now combining both of them both the k and the k prime points. So, it is E square by h cross square v F square minus e h cross B psi B k equal to px square plus py square by 2 m psi B k equal to half let us call it a k tilde y minus y 0 square plus py square by 2 m psi b k and so on. As where your k tilde equal to nothing but E square b square by m and y 0 equal to px over e B or h cross k x over e B. So, we are doing a similar analysis and so on and then here what we get is a 1 over m e square divided by h cross square v F square equal to 2 n plus half h cross h cross omega b plus h cross omega b which gives you what 2 n h cross omega b by m.

So, n is equal to 0 1 2 etcetera and so on. And then of course what we get is this is equal to sine of n root over of n and that is why we wrote it earlier with a mod of n and this is equal to a v F and 2 h cross e B square root divided by m okay. So, this is the thing and you can write it as h cross omega tilde sine of n this is what we have written and a mod of n okay. So, this is the Landau levels and so on. And then there are a few comments that can be made here. So, basically the lowest Landau level is of course, n equal to 0 and this lowest Landau level is somewhat special and because it receives contribution from only one sub lattice say the A sub lattice for the Dirac point at K and from the other sub lattice that is B sub lattice at the other Dirac point that is K prime.

$$\frac{1}{2m}\left(\frac{E^2}{\hbar^2 v_F^2} - e\hbar B\right)\psi_B^{\vec{K}} = \left(\frac{\tilde{p}_x^2 + \tilde{p}_y^2}{2m}\right)\psi_B^{\vec{K}} = \left(\frac{1}{2}\tilde{K}(y-y_0)^2 + \frac{\tilde{p}_y^2}{2m}\right)\varphi_B^{\vec{K}}.$$

$$\tilde{K} = \frac{e^2B^2}{m}, \quad y_0 = \frac{p_x}{eB} = \frac{\hbar k_x}{eB}.$$

$$\frac{1}{m}\frac{E^2}{\hbar^2 v_F^2} = 2\left(n+\frac{1}{2}\right)h\,\omega_B + \hbar\omega_B = \frac{2n\hbar\omega_B}{m}.$$

$$E = \mathrm{sgn}(n)\sqrt{n}\,v_F\sqrt{\frac{2\hbar eB}{m}}.$$

$$\boxed{E = \hbar\tilde{\omega}\,\mathrm{sgn}(n)\sqrt{|n|}.}$$

Whereas, all the other Landau levels that is n naught equal to 0 Landau levels they receive contributions from both the sub lattices and that is why the n equal to 0 is actually special here whereas, in the 2D electron gas there is no nothing special about any of the Landau levels ok. So, let me write the wave functions at the K and K prime points. So, it is a psi n k and k this is at a k point. So, this is equal to some a n divided by some root over L exponential minus i k x. So, let us call it as a L x and now then you have a 0 0 sin of n minus i and this is a sin n minus 1 k and it is psi n k.

So, just from the one sub lattice here for the other one it is equal to k and k prime this is equal to some a n by root over L x exponential minus i k x x and again you have a psi n k S g n minus i psi n minus 1 k and a 0 0 that is a structure of that and these a n is equal to 1 for n equal to 0 and this is equal to 1 by root 2 for n not equal to 0 okay. These are these things and just to make sure the definition of this. So, S g n of n equal to 0 for n equal to 0 and it is actually the sin. So, n by mod n for n not equal to 0 okay and each of these psi's well we are using the same psi.

Wave functions

$$\underline{\bar{\Psi}}^{\vec{K}}_{n,k} = \frac{A_n}{\sqrt{L_x}} e^{-ik_x x} \begin{pmatrix} 0 \\ 0 \\ \mathrm{sgn}\,(-i)\, \psi_{|n|-1,k} \\ \psi_{|n|,k} \end{pmatrix}$$

$$\underline{\bar{\Psi}}^{\vec{K}'}_{n,k} = \frac{A_n}{\sqrt{L_x}} e^{-ik_x x} \begin{pmatrix} \psi_{|n|,k} \\ \mathrm{sgn}(n)(-i)\, \psi_{|n|-1,k} \\ 0 \\ 0 \end{pmatrix}$$

$$A_n = 1 \quad \text{for} \quad n = 0$$
$$\quad = \frac{1}{\sqrt{2}} \quad \text{for} \quad n \neq 0$$

$$\mathrm{sgn}(n) = 0 \quad \text{for} \quad n = 0$$
$$\quad = \frac{n}{|n|} \quad \text{for} \quad n \neq 0.$$

$$\psi_{n,k} = \exp\left[-\frac{1}{2}\frac{(y-y_0)^2}{l_B^2}\right] H_n\left(\frac{y-y_0}{l_B}\right)$$

So, let me write it as a big psi here. So, the small size are n k are these exponential minus which are the Gaussians it is y minus y 0 square divided by l b square and the H n y minus y 0 divided by L b that is a Hermite polynomial okay. So, these are the Landau levels and their structures for this graphene you have done an extensive study we first done it for one k point any of the k points rather we have not distinguished the Hamiltonian we have simply taken it as v F sigma dot k, but there is a sign that you know at one k point with respect to the other. So, if you take that into account the k and k prime as well as the a and b becomes a 4 by 4 problems slightly more complicated, but you can still solve it. And now will not go and do calculation using the Kubo formula, but the I will tell you the results that comes.

So, it is integer quantum Hall effect in graphene okay. So, the integer quantum Hall effect is given by the sigma x y that is a Hall conductivity is 4 n plus half e square over H okay that is the formula. So, this is called as a half integer quantum Hall effect. So, this can be written as nu e square over H okay, where nu has values which are minus 10 minus 6 minus 2 plus 2 6 10 etcetera okay. So, these are the quantization of the plateaus and these plateaus are quantized with these nu's. So, n is actually an integer and these are positive and negative for the electron and holes and these factor of 4 that you see here at the front it takes into account the degeneracy the pseudo spin degeneracy which is A and B sub lattices as well as the degeneracy corresponding to the 2 valleys that is the 2 direct points okay.

So, your nu is equal to nothing but plus minus 4 n plus half and even though we call it as a integer quantum Hall effect it is actually a half integer quantum Hall effect and because of this half factor that is there if you note that your nu equal to minus 2 and nu equal to 2 okay. So, a minus 2 and 2 so, these 2 values they correspond to empty and filled levels with eigenvalue E n equal to 0 that is the lowest Landau level. So, E n equal to 0. So, these for n equal to 0 Landau level n equal to 0 Landau level for n equal to 0 Landau level the valid degeneracy. So, valid degeneracy is called as a iso spin at times and the sub lattice is called as a pseudo spin is same as same as the sub lattice degeneracy and this is stated in a slightly different fashion when I say that the n equal to 0 Landau level actually receives contribution from only one sub lattice and not from both, okay.

So, these Landau quantization for these non relativistic dispersion that you have learnt earlier it produces equidistant Landau levels whereas, here for the relativistic case there are non equidistant okay. And as opposed to the usual degeneracies or rather these usual n values which take usually positive numbers or including 0 for the harmonic oscillator which we have seen again from the 2 DEG here they are entitled to take. So, these take you know a negative values as well okay. So, these new values will dictate the values of n. So, n actually corresponds to the Landau level index the same as the ones that we have derived and nu is equal to plus minus this is the quantization of the plateaus.

So, okay. These correspond to emptying and filling of the LL with n equal to 0 with n equal to 0 which means en equal to 0 okay. So, this is 1 and this is 2 okay. So, this is pretty much the quantum Hall effect in graphene and it has been seen experimentally and all these integers were confirmed from the plateaus. Let me show you the plot.

Integer quantum Hall effect in graphene.

$$\sigma_{xy} = \pm 4 \left(n+\frac{1}{2}\right)\frac{e^2}{h} = \nu \frac{e^2}{h}$$

$\nu = \ldots\ -10,\ -6,\ -2,\ 2,\ 6,\ 10\ \ldots$

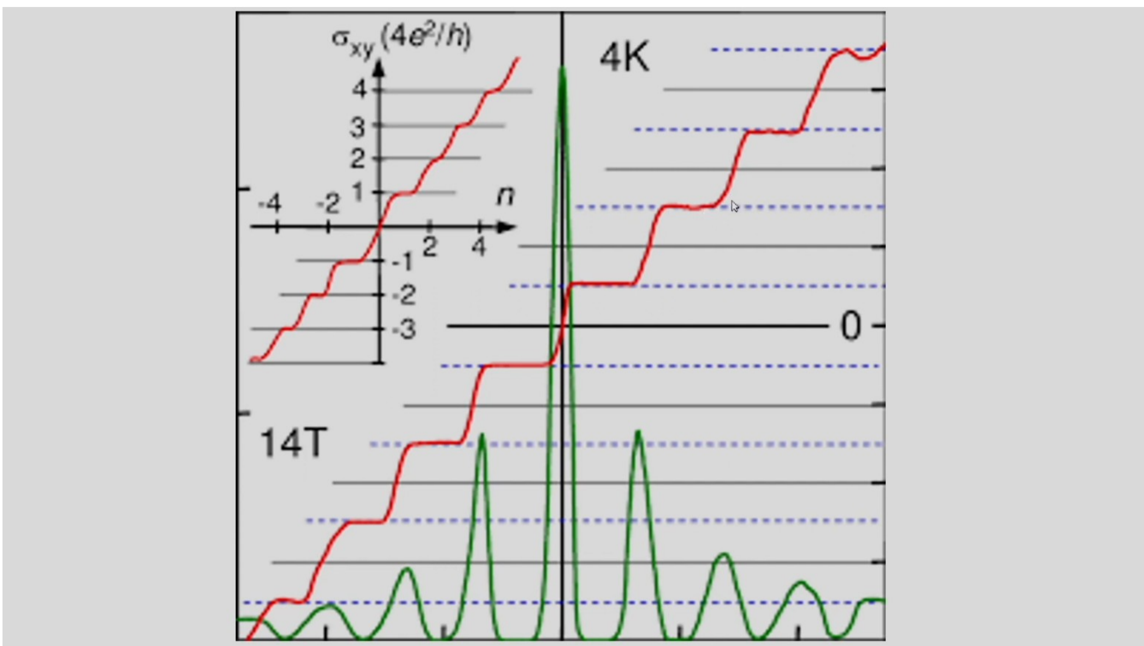
n : integer

$\boxed{\nu = \pm 4\left(n+\frac{1}{2}\right)}$ $\longrightarrow$ Quantization of the plateaus.

1) $\nu = -2$ and $\nu = 2$. $\Rightarrow$ Correspond to emptying and filling of the LL with $n=0$, $E_n = 0$.

2). For $n = 0$ Landau level, the valley degeneracy is same as the sublattice degeneracy.

So, this is the plot that one gets for the quantum Hall effect. So, these red plots are the Hall effect and the green plots are the magneto resistance plots or rather magneto conductance plots and the plateaus are not at these line black lines, but they are shifted by this half integer and seen at these blue lines okay. And they are not only at 0, but they these blue lines also go beyond or rather below 0 and these are like the only Hall effect in in the left is the Hall effect this red line is being shown and you can clearly see the plateau structure and scale is not set by e square by h as we have seen earlier it is set by 4 e square by h okay. And so, this is the quantum Hall effect in graphene and if you want to see the difference between non relativistic. So, this is non relativistic 2DEG and this is a relativistic that is graphene in chiral massless fermions that is graphene where you see the density of states is plotted as a function of energy and you see that the density of state is bunched here and here. So, they are almost they are overlapping in this particular region whereas, there is a significant gap here in this energy okay.



So, this distinguishes the quantum Hall effect seen in graphene and that in 2D electron gas. Thank you.