Dynamics of Classical and Quantum Fields: An Introduction

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Electromagnetic Fields
Lecture - 09
The Relativistic Electromagnetic Field

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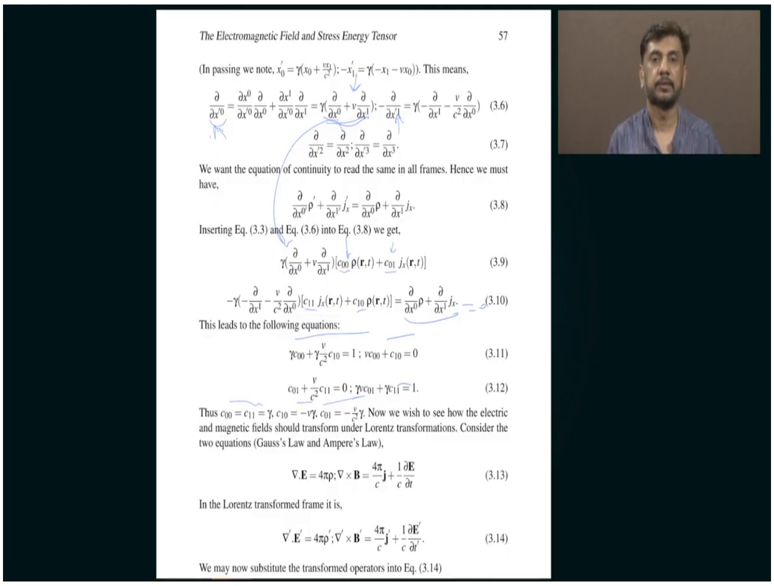

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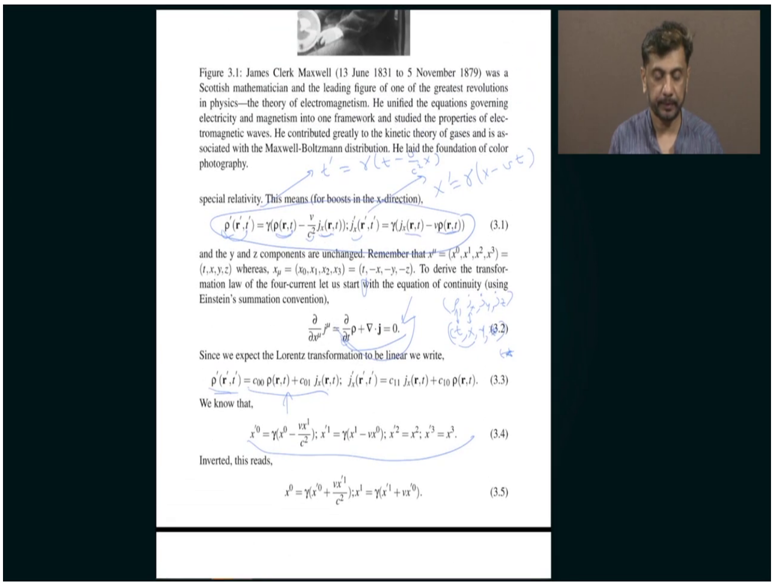

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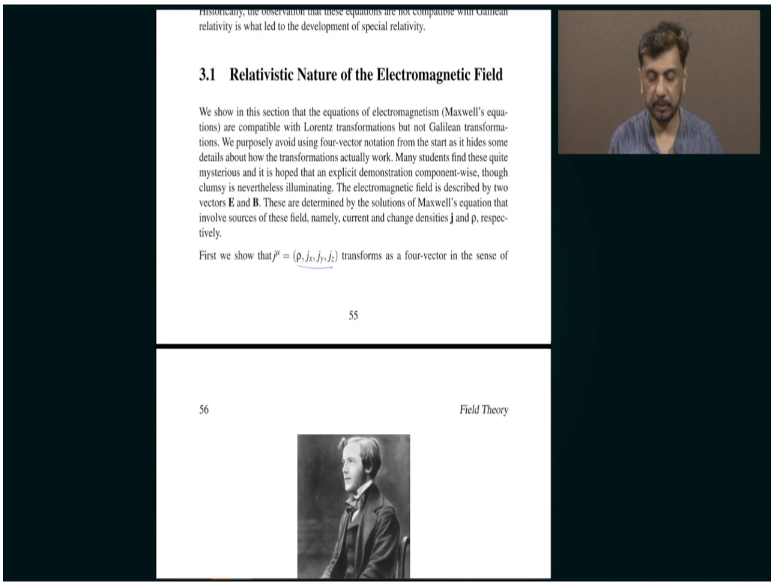

So in today's class we are going to continue our discussion of the Relativistic nature of Maxwell's equation. So, if you recall in the last class I had shown how the collection of density and components of current form a relativistic 4 vectors. So, if you remember I had I had pointed out that you can construct this object which has 4 components. Namely; the first components is the density the other three components are the xyz components of the current density.

So, you have particle density and current density. So, by demanding that the equation of continuity be the same in both the original reference frame and the Lorentz boosted reference frame. You will be able to conclude that this is possible only if the currents and densities transform in this way and then we just derived that this is nothing but c 0 is gamma which is you know the parameter that determines length contraction in relativity.

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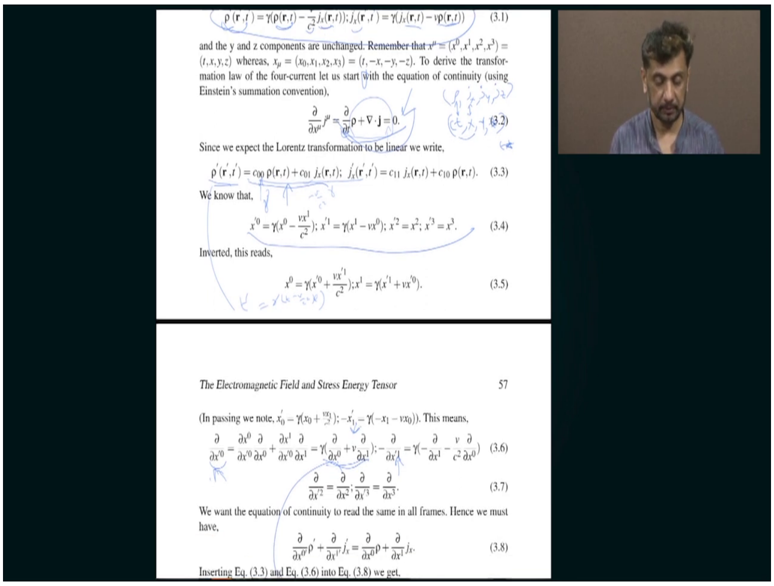

And this is going to be v by c squared minus v by c squared gamma. So, this is very closely reminiscent of you know t dash going to gamma into t minus v by c square into x. So, in other words t is analogous to rho and x is analogous to j x. So, in other words rho and j transform exactly the same way as t and x transform under Lorentz transformations ok. So, t and rather rho and j x transform the same way as t and j x transform under Lorentz transformation. So, now that is so, much for currents and densities.

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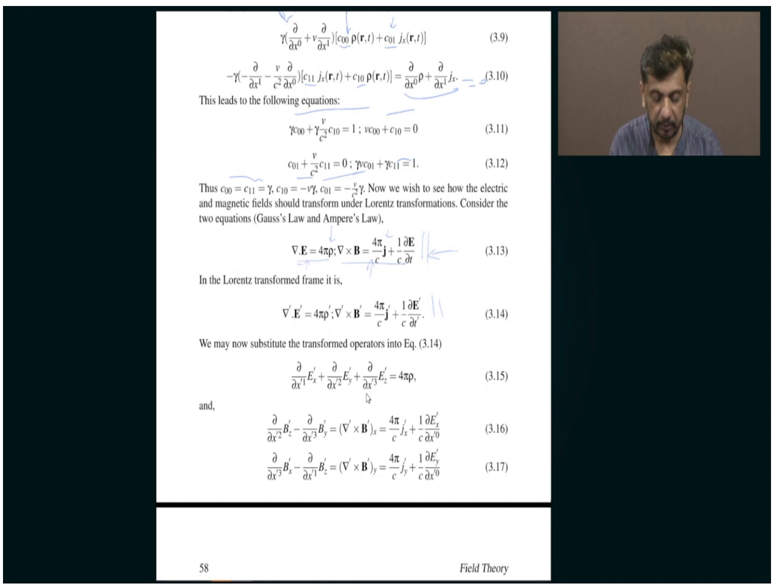

Now, I am going to go and prove to you that the rest of Maxwell's equations are also consistent with special relativity. So, in order to do this you look at the Maxwell's equation. So, if you must know that there are four Maxwell equations two of which involve sources the other two do not involve sources and they refer to the ones that do not involve sources are basically the ones that you know the Faraday's law and basically the other one which tells you the fact that magnetic field does not diverge ok.

So, the point is that the source Maxwell equations with sources are the Gauss law and Ampere's law. So, this is Gauss law the first one is Gauss law and the other is ampere law. Now I am remember that I have just proved to you that the sources namely rho and j transform as components of a 4 vector under Lorentz boosts. So, now, the question is given that information and given these two Maxwell's equation the question is a how do the electric and magnetic fields transform under Lorentz transformation.

So, I am going to postulate that these equations have the same form under Lorentz transformation. So, in other words this equation looks exactly the same when I replace the gradients with the corresponding gradients in the boosted frame and the electric field so on so forth. So, now you see I go ahead and explicitly write down the components and.

So, it is going to be a little lengthy because I am going to work it out component by component and when I do that you see that the first one is a scalar equation, but the second one is a vector equation. So, that the second one splits up into three separate equations for each of the components ok.

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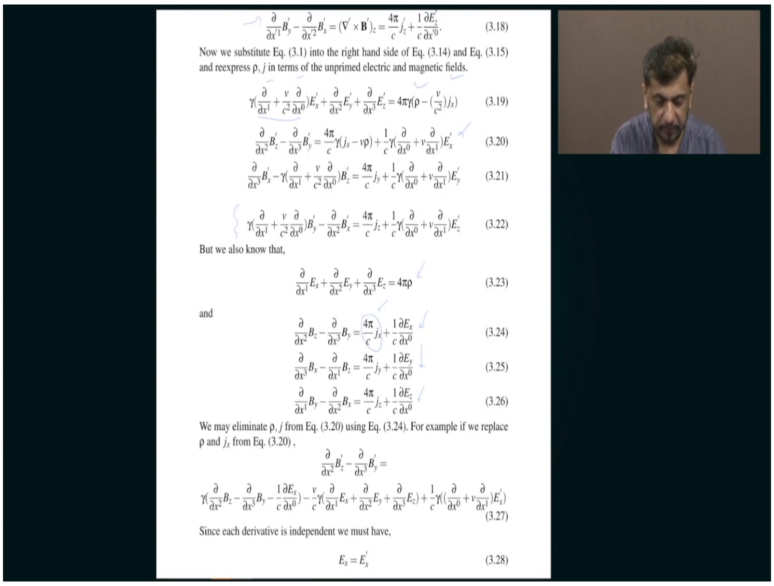

So, now keep in mind that j dash x is related to j x and rho. So, which is what I have written here. So, I have been able to write and then the gradients themselves can be written because I know how the positions and times change under Lorentz transformation. So that means, the gradients are also correspondingly they get linearly combined in this fashion under Lorentz transformation.

So, when I do all this I put them all together its going to start to look like this ok. So, so these three equations can be made to for example, this equation can be made to look like this the one involving the derivatives of B dash can be made to look like this, but then keep in mind that the original in the original reference frame these were the equations ok.

So, then you can eliminate rho and j from 3 2 by using 324. So, you see. So, I am going to eliminate rho and j from this equation by looking at by using this. So, 4 pi by cj x is given by the rest of it. So, wherever there is 4 pi by cj x I can replace it by the Bs themselves. So, then you stare at this then you compare the two sides and then you will immediately conclude. So, you see what I have done right.

So, basically what I have done is that I have taken the j x which is from here and then I have substituted the corresponding j x into this ok. So, then I get this. So, I have eliminated j x as it were. So, so remember that there is only this source right and similarly rho also I have eliminated that way ok by using this. So, I write 4 pi rho as d by dx of ex and so on so forth. So, that is what I have done here right.

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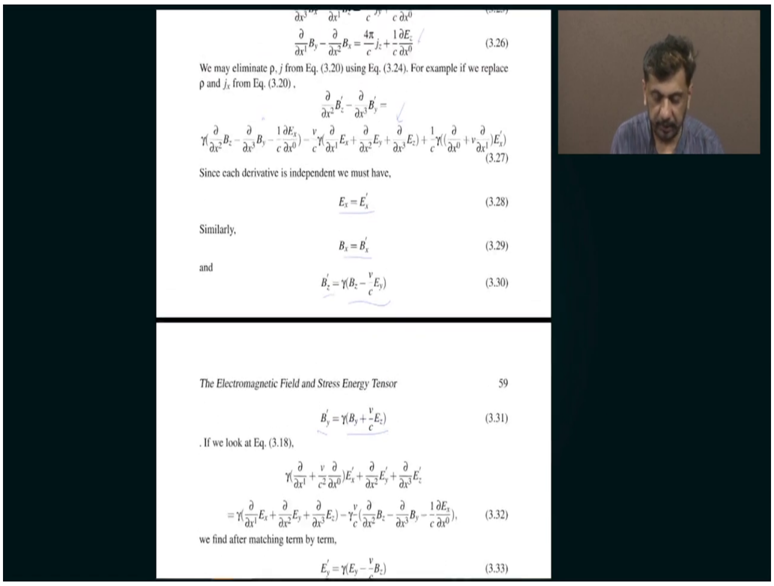

So, having done that you see you simply compare the two sides. So, then you will conclude that this is valid only when E x is same as E x dash B x is same as B x dash, but B z dash is given by this interesting formula if gamma into B z minus v by c into ey ok and then B dash y is similarly this ok. So, similarly if you look at 3.18 ok. So, which is your z component and then you again eliminate the sources and then you look at the different terms.

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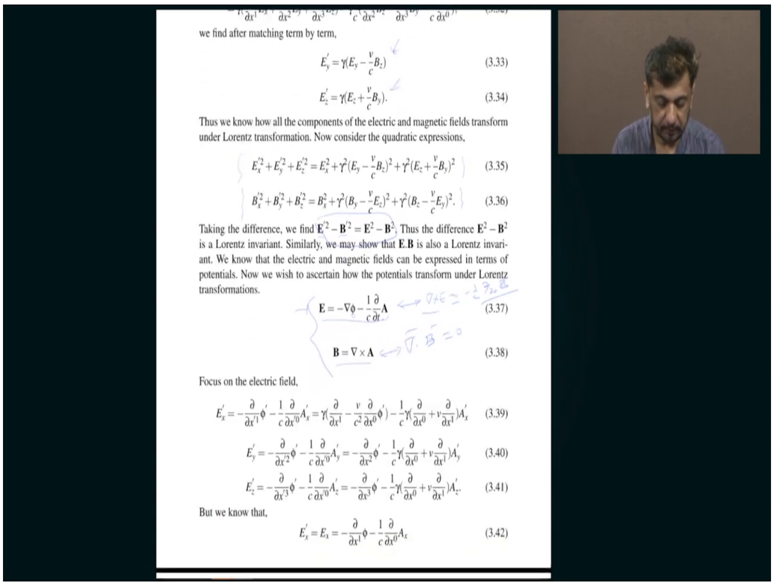

So, you will get the rest of the transformation. So, this will tell you all the how the B is transform B dash how does B dash transform B dash x B dash y B dash z. So, B dash x B dash z B dash y. So, we know how B transforms, but here from equation 3.20 you only know you know how all the Bs transform, but you know only how one of the x components of electric field transform only one of the components of the electric fields how they transform.

But if you want to know the rest you just look at any one of the other ones like 3.18 and they do the same thing eliminate sources then you will conclude that the other components transform in a very similar way. So, this is pretty much the whole story; that means, now we have successfully told you how the components of the currents and the density transform under Lorentz transformation. So, now, we have also successfully said that how the electric and magnetic fields transform under Lorentz.

So, now having done this we are now ready to make further interesting observations about the relativistic nature of the electromagnetic field. So, one of the important observations is that if you square the electric field you will see that in the primed reference frame and the unprimed reference frame they of course, are going to be related the square of the electric field in the primed reference frame is related in a complicated way to the electric fields and the magnetic fields in the unprimed reference frames.

But however, when you take the difference between the so, I have to also remind you that I am working in CGS units. So, that in CGS units the electric and magnetic fields have the same units same dimensions. So, that is why I am able to do this ok. So, if I subtract these two that will be the square of the electric field in the primed reference frame minus the square of the magnetic field in the primed reference frame.

You will see that the complicated dependence on the right hand side especially the dependence on the boost factors namely the relative velocity between the reference frames disappears and you have this very beautiful result which says that E squared B squared is independent of which reference frame you are looking at. So, in other words is a Lorentz invariant.

So, similarly you can also show that E dot B is a Lorentz invariant ok. So, if E dot B has a certain value in a certain reference frame it has the same value when you move to a different reference frame which is moving relative to this frame. So, this sort of completes the proof and description of the relativistic nature of the electromagnetic field.

So, remember that it is we have done a thorough job because we have also included sources. So, it is the most general description of the electromagnetic field. But now we can go one step further and introduce certain quantities which are called potentials. So, see the way electric and magnetic field transforms they are not exactly similar to 4 vectors, but they somewhat resemble 4 vectors, but not quite.

However you see the density and currents are exactly like 4 vectors. So, if you go back and see this rho dash transforms exactly the way t dash does. So, remember that t dash is gamma t minus v by c squared x. So, similarly rho dash is gamma rho minus v by c square j x. So, its a as if rho is interchangeable with t and rho dash is interchangeable with t dash and j x is interchangeable with x.

So, the bottom line is that you see that rho and j x j y j z form a 4 vector, but; however, E x E y E z of course, there is no 4 fourth component at all in the case of electric and magnetic field there six spatial components namely E x E y E z and B x B y B z there is no analogous time component. So, it is hardly surprising that electric and magnetic fields do not transform as 4 vectors.

However, we want to write electric and magnetic fields in such a way in terms of other things which we can identify as 4 vectors. So, we want to write the electric and magnetic field in terms of quantities which finally, transform like 4 vectors. So, to do that we introduce what are called electric basically we introduce what are called scalar potentials and vector potentials. So, you will see that the scalar potential transforms as the time component of a 4 vector the vector potential transforms as the space component of the 4 vector.

So, that is the reason why we introduce potentials. So, it is to make the analogy with special relativity as close to the special coordinates as possible. So, the electric field can always be written like this because see I choose to introduce quantities called phi and A which obey this. So, you will see that this sort of identification immediately solves the source free Maxwell's equation.

So, namely this no divergence of magnetic field that is the lack of magnetic monopoles is obeyed by this correspondence and the Faraday's law which basically tells you that curl of E is minus 1 by c d by dt of B is automatically. So, if you take curl on both sides you get back this Faraday's law. So, these are the source free Maxwell equations that are automatically obeyed by this choice.

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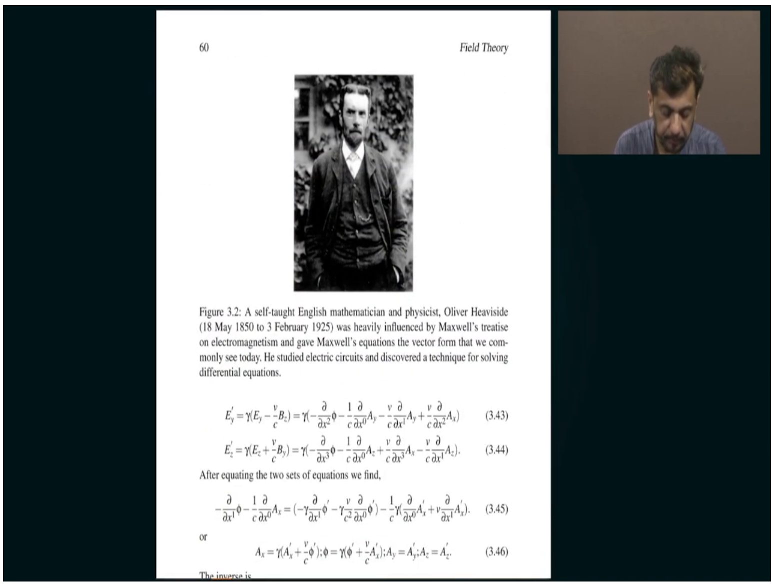
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Figure 3.2: A self-taught English mathematician and physicist, Oliver Heaviside (18 May 1850 to 3 February 1925) was heavily influenced by Maxwell's treatise on electromagnetism and gave Maxwell's equations the vector form that we commonly see today. He studied electric circuits and discovered a technique for solving differential equations.

$$E_y' = \gamma(E_y - \frac{v}{c}B_z) = \gamma(-\frac{\partial}{\partial x^2}\phi - \frac{1}{c}\frac{\partial}{\partial x^0}A_y - \frac{v}{c}\frac{\partial}{\partial x^1}A_y + \frac{v}{c}\frac{\partial}{\partial x^2}A_x) \quad (3.43)$$

$$E_z' = \gamma(E_z + \frac{v}{c}B_y) = \gamma(-\frac{\partial}{\partial x^3}\phi - \frac{1}{c}\frac{\partial}{\partial x^0}A_z + \frac{v}{c}\frac{\partial}{\partial x^3}A_x - \frac{v}{c}\frac{\partial}{\partial x^1}A_z). \quad (3.44)$$

After equating the two sets of equations we find,

$$-\frac{\partial}{\partial x^1}\phi - \frac{1}{c}\frac{\partial}{\partial x^0}A_x = (-\gamma\frac{\partial}{\partial x^1}\phi' - \gamma\frac{v}{c^2}\frac{\partial}{\partial x^0}\phi') - \frac{1}{c}\gamma(\frac{\partial}{\partial x^0}A_x' + v\frac{\partial}{\partial x^1}A_x'). \quad (3.45)$$

or

$$A_x = \gamma(A_x' + \frac{v}{c}\phi'); \phi = \gamma(\phi' + \frac{v}{c}A_x'); A_y = A_y'; A_z = A_z'. \quad (3.46)$$

The inverse is

So, now you go ahead and ask yourself how do phi and A transform under Lorentz transformation. We can answer that of course, because we already know how E and B transform under Lorentz transformation. We have found that E and B do not transform well even though the transformation laws are simple they still do not transform like 4 vectors.

However we expect now that phi and a should transform like 4 vectors namely phi is the time component of a certain 4 vector and A is the special component of that 4 vector. So, to prove this let us go ahead and find out how E transforms under Lorentz transformation when expressed in terms of the potentials. See when expressed in terms of the potentials you see that as usual you write down the gradients and the time derivatives also in terms of a the Lorentz transformed versions.

And then now you go ahead and. So, you know that the x component of the electric field does not transform at all, but the y component transforms in this peculiar way. So, the y component transforms it gets mixed up with the z component of the magnetic field. So, when you do all that and you insert it into your earlier transform transformed electric field in terms of the potentials then you will be successful in proving this transformation for both the inverse and the forward and backwards transforms.

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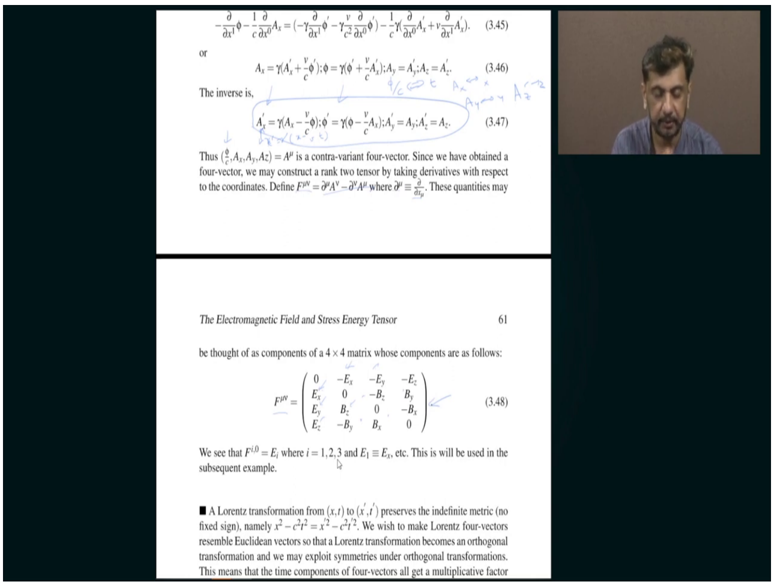

So, in other words you will be successful in proving that the vector potential in the transformed frame transforms like this whereas, the scalar potential transforms like this. So, you can see that this is certainly reminiscent of a what we would normally associate with. So, this phi by c is analogous to time and A x is analogous to x and A y is analogous to y and A z is analogous to z. So, you can see that. So, if you think of this as x dash I mean analogously.

So, this is going to be x dash is gamma x minus v phi is phi by c is t. So, you will get back your familiar result that x dash is gamma into x minus v t. So, you can see that from here the phi by c and ax a y a z are transforming exactly like the components of 4 vector. So, now, you see so, I have in this description purposely avoided using you know these field tensor type of ideas which makes all the proofs very compact. So, if you write down Maxwell's equation in terms of what are called field tensors the relativistic nature of Maxwell's equations becomes so obvious that there is nothing left to prove.

But I find that kind of an approach is somewhat opaque and it you know obscures some of these details of how these transformations work. So, I have purposely explicitly pointed out how the transformations work. So, that then you can go ahead and confidently work it out using the more you know concise 4 vector notation and you can be confident that you have understood the underlying meaning of what is happening.

So, now that is exactly what we are doing now you see I am going to define the 4 vector in this way. So, where these are my derivatives the contravariant derivatives will involve the derivatives respect to the corresponding covariant coordinates and this is how I define my. So, the field tensor is called the field tensor and basically a collection of electric and magnetic field components arranged in such a way that the field tensor is fully anti symmetric. So, in other words f mu nu is minus f nu mu.

So, therefore, all the diagonal components are 0 and the off diagonal components are negative of each other. So, basically you see that there are. So, for skew symmetric 4 by 4 matrix the number of independent components are only 6 because these are the only independent component. This once you specify this is already known because its the negative of that. You specify this is already known, this is specify this is already known.

Similarly, if you specify this is known specify this is known and so on. So, you have totally six. So, that is perfect because we know that we need six. So, ex ey ez is three and B x B y B z is another three. So, put together six we really need all six to describe a field tensor, but the point is that the field tensor is really a tensor it is not really a 4 vector or anything.

So, that is why we had to define it in this peculiar way and that is why it was not at all obvious how the I mean. So, it is only when you replace the or express the electric and magnetic field in terms of potential then only you will be successful in linking the electric and magnetic fields to 4 vectors because through the potentials the potentials are 4 vectors. But the electric magnetic field themselves are not 4 vectors they are components of a rank 2 tensor ok.

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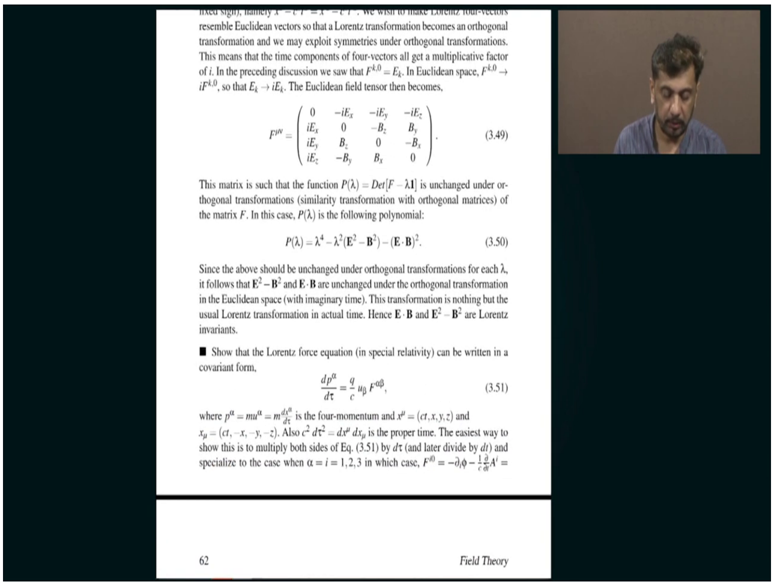

resemble Euclidean vectors so that a Lorentz transformation becomes an orthogonal transformation and we may exploit symmetries under orthogonal transformations. This means that the time components of four-vectors all get a multiplicative factor of i. In the preceding discussion we saw that $F^{k,0} = E_k$. In Euclidean space, $F^{k,0} \rightarrow iF^{k,0}$, so that $E_k \rightarrow iE_k$. The Euclidean field tensor then becomes,

$$F^{\mu\nu} = \begin{pmatrix} 0 & -iE_x & -iE_y & -iE_z \\ iE_x & 0 & -B_z & B_y \\ iE_y & B_z & 0 & -B_x \\ iE_z & -B_y & B_x & 0 \end{pmatrix}. \quad (3.49)$$

This matrix is such that the function $P(\lambda) = Det[F - \lambda \mathbf{1}]$ is unchanged under orthogonal transformations (similarity transformation with orthogonal matrices) of the matrix F. In this case, $P(\lambda)$ is the following polynomial:

$$P(\lambda) = \lambda^4 - \lambda^2(\mathbf{E}^2 - \mathbf{B}^2) - (\mathbf{E}\cdot\mathbf{B})^2. \quad (3.50)$$

Since the above should be unchanged under orthogonal transformations for each λ, it follows that $\mathbf{E}^2 - \mathbf{B}^2$ and $\mathbf{E}\cdot\mathbf{B}$ are unchanged under the orthogonal transformation in the Euclidean space (with imaginary time). This transformation is nothing but the usual Lorentz transformation in actual time. Hence $\mathbf{E}\cdot\mathbf{B}$ and $\mathbf{E}^2 - \mathbf{B}^2$ are Lorentz invariants.

■ Show that the Lorentz force equation (in special relativity) can be written in a covariant form,

$$\frac{dp^\alpha}{d\tau} = \frac{q}{c} u_\beta F^{\alpha\beta}, \quad (3.51)$$

where $p^\alpha = mu^\alpha = m\frac{dx^\alpha}{d\tau}$ is the four-momentum and $x^\mu = (ct, x, y, z)$ and $x_\mu = (ct, -x, -y, -z)$. Also $c^2\, d\tau^2 = dx^\mu\, dx_\mu$ is the proper time. The easiest way to show this is to multiply both sides of Eq. (3.51) by $d\tau$ (and later divide by dt) and specialize to the case when $\alpha = i = 1,2,3$ in which case, $F^{i0} = -\partial_i\phi - \frac{1}{c}\frac{\partial}{\partial t}A^i =$

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So, in my book I have several exercises which I will encourage you to do on your own. So, perhaps I will assign these exercises to you and some of the tutorials that you will be encountering shortly. So, I am going to skip these assignments and well these are not really assignment these are worked out examples, but these are also things that you should try and do on your own ok.

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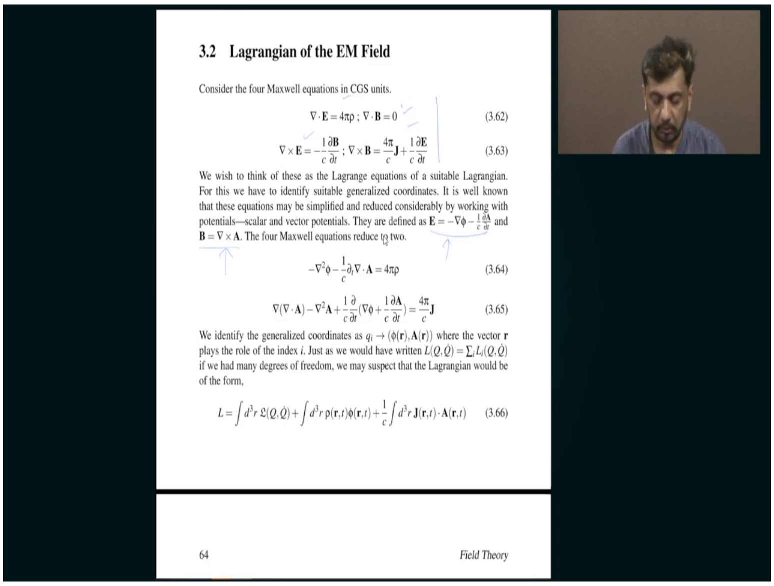

But just to maintain the continuity of my presentation I will skip the examples because those are meant to illustrate certain specific points which you should try to study on your own. So, now, let me go to the basic promise I made in the beginning namely that every dynamical system the equations of every dynamical system can be thought of as the Euler Lagrange equation of a suitable Lagrangian.

So, in other words for example, Newton's second law is can be thought of as basically some consequence of some Lagrangian exactly in the same way I am going to see if I can think of Maxwell's equations themselves as a consequence of a suitable as the Euler Lagrange equation of a suitable Lagrangian ok. So, the question is how would I do that. So, to do that I am going to first write down these four Maxwell's equation the first two are the ok.

So, in other words the second and third are the source free Maxwell's equation the first and fourth are the Maxwell's equation with sources. By the second and third Maxwell's equations which do not have sources are automatically obeyed by re expressing the electric and magnetic fields in terms of the corresponding potentials. So, when I do that and I insert it back into the sourced Maxwell equation. So, when I so, this automatically solves my source free Maxwell's equation, but then I am going to insert it into the Maxwell equation with sources ok. So, I get this result ok.

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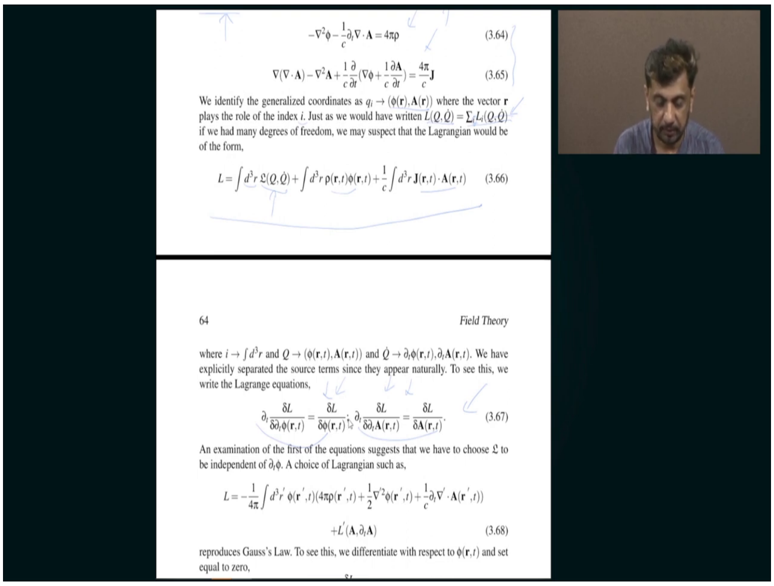

So, the claim is that these equations namely 3.64 and 3.65 are basically the Euler Lagrange equation of some Lagrangian ok. So, but to do that you see I have to the so the claim is that there is some Lagrangian whose Lagrange equations are these 2 sourced Maxwell equation. But then in order to make this claim of course, I have to first identify the generalized coordinates. So, it is clear what the generalized coordinates are because you see this most time derivatives of the potentials.

So, therefore, the generalized coordinates are likely to be the scalar and vector potentials themselves. So, I am going to make that claim and then I am going to say that the Lagrangian should exist which has that property ok. And besides remember that this has the form. So, you see this is my generalized coordinate. So, what do I mean by this what is r? r takes on the role of some index i.

So, so you remember in a dynamical system with finite number of generalized coordinates it would be labeled as qi where i is discrete index like 1, 2, 3. So, you have a finite number of degrees of freedom in that situation, but here we are talking about a field. So, a field not only has an infinitely many degrees of freedom it also has a continuously infinite number of degrees of freedom so; that means, that i gets replaced by a continuous index called r vector ok.

So, r vector plays the role of the index i. So, just as we would have written for a system with finite number of generalized coordinates we have written the Lagrangian as the sum over all the you know different coordinates. So, the Lagrangian due to the time derivatives of a you know say if you are talking about r theta phi you would have mr dot square plus r square theta dot square plus r squared sin squared theta phi dot squared. So, like that we would have written it separately as the dot product of the first generalized coordinates squared second etcetera etcetera.

So, so that is what you were written it if there were a finite number of generalized coordinates. But now that there are infinitely many of them this summation over i gets replaced by an integral over r because remember that i gets replaced by r, but then because there are sources.

So, this is if there were no sources by I mean by implication that this script L corresponds to the Lagrangian of a system with no sources, but if there are sources we suspect that it is going to look like this. But now I am going to postulate that. So, in other words I have to now figure out what this is in order for the Euler Lagrange equations of 3.66 to be exactly the same as 6 4 and 6 5 ok.

So, obviously, there are two equations namely 6 4 and 6 5 and we expect 2 Lagrange equations because one is with respect to phi the other is with respect to A. So, the Euler Lagrange equation for the phi is basically this and the Euler Lagrange. So, it is a derivative with respect to phi dot and d by d t of d L by d phi dot is equal to d L by d phi. So, similarly d by d t of d L by d a dot is equal to d L by d A.

So, these are the Euler Lagrange equations of Lagrangian which we still do not know what it is, but whatever it is it has to be constructed in such a way that the equations that you obtained 3.67 should be identical that there are two equations and 3.67 the one is the first one the second one. So, these 2 have to be respectively identical to 6 4 and 6 5 ok.

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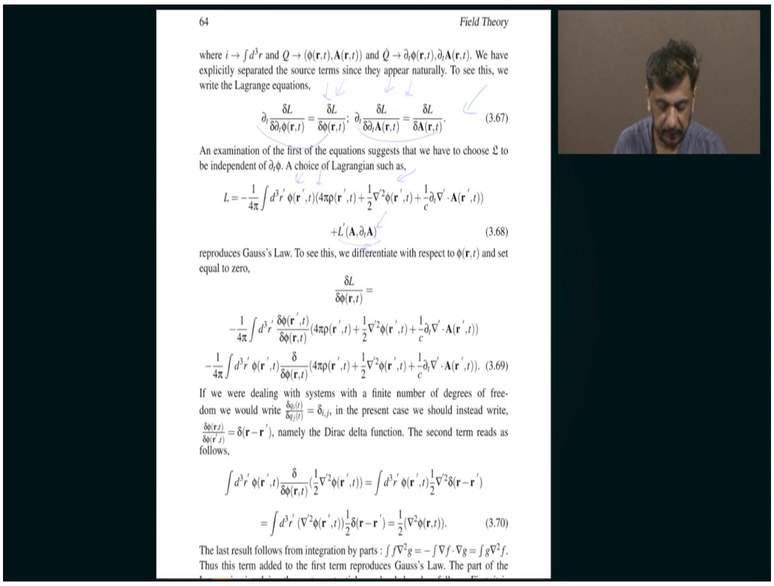

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where $i \to \int d^3r$ and $Q \to (\phi(\mathbf{r},t), \mathbf{A}(\mathbf{r},t))$ and $\dot{Q} \to \partial_t \phi(\mathbf{r},t), \partial_t \mathbf{A}(\mathbf{r},t)$. We have explicitly separated the source terms since they appear naturally. To see this, we write the Lagrange equations,

$$\partial_t \frac{\delta L}{\delta \partial_t \phi(\mathbf{r},t)} = \frac{\delta L}{\delta \phi(\mathbf{r},t)}; \ \partial_t \frac{\delta L}{\delta \partial_t \mathbf{A}(\mathbf{r},t)} = \frac{\delta L}{\delta \mathbf{A}(\mathbf{r},t)}. \quad (3.67)$$

An examination of the first of the equations suggests that we have to choose $\mathfrak{L}$ to be independent of $\partial_t \phi$. A choice of Lagrangian such as,

$$L = -\frac{1}{4\pi} \int d^3r' \ \phi(\mathbf{r}',t)(4\pi\rho(\mathbf{r}',t) + \frac{1}{2}\nabla'^2\phi(\mathbf{r}',t) + \frac{1}{c}\partial_t \nabla' \cdot \mathbf{A}(\mathbf{r}',t)) + L'(\mathbf{A}, \partial_t \mathbf{A}) \quad (3.68)$$

reproduces Gauss's Law. To see this, we differentiate with respect to $\phi(\mathbf{r},t)$ and set equal to zero,

$$\frac{\delta L}{\delta \phi(\mathbf{r},t)} = -\frac{1}{4\pi} \int d^3r' \ \frac{\delta \phi(\mathbf{r}',t)}{\delta \phi(\mathbf{r},t)}(4\pi\rho(\mathbf{r}',t) + \frac{1}{2}\nabla'^2\phi(\mathbf{r}',t) + \frac{1}{c}\partial_t \nabla' \cdot \mathbf{A}(\mathbf{r}',t)) - \frac{1}{4\pi} \int d^3r' \ \phi(\mathbf{r}',t)\frac{\delta}{\delta \phi(\mathbf{r},t)}(4\pi\rho(\mathbf{r}',t) + \frac{1}{2}\nabla'^2\phi(\mathbf{r}',t) + \frac{1}{c}\partial_t \nabla' \cdot \mathbf{A}(\mathbf{r}',t)). \quad (3.69)$$

If we were dealing with systems with a finite number of degrees of freedom we would write $\frac{\delta q_i(t)}{\delta q_j(t)} = \delta_{i,j}$, in the present case we should instead write, $\frac{\delta \phi(\mathbf{r},t)}{\delta \phi(\mathbf{r}',t)} = \delta(\mathbf{r} - \mathbf{r}')$, namely the Dirac delta function. The second term reads as follows,

$$\int d^3r' \ \phi(\mathbf{r}',t)\frac{\delta}{\delta \phi(\mathbf{r},t)}(\frac{1}{2}\nabla'^2\phi(\mathbf{r}',t)) = \int d^3r' \ \phi(\mathbf{r}',t)\frac{1}{2}\nabla'^2\delta(\mathbf{r} - \mathbf{r}') = \int d^3r' \ (\nabla'^2\phi(\mathbf{r}',t))\frac{1}{2}\delta(\mathbf{r} - \mathbf{r}') = \frac{1}{2}(\nabla^2\phi(\mathbf{r},t)). \quad (3.70)$$

The last result follows from integration by parts : $\int f\nabla^2 g = -\int \nabla f \cdot \nabla g = \int g \nabla^2 f$. Thus this term added to the first term reproduces Gauss's Law. The part of the

So, the question is how would you achieve that? So, the way we achieve that is by choosing the rest of the Lagrangian in this way ok. First of all look if you look at 6 4 it does not have d by d t of phi it only has d by d t of A. So, that implies that the Lagrangian should be independent of d by d t of phi ok. So, which is why I have chosen it to be this. So, you will see that this will now involve. So, the all the phi dependence has been extracted. So, this is all there is to the phi dependence.

Now, I have to convince you that this is. In fact, correct because of course, I have not told you what this is, but this I will tell you later, but even not knowing what this is because the phi dependence has been extracted sufficient for us to reproduce at least 3.64. So, to do that find the derivative of the Lagrangian with respect to phi then you will see that it is basically equal to one half of yeah.

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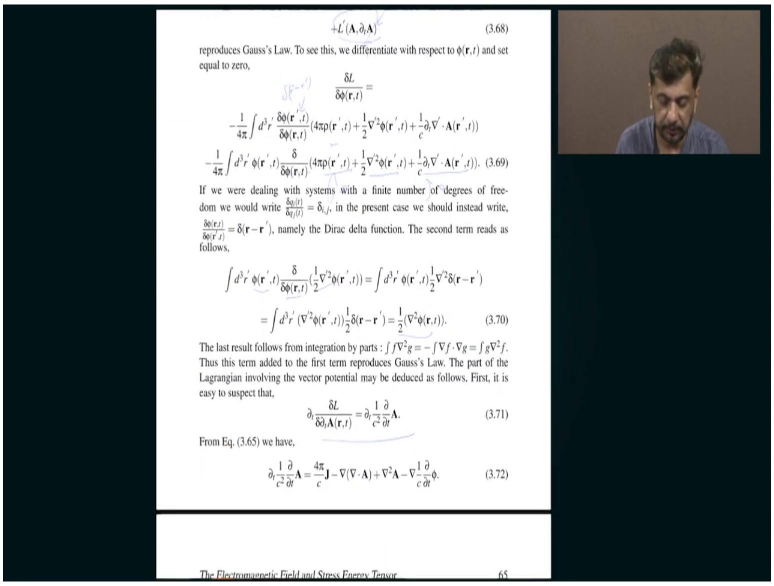

So, it is basically going to be equal to. So, this is going to be a Dirac delta function ok. And whereas, this is of course, source so, it is unrelated so, that is 0 and this is going to give you. So, this sort of thing is being evaluated here. So, this is all 0 because these two do not under relate it to phi. So, this is all 0 ok only this matters.

So, when you do that you get this result ok. So, I I have skipped some steps. So, bottom line is that if you go ahead and insert this in your Euler Lagrange equations here you will exactly get this ok. So, I have skipped a few steps which you have to fill in, but bottom line is that this particular choice is sufficient for you to reproduce the first of these equations 3.64. So, similarly you can figure out the rest of the Lagrangian. So, see that I have not told you what this is. So, I have to fix that as well.

So, based upon the rest of these observations. So, you can. So, this equation has to reproduce 3.65. So, the question is how would you select L dash. So, that it does that.

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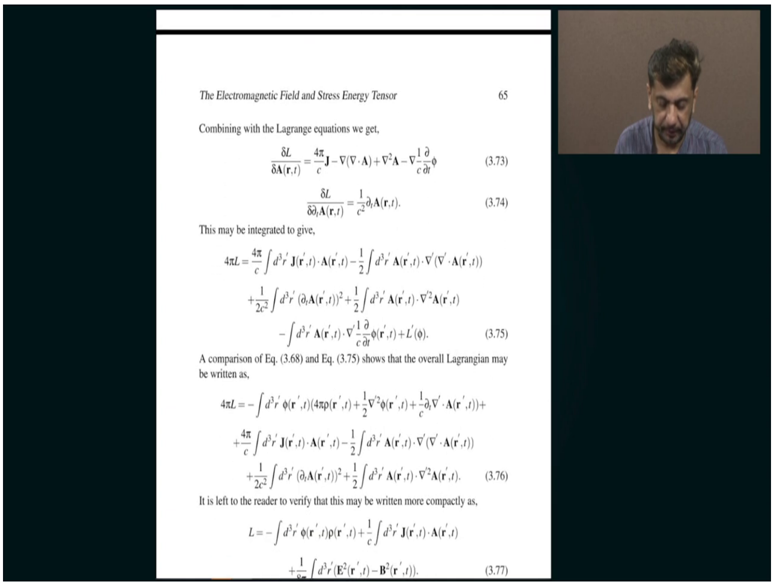
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Combining with the Lagrange equations we get,

$$\frac{\delta L}{\delta \mathbf{A}(\mathbf{r},t)} = \frac{4\pi}{c}\mathbf{J} - \nabla(\nabla\cdot\mathbf{A}) + \nabla^2\mathbf{A} - \nabla\frac{1}{c}\frac{\partial}{\partial t}\phi \quad (3.73)$$

$$\frac{\delta L}{\delta \partial_t \mathbf{A}(\mathbf{r},t)} = \frac{1}{c^2}\partial_t \mathbf{A}(\mathbf{r},t). \quad (3.74)$$

This may be integrated to give,

$$4\pi L = \frac{4\pi}{c}\int d^3r'\, \mathbf{J}(\mathbf{r}',t)\cdot\mathbf{A}(\mathbf{r}',t) - \frac{1}{2}\int d^3r'\, \mathbf{A}(\mathbf{r}',t)\cdot\nabla'(\nabla'\cdot\mathbf{A}(\mathbf{r}',t))$$
$$+\frac{1}{2c^2}\int d^3r'\,(\partial_t\mathbf{A}(\mathbf{r}',t))^2 + \frac{1}{2}\int d^3r'\,\mathbf{A}(\mathbf{r}',t)\cdot\nabla'^2\mathbf{A}(\mathbf{r}',t)$$
$$-\int d^3r'\,\mathbf{A}(\mathbf{r}',t)\cdot\nabla'\frac{1}{c}\frac{\partial}{\partial t}\phi(\mathbf{r}',t) + L'(\phi). \quad (3.75)$$

A comparison of Eq. (3.68) and Eq. (3.75) shows that the overall Lagrangian may be written as,

$$4\pi L = -\int d^3r'\,\phi(\mathbf{r}',t)(4\pi\rho(\mathbf{r}',t) + \frac{1}{2}\nabla'^2\phi(\mathbf{r}',t) + \frac{1}{c}\partial_t\nabla'\cdot\mathbf{A}(\mathbf{r}',t)) +$$
$$+\frac{4\pi}{c}\int d^3r'\,\mathbf{J}(\mathbf{r}',t)\cdot\mathbf{A}(\mathbf{r}',t) - \frac{1}{2}\int d^3r'\,\mathbf{A}(\mathbf{r}',t)\cdot\nabla'(\nabla'\cdot\mathbf{A}(\mathbf{r}',t))$$
$$+\frac{1}{2c^2}\int d^3r'\,(\partial_t\mathbf{A}(\mathbf{r}',t))^2 + \frac{1}{2}\int d^3r'\,\mathbf{A}(\mathbf{r}',t)\cdot\nabla'^2\mathbf{A}(\mathbf{r}',t). \quad (3.76)$$

It is left to the reader to verify that this may be written more compactly as,

$$L = -\int d^3r'\,\phi(\mathbf{r}',t)\rho(\mathbf{r}',t) + \frac{1}{c}\int d^3r'\,\mathbf{J}(\mathbf{r}',t)\cdot\mathbf{A}(\mathbf{r}',t)$$
$$+\frac{1}{8\pi}\int d^3r'(\mathbf{E}^2(\mathbf{r}',t) - \mathbf{B}^2(\mathbf{r}',t)). \quad (3.77)$$

So, the rest of it is basically recovered in a similar way. So, you compare the phi equation with the answer and the A equation with the answers and then you will be able to a successfully show that the Euler Lagrange equations of the this Lagrangian. So, this is the final answer. So, if you select this to be your Lagrangian you can show. So, even if you did not follow the constructive derivation of that you see I have actually tried to argue how to construct the Lagrangian from Maxwell's equation.

So, even if that constructive proof of the Lagrangian is something you did not follow you can certainly do the reverse that is assume that this is the Lagrangian and then try to derive the Lagrange equations or the Euler Lagrange equation of this Lagrangian assuming phi and A are your generalized coordinates.

In which case you are guaranteed to obtain the Maxwell equations right. So, the source Maxwell equations ok. So, that completes the Lagrangian description of the electromagnetic field because we have successfully written down a Lagrangian whose Lagrange equations are precisely the Maxwell's equations. So, the question is. So, this is as far as the Lagrangian formalism is concerned.

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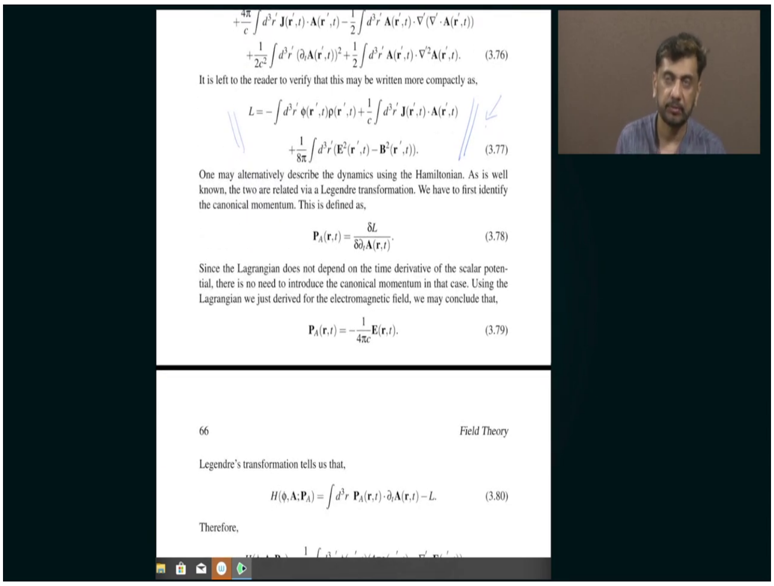

So, in the next class I am going to discuss the Hamiltonian formulation of the electromagnetic field. Because you see I have told you repeatedly that both have their advantages the Lagrange formalism is useful because it is the first example where you study a dynamical system using generalized coordinates paying attention to constraints without knowing all the components of the forces. So, it first teaches you.

How to bypass having to know all the constraint forces, but then the Hamiltonian approach is advantages in a for a different reasons one is of course, that quantum mechanics traditionally described in terms of Hamiltonian's although there is no reason why it should because Lagrangian also are equally useful in doing quantum mechanics I am going to discuss that a bit later.

But the more important technical reason is that symmetries are naturally described in terms of flows in the Hamiltonian language. So, dynamical symmetries and other kinds of symmetries encountered in Noether's theorem are described as flows which appear naturally in the context of Hamiltonian mechanics. So, it is important to be able to describe a dynamical system both in terms of Lagrangian as well as in terms of Hamiltonian's.

So, now that we have successfully described the Lagrangian formulation of Maxwell's equation we should be able to go ahead and now describe the Hamiltonian analogue of the same system. So, that is something I am going to do in the next class. So, until then I take care. So, I am stopping now.