Quantum Hall Effect

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Week-04

Lec 10: Tight binding model, Hofstadter Butterfly

So, let me rewind some of the discussion that we have already done. We know that the Hall plateaus are quantized and we have taken a 2D electron gas and have put a perpendicular magnetic field and when you calculate the Hall conductivity the conductivities or the resistivity either of them is found to be quantized at values that are integral multiples of h over e square or e square over h. Let us just talk about the conductivity for the moment. So, it is integral multiple of e square over h and these integers are very robust they do not go away with disorder and the system as I said does not possess translational symmetry or a time reversal symmetry still these plateaus continue to exist. They are resilient to these external conditions that we have spoken about. This is of course, a fact we need to understand deeper that is why do they survive even though the symmetry protections are not there and in fact, it is because the symmetry protections are not there the plateaus actually arise and some of the physical reasons we have seen that there are disorder that plays a role in broadening the Landau bands and as you know the Landau levels move or as a magnetic field is increased the Landau bands kind of successively cross the Fermi level and that is where they spend some time while crossing the Fermi level and one sees that there is a plateau that arises.

Now, we have said that these quantization of the plateaus or in the Hall conductivity are related to a topological invariant, but we have not elaborated it as yet we are going to do that, but in order to understand that statement quite you know convincingly what we do is that we change the system and we go to a system which has translational invariance such as a crystal lattice and we want to see this in a crystal lattice what happens. Suppose we put a simple lattice such as you know two dimensional square lattice in a magnetic field then what happens. Now, it is you cannot take a 1D lattice for you know discussing magnetic field because we need to understand that there should be an area for the flux to penetrate in order to you know for that magnetic field to have any effect a 1D lattice of course, does not enclose such an area and that is why we have to go to 2D and the simplest of the 2D lattice is a 2D square lattice a uniform square lattice and that is what let us see that the as I said the idea is to understand the quantization of the Hall conductivity in terms of a topological invariant this will yield sort of mathematical way to arrive at that topological invariant that we just talked about and this will make the calculations more clear and otherwise in a 2D electron gas a dirty system and with no translational invariance it is difficult to or rather it is not possible to do a calculation we

just have to say that the topological invariant is the coefficient that sits in front of the Hall conductivity and remember Hall conductivity has been found using Kubo formula the formula that we have derived. Now in order to talk about these periodic potential let us understand the periodic potential I have already shown that an electron is actually moving in a periodic lattice.

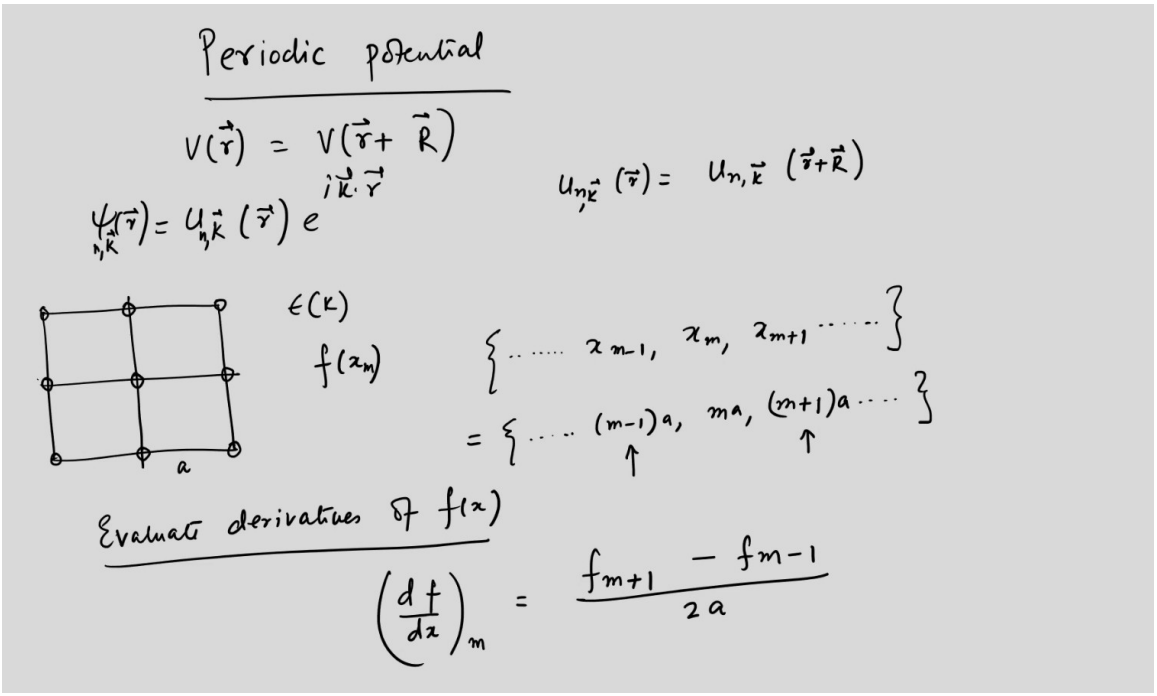
So, it experiences a potential which is periodic in nature that is it is of the form V of r equal to V of r plus r let us write a vector on everything where r is the periodicity. So, capital R is the distance between two successive ions and they are offering a potential to the to the electron that is you know passing through it and we have drawn a 1D lattice, but of course this can be done in any dimension alright. The solution of this problem that is if you want to solve this one particle problem in presence of a periodic potential this is called as Cronig-Penny model solving the Schrodinger equation in presence of such a potential gives you the energy quantization for a you know a problem which is a continuum problem that is when the electrons have a k square type dispersion a parabolic dispersion h cross square k square over 2m. But in a lattice the wave functions are given by a psi of r is equal to uk of r and exponential i k dot r and this is the wave function. So, this is a function of course k and if you want to put a band index because there are multiple bands that are going to form and so we can put a n here and we can put a n here where n denotes the band.

So, it is the first band or a second band and so on. So, it is basically the band index and these u n k of r it captures the periodicity of this potential that we have talked about. So, k is actually a vector. So, you can write down it as a vector and so this is like r plus r and this goes by the name Bloch's theorem. But this is only half the problem we need to also know about the energy eigenvalues for these periodic potential and we can we can know it in a number of ways.

And one of the things that are done in courses on solid state physics or they are taught in that course on solid state physics basic solid state physics where you learn a tight binding approximation okay. So, tight binding approximation means that the electronic wave functions are tightly bound to the ionic core and it has very little overlap with the neighboring ions, but enough just to make the electron move from one position of one ionic core to the next alight. So, let me give you a very simple derivation of the this dispersion the electronic dispersion and in this particular case we just talk about a square lattice. So, a square lattice is a simplest thing. So, we draw a square and let us just draw this and every junction of this vertical and the horizontal lines there is a lattice side that are present and so this is that lattice say the lattice constant is a and we are going to write down the tight binding dispersion which is epsilon as a function of k of course, this is a real space structure we are going to derive the dispersion in the momentum space which is a function of k.

And in order to do that let me sort of resort to technique or formalism which is like discretizing the Hamiltonian into a grid which is consisting of these lattice points okay. You can follow the derivation of this tight binding method for a general lattice say for example, a three dimensional simple cubic lattice from Ashcroft okay. Let me show another way of doing it let us say we have a function on a 1D grid okay. So, there is a function f of x that resides on this grid and these 1D grid is I am just talking about a 1D function, but it can be generalized to more than one dimension. There are these sides which are like so these are is no minus sign there is just a dot dot and then it is x m minus 1 x m and x m plus 1 and so on.

So, this is the grid points and this function is defined on those grid points. So, I can write it as x m where m is any of the sides that you see written out here okay. So, what we can do is that since we are talking about a lattice your x m and x m minus 1 or x m and x m plus 1 they are separated by a lattice constant a. What I mean by that is that you can write down this as you know. So, this is like a m minus 1 a m a and m plus 1 and so on okay.



This really denote the coordinates in a one dimensional array. So, what I want to do is that I want to evaluate derivatives. And you may ask this question that why do we want to evaluate derivatives and the simple answer is that the second derivative which is d 2 f

d x square in this particular case actually denotes the kinetic energy because your kinetic energy is written as minus h square h bar square over 2 m, m being the particle mass the mass of the particle that moves on this 1 d lattice and into d 2 d x 2. So, we want to discretize a d 2 d x 2 on a lattice which gives you more you know intuitive way of connecting a continuum system and to that of a lattice okay. So, this you know from your elementary knowledge of derivatives.

So, it is a d f d x since we are talking about 1 d I can take the liberty of writing the full derivative otherwise it could be you know if you are talking about a two dimensional system. So, then it will be like a del f del x and so on. So, this is equal to f m plus 1 minus f of m minus 1. So, we are taking the two point difference formula we are not taking f m plus 1 and f m or f m minus 1 and f m, but we are taking this site and this site and then dividing it by 2 a and this is the formula for the derivatives I mean the first derivative so to say. Let me write down the second derivative the second derivative which is d 2 f d x 2 of course, this is a m and this is written as f m plus 1 minus twice of f m this is called as the three point formula and so on and then f m minus 1 divided by a square and these f's are the values of the function and a is of course, the lattice constant or the distance between two successive points.

So, we want to calculate the matrix element of the operator either these d 2 f d x 2 or d f d x let me show for both between two sites m and n. So, it is a m d d x of n m n actually denote sites. So, this is like m plus 1 n minus m minus 1 n ok. So, where f of m we are using that f of x m. So, this is the first derivative or this the matrix elements of the operator d d x and this is nothing, but using the relations of the orthogonality relations this is nothing, but delta n m plus 1 otherwise it will go to 0 if n is not equal to m plus 1 because these are kets.

So, we are taking the basis to be the side basis m and taking the expectation value of these operator d d x and d 2 d x 2 in this basis which is given by this. So, let me write that this is the basis of the problem which is the side basis minus delta n m minus 1 divided by 2 a okay. So, this is the d d x and similarly we can do it for a double derivative which is d 2 d x 2 which is this and this is equal to again we can write it down. So, delta n m plus 1 you can check that minus delta n m plus delta n m minus 1 and a square. So, just to tell you that delta n m this is called as a Kronecker delta okay.

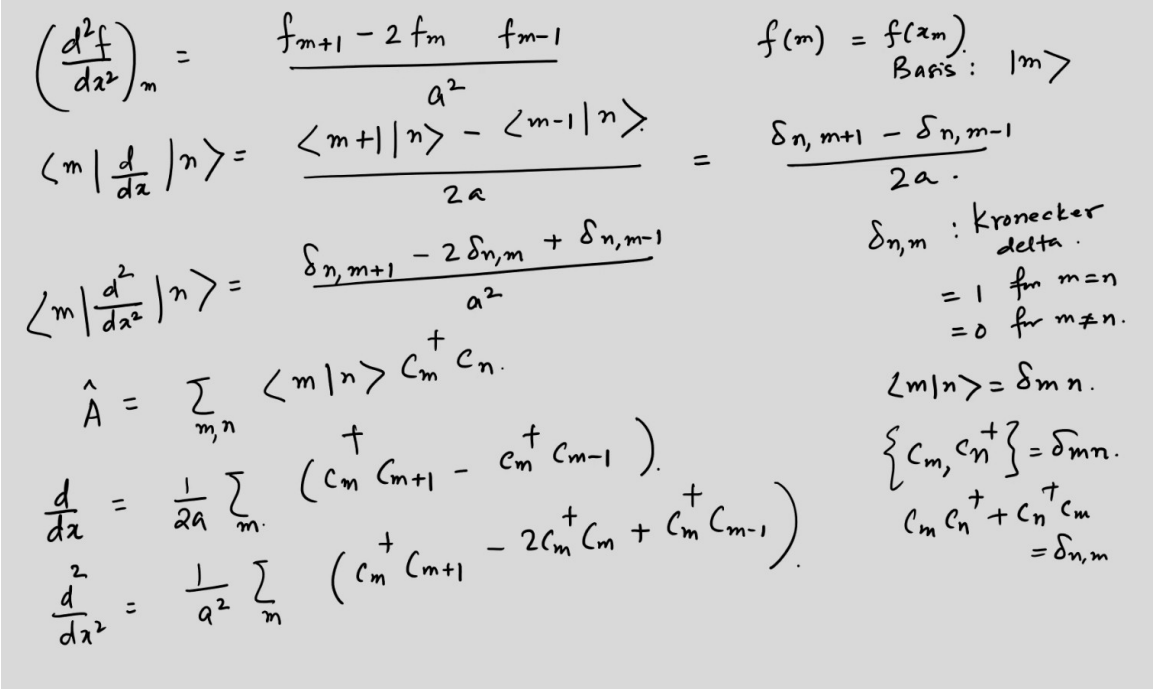
So, what is meant by Kronecker delta that is the definition is. So, if this is equal to 1 for m equal to n it is equal to 0 for m not equal to n, that is when m and n becomes same then this it gives you a value 1 otherwise it gives you a value 0 okay. So, this is what it

means and where it comes from is from the orthogonality relation like this which is delta m n. So, these have matrix elements only between these states which are given by. So, single derivative only has matrix elements between the neighboring sides either the left neighbor which is this is m plus 1 is the right neighbor and m minus 1 is a left neighbor.

And similarly this has the d 2 d x 2 has matrix elements in between these 3 states that is n equal to m n equal to m plus 1 n equal to m minus 1. And you can understand that if I want to write down the double derivative in 2 dimensions that is if I want to write d x 2 plus a d 2 d y 2 and then I have to introduce another variable say m n for x direction I mean m will denote for this x direction the coordinates in the x direction and n will denote coordinates in the y direction and then you have to have m prime and n prime in order to take the these matrix elements all right. Let me write down a general operator a and which has a form which is like m n and this is like m n and C m dagger C n ok. I am introducing this fermionic operators I mean this could be fermionic this could be bosonic depending upon what is the context of the problem. Let us say we are talking about electrons because that is under our discussion now we are focusing on electronic transport.

So, these are electronic operators and they have you know the anti commutation relations between them. So, which are given by a C m and a C n dagger this is equal to delta m n and so on. So, this means that this anti commutation relation means that there is a plus sign between the 2 that is C m C n dagger plus C n dagger C m this is equal to delta n m or m n does not matter. This tells you that that we are really talking about the second quantized operators to describe the electron. So, this could be a Hamiltonian I have just written any operator it could be the Hamiltonian of the problem.

So, if you use those definitions that we have just written above. So, then this is equal to 1 1 over 2 a which can come out of the integral and this is equal to like C m C m plus 1 and minus C m dagger. So, this is C m dagger C m plus 1 and this is a C m dagger C m minus 1 and so on okay. So, this is the d dx operator and similarly the d 2 d x 2 operator can be written in a similar fashion which has a 1 over a square in the denominator and you have a a and then you have a C m dagger C m plus 1 minus 2 C m dagger C m plus C m dagger C m minus 1 and I am just following these definitions that we have said. So, these we are discretizing the space.



So, that from a continuum system we are going into a lattice. So, we are using this discretization scheme in order to understand that how a kinetic energy which was written as simply as you know minus h square by 2 m and d 2 dx 2 how that can be written in a lattice. So, that is the idea behind this and we are sort of proceeding in that direction okay. Just like what we wrote so, h equal to p square over 2 m and in 3 dimension this is you know it is equal to minus h square over 2 m and a d 2 dx 2 plus a d 2 dy 2 plus a d 2 dz 2 okay. That is the operator and if you have no potential that is the only term that is there of course, in presence of a periodic potential you will have a periodic term which is V of r.

So, right now we are not talking about this V of r we are a focus on this now all right okay. So, of course, here what comes is that so, you have this h cross square over 2 m and then we have this n and let me just write the only in one dimension. So, this this and then n because that will give me the energy that is the expectation value. So, I am taking it between the same states and this will give me equal to say a t nm which is t nm is the hopping amplitude from site n to site m okay. So, that is your t nm if this is equal to so, you make a an approximation that this is equal to t if n equal to m plus a delta I mean a x cap where x cap is the unit vector in the x direction and as I said that n and m are just the site indices.

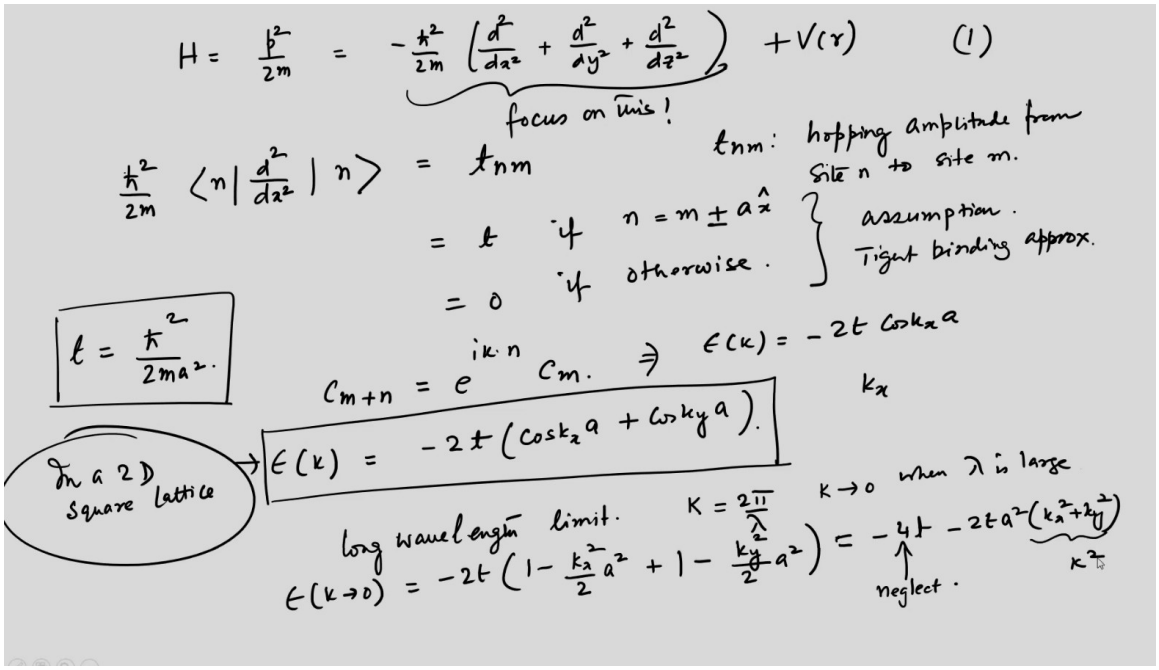
So, if n happens to be m plus 1 unit that is 1 lattice spacing either in the plus direction or in the minus direction and so on. So, this is equal to a plus and minus in 1 d or it is this thing if otherwise. So, you make this approximation or let us say this is the assumption and which is also the called as the tight binding approximation. I remind you that I have said that the approximation is that the electronic wave functions are strongly bound to the sites and they have very little overlap with the neighboring sites, but just enough to give you a hopping from or a jump of the particle from one site to its neighboring site okay. So, this is of course, your T is equal to nothing, but equal to in our language it is like H over 2 m a square and so on.

And now, in addition to this if you consider these C m and C n operators we can do a Fourier transform of this operators is just like the restatement of the Bloch's theorem for the operators and in that case your C m plus n is equal to exponential i k dot n and let me write it in since I am writing in one dimension let us not you know write with a vector sign because it is a one dimensional problem. So, this is equal to C m and if that is true then of course, your epsilon k or the energy that comes from for this Hamiltonian. So, epsilon k is nothing, but the energy or the eigenvalue of this Hamiltonian that is written in equation 1, but without the V of r there is no V of r there and this is equal to a minus 2 t including the minus sign now it is equal to cosine k x a plus a cosine k y a ok. In the sense that we are only dealing with the kinetic energy term and so, these k x etcetera these are two dimensional. So, I have now written it in two dimension in a 2D square lattice in one dimension simply it gives you a minus 2 t cosine k x a I just to go went one step ahead and in a square lattice with lattice constant a I write this as this is the energy dispersion.

So, this is called as a tight binding energy dispersion. So, this was a missing link in the problem where you are talking about the electrons present in a periodic potential. So, this is the kinetic energy this is how the kinetic energy behaves and if you look at it carefully your the long wavelength limit of this dispersion exactly looks like the continuum problem. So, if you take a long wavelength limit which means that your k actually can be written as 2 pi over lambda and which means lambda is large. So, k goes to 0 when lambda is large.

So, that is the long wavelength okay. So, if k goes to 0 I can do a. So, k going to 0 limit of this is minus 2 t into 1 minus k x square over 2 this is the cosine function that that is how the cos behaves as you take the k x going to 0 limit k x of course, a square and a plus a 1 minus k y square by 2 a square and so on. So, this I take away this. So, this is equal to 4 t minus 4 t 4 t because there is a 1 and 1 that will make 2. So, 2 and 2 4 t and then there is a 2 t and a square and the k x square plus a k y square if I drop this term neglect what I mean is that it is a constant term which can be neglected.

So, this really looks like the k square dispersion which you are most familiar with it is a free particle dispersion in k space which is h cross square k square over 2 m, okay.



So, the dispersion of electrons in 2 dimension in a 2 dimension square lattice is given as this minus 2 T cosine k x a plus a cosine k y a. Now of course, the system has translational invariance we will come back to this figure you may be seeing this this is one of the important discussions that will have to be done and this called as the Hofstadter butterfly, but I will come back to this alright. So, now of course, one of the chief ingredients of our discussion is the magnetic field. So, we will let us you know do a square lattice in a magnetic field okay.

And we write down again the Hamiltonian. So, it is in a square lattice. So, we have the Hamiltonian the tight binding Hamiltonian. Now we are not writing m n m was written because it was a single this thing let me write it as i and j which means the same thing. And when I use this symbol i j with an angular bracket it means that they are nearest neighbor.

So, j is a nearest neighbor of i. So, this is equal to in principle you can have t i j to be inside which means that it can be different the hopping amplitudes can be different for each of the links in a square lattice I have already shown you a square lattice. So, this is a C C j dagger C i and now a magnetic field has to be included. And how do you include magnetic field you have seen that in a continuum system it is included as e plus p plus e A. So, it enters through the momentum of the particle. So, the magnetic field enters

through the vector potential and the vector potential actually renormalizes the value of the momentum it makes the mechanical momentum to be the canonical momentum which is nothing, but p plus e A.

But in a lattice it enters just like this that t the hopping amplitude it is a t into exponential some i and some phi phi some phase and this phase contains the information about the magnetic field through the vector potential. So, this phi is nothing, but it is related to the line integral of the vector potential and if you use the Stokes theorem which says that the line integral of a vector potential is actually or rather any vector is can be written as curl of the vector and over this d s. Now this is of course, a closed integral and so this s is the surface of the closed contour that you have considered. So, this curl is nothing, but B.

So, this becomes a B dot d s. So, this phase that you see involves the flux that threads through the lattice. So, it the magnetic field has a flux which is given by the b into the area and that whatever flux threads sample or the system is given by this that appears the phase. So, there is a Hermitian conjugate that is very important we have not written it earlier, but because of the Hermitian conjugate the Hamiltonian becomes Hermitian and it gives real eigenvalues. If you do not have that then you have problems. Now in presence of the magnetic field we do not have translational invariance and the reason that we lose translational invariance is because the hopping now becomes a function of i j that is these sites because as the electron hops it picks up a phase and then it picks up another phase.

I mean that phase keeps growing as you keep as the electron or as the particle keeps hopping from one side to another. Then of course the Hamiltonian which only comprises of the kinetic energy or the hopping term then is not same from going from one side to another and if it is not same then you cannot Fourier transform and write it epsilon as a function of k. But luckily there is something else that happens we would be able to formulate a magnetic Brillouin zone or magnetic unit cell which of course does not have the periodicity, but that unit cell is repeated which means that what I am trying to say is the following exponential i phi is a phase right. So, phi is a phase which is just like an angle.

Now if the phi changes from phi to phi plus 2 pi. So, if the angle changes to 2 pi then of course the phase does not change at all because exponential i phi plus 2 pi is equal to same as exponential i phi because exponential 2 pi i equal to 1 which is cos 2 pi plus i sine 2 pi. So, because of this even if the electron hoppings are different, but it is only different over certain dimension and we can take that as a unit cell. See previously the unit cell was just comprising of one atom which we have you know shown here. So, here I mean one lattice point you can frame a unit cell there and you can translate it

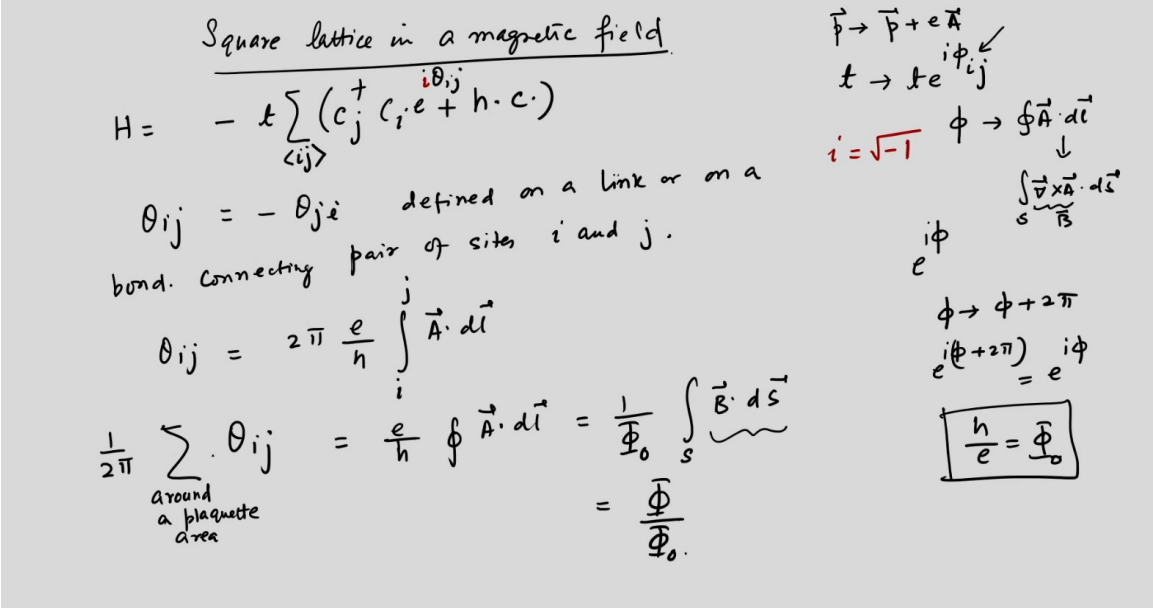
everywhere in order to generate the lattice if you have a bipartite lattice then of course you can you have to have two atoms per unit cell and so on. I am just doing a general discussion, but here you cannot generate the lattice because the Hamiltonian being it changes as you go from one side to another.

So, this picks up a phase which we let us write it as theta i j okay. So, as it goes from i to j it picks up a phase. So, there is a i here and the reason that I am writing it with a red ink is because this is equal to so, i in red is root over minus 1 whereas, the i in black is not a other side indices okay. You do not have to you know carry on this ambiguity for too long will sort of and if you if it bothers you please help yourself to write with m and n okay which is going to be fine alright. And this phase factor theta i j equal to minus of theta j i.

So, basically it is defined on a link or on a bond, okay. So, it connecting pair of sites i and j alright. So, what is then a theta i j? Theta i j equal to 2 pi e over h and from i to j now these are of course, sites and this vector potential dot dl okay. And so, A is the vector potential. So, now, what it can be done is that this 1 over 2 pi this 2 pi if I take it down and then I sum over all the theta i j's in going from these.

So, this as I go in one this thing so, I go like this then I go like this then I go like this and then I go like this. Why do I go in a particular plaquette this like called a plaquette? I go in a placket because in order to understand that what is the flux that thread this is related to these hoppings would be related to the flux that threads that particular these area which is shaded like this okay. So, in order to do that let us let us sum over all this around an around a placket area let us write it like this. And this is equal to nothing, but e over h and then integral I mean the closed integral a dot dl and so, this is h. So, e over h you know h over e is just to remind you this I have done several times this is the flux quantum let us call it a phi naught which has a value which I have told also a number of times okay.

So, this is equal to 1 over phi 0 because e over h is inverse of h over e and then there is a surface and then this is equal to b dot ds okay. So, as I just said that these phases are related to the flux. So, b dot ds is a flux that threads one placket of that square lattice which I have shown and phi 0 is of course, a flux quantum around this placket.



So, we know this. So, this is actually the phi. So, let me write this as phi over phi 0 and that is equal to 1 over 2 pi and then sum over all this theta ij and so on. So, suppose this phi that you have here this flux b dot ds. So, this phi which is let us call it as a 1 over 2 pi and then this theta ij over a plaquette I am just in shorthand I am just writing a placket that is around a plaquette. So, this becomes equal to or rather you know this phi over phi 0 it becomes a fraction like this. So, p by q is called as a rational fraction when p and q are co-prime integers and what I mean by co-prime integers is that basically they have no common quotient that is a one cannot be divided by the other.

So, just say 1 by 3, 2 by 3, 3 by 5 and so on 3 by 7 and so on. So, there is no common factor and that is why they are called co-prime and this is a rational fraction. So, if it is in this form because phi by phi 0 is a fraction and if this fraction happens to be a rational fraction in that case then the spectrum the spectrum that I have shown this is quite important splits into q sub bands. And so, basically somebody called D Hofstadter he studied this problem for the first time and that is why these quantization of the electronic dispersion which is minus 2 t cosine k x a plus cosine k y a which is which just saw. Now that further splits into these sub bands and it forms a fractal structure.

I will show you the picture, but you will have to you know calculate work out that and calculate this thing. And so, suppose now our job is to take a specific choice of magnetic field that is easy because we have several times said that we want to take a B which is B Z cap and then of course, we have one of the choices which is a Landau gauge. So, let us write a Landau gauge. So, this is equal to B X Y cap or you can take B Y X cap.

Now because of this let us see what happens. Let me draw this square lattice once again and you will see that just bear with me till I draw the lattice. These are the bonds along with which the electron hops or the links you can call them a links or bonds and so on so forth. Well, I think this should be enough for our discussion. So now, what happens is that you see the hopping in the x direction is unchanged because the gauge is in the y direction. So, the theta or the phase that the electron picks up will only be affecting the movement in the or hopping along the y direction.

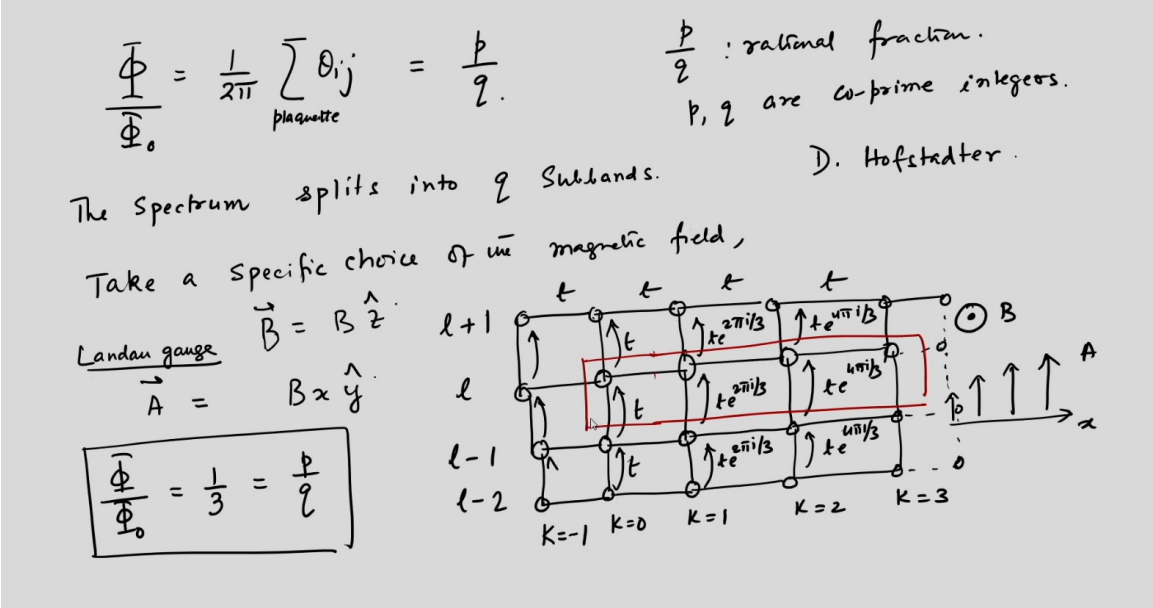
So, the x direction still remains as t t t and t. Now let me take for l equal to 1. So, these are different rungs and different legs legs and rungs. So, l stands for leg and this l minus 1th leg this l minus 2th leg this l plus 1th leg and so on. So, this is a k equal to minus 1 rung and k equal to 0 rung and k equal to 1 and k equal to 2 and k equal to 3 and so on. Now, this hopping are like t in the x direction in all the x direction is t because the the gauge is particularly taken in the y direction because it is B x y cap and because of that these hopping along these y directions they will pick up a phase this directions it will pick up a phase and they will pick up a phase depending upon this phi by phi 0.

In a particular case let us take phi by phi 0 a rational fraction a specific rational fraction one-third which is simple. So, then what will happen to these hopping? So, this hopping will be t into exponential 2 pi i by 3 t exponential 2 pi i by 3 t exponential 2 pi i by 3 and so on. So, this will be at k equal to 0 will be still t of course, because this is we are talking about. So, this will be like t exponential 4 pi i by 3 t exponential 4 pi i by 3 t exponential 4 pi i by 3 and so on and similarly there will be t exponential 4 pi i by 3 along these bonds. So, t exponential 4 pi i by 3 and t exponential 4 pi i by 3 and so on and all the horizontal hopping are of course t.

Now you see your B is in the z direction that is it is coming out from the board okay. So, you see the point dot of the arrow and the this thing is in the y direction of course, and but it is increasing in the x direction like this and so on. So, this is the A this is the x direction because it is a B into x and it is increasing in the y direction. So, this is not the direction this is x because as x increases B increases ok. So, we have particularly taken the p by q equal to one-third and have done this.

Now you see why I have drawn it till this and not drawn in beyond this that is not one more sequence of sites and the reason is the following that you see the next thing if suppose there is one here and here and here let me sort of show it by some the hopping would have been 6 pi i by 3 in that along the y direction and 6 pi i by 3 is nothing, but 2 pi i. So, that will be the same hopping that we have started with here and so on okay. I mean just check that whether everything has been written correctly, but when the hopping becomes 6 pi i by 3 it becomes equal to 2 pi i and which is same as t. So, we are able to identify a magnetic you know unit cell which I mean this is the magnetic unit cell

for this choice of flux which repeats. So, it starts with ok maybe I will sort of take this thing also because it starts with t in the vertical direction and then it sort of carries on.

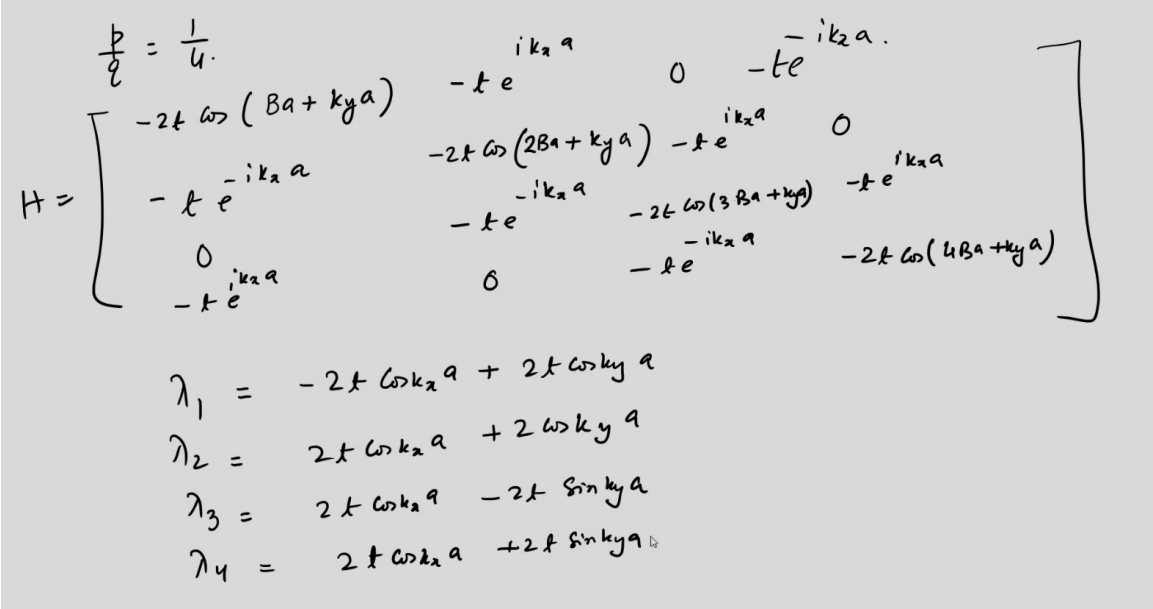


So, it is t in the vertical direction is t 2 pi i by 3 and then it is t 2 4 pi i by 3. So, possibly we will just remove this and check what they are in this particular. So, it is k equal to 1. So, it should have been minus 2 pi i by 3 and so on might have written it wrong, but check what the d and. So, this is the unit cell of this problem and now even if the system has lost translational invariance because of the presence of the magnetic field we still are able to calculate or rather take a unit cell and diagonalize that Hamiltonian okay.

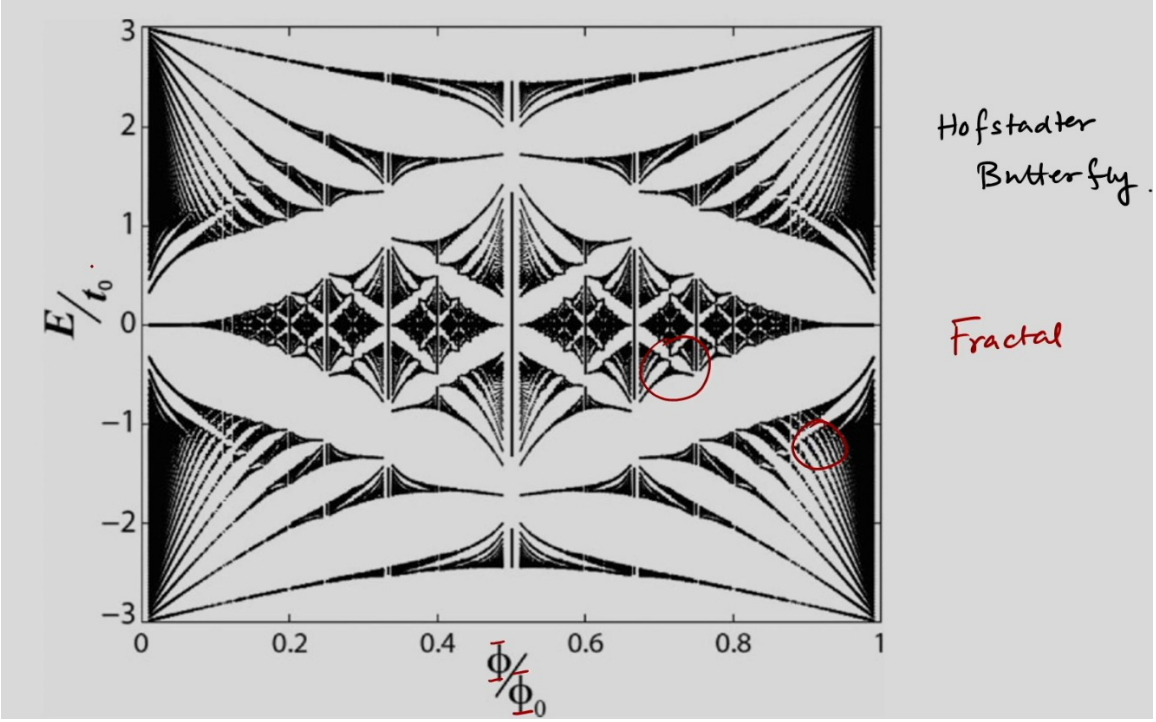
And one such exercise one can do in which I have taken a slightly different unit cell that is I have taken p by q equal to 4 that is 1 over 4. So, written down a Hamiltonian which is so, this there is a minus 2 t cosine of say b a plus k y a and so, this is including all of that. So, t exponential i k x a 0 t minus i t k x a and there is a t e to the power minus i k x a then there is a minus 2 t cosine this all minus signs cosine b a plus a 2 b a rather k y a and minus t exponential i k x a and a 0 and a 0 here. So, let me show you a specific example where we deviate from the earlier example and take another fraction another rational fraction for this phi over phi 0 which is of the form p by q and let me take it as 1 over 4 which is another rational fraction that we talk about. Now that of course, the unit cell the magnetic unit cell will come comprise of 4 terms or rather 4 lattice sides and hence we can write down a 4 by 4 matrix.

So, the matrix one can check that the matrix looks like this and it is a 4 by 4 because there are terms which are on-site terms and then there are terms which are across the

sides and so on. So, these are terms and then when you solve this one gets a 4 eigenvalues and let us write down the 4 eigenvalues let us write them as so it is minus 2 t cosine k x a plus 2 t cosine k y a lambda 2 equal to 2 t cosine k x a plus 2 t cosine k y a. So, this is k y a and lambda 3 equal to 2 t cosine k x a minus 2 t sine k y a and lambda 4 equal to 2 t cosine k x a plus 2 t sine k y a and so on. These are the 4 eigenvalues which you can solve using either Mathematica or MATLAB or Python and once when you do this and plot the energy. So, these are the energies of these magnetic unit cell and when you do that for a large number of values of the magnetic field you get a spectrum which looks like this.



This is called as the Hofstadter butterfly. This is actually fractal in nature because there is a self similar structure. So, if you read on fractals basically a small part of that a small part of that looks like the whole picture that you see here. So, whether you see it here or you see it here everywhere there is a structures a self repeating and this is called as a fractal and these fractal structures or these in the energy as a function of phi by phi 0 this is the same phi by phi 0 that we have talked about. So, these for different values of the flux. So, we have shown for one value of phi by phi 0 all these results if you do it as a function of phi over phi 0 then the E in terms of this t is or t naught is the scale of the problem and one sees a fractal nature.



And so, our job is not to really dwell on the Hofstadter butterfly, but this one of the things that arise for these energy spectrum in presence of a magnetic field where the tight binding dispersion itself splits up each of the bands split up if there are multiple bands for a given problem they split up into these self similar structures as a function of phi over phi 0 or p by q you see all those fractions were taken. It is another piece of information that the quantization of the Hall conductivity can be captured from these gaps in the spectrum through a formula called as a strata formula will not worry too much about that because we have other ways of calculating the Hall conductivity and so on. So, we will leave that and carry on the discussion will not do a sort of completely elaborate study of the spectrum in case of graphene, but we want to see how the situation evolves when you have graphene. So, we will talk about graphene and graphene in magnetic fields. Thank you.