Dynamics of Classical and Quantum Fields: An Introduction
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Symmetries
Lecture - 08
Symmetries in Field Theories

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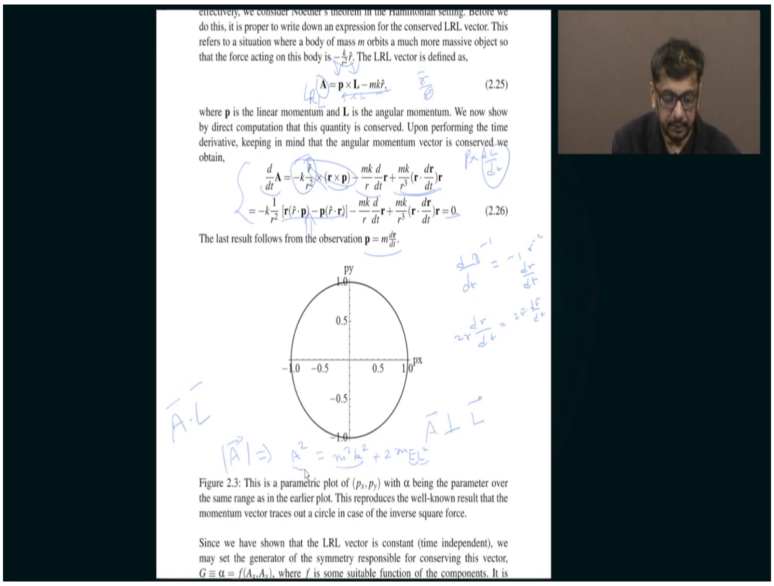

Ok. So let us continue our discussion. So, if you remember that in the last class I try to wind up this discussion of dynamical symmetries and the importance of this Runge Lenz vector. So, basically I showed that, this if you recall this Runge Lenz vector is defined in this peculiar way, that is defined as the linear momentum, cross product with angular momentum and you subtract out the scalar term in the direction of r hat, which is r hat is the direction connecting the origin to the mass that is going round and round the origin.

So, this k is basically the constant that appears in the force. So, so we assume inverse square force and strength of that force is k. So, there is a inverse square attractive force and the strength of the force is k. So, it is k by r squared. So, that is the k there. So, this is the Runge Lenz vector which I just showed in the last lecture that, it is conserved by explicitly evaluating the derivative with respect to time and then you are able to show that it is conserved. So, I repeatedly pointed out in the last lecture that, the Runge Lenz vector is independent of the angular momentum.

That is of course, true for the large part, but I have to qualify that with the following statement that, while its direction is independent of, in fact not only independent, it is actually perpendicular the direction of A is perpendicular to the direction of L. And why is that? Because you take dot product of A and L, you will see that p cross L dot l is 0; but then you will be left with the other term which is minus m k r cap dot l, but remember that r cap is the unit vector connecting the origin to the mass and then that is in the orbit of the motion of the planet or whatever you are talking about.

So, it is in the orbit and the angular momentum is by definition the vector which is perpendicular to that orbit. So, therefore, that is also 0. So, as a result the A vector by construction is perpendicular to the L vector. So, therefore, A and L are independent in the sense of directions, so they are they are mutually perpendicular. However, the magnitude of L is actually already. So, the direction of A is a new conserved quantity, where the magnitude of A is not a new conserved quantity.

So, the magnitude of A is relatable to conserved quantities that we have already encountered, namely the angular momentum and the energy. And the way you write this is A squared is, you can show that A squared is going to come out as m squared k squared plus 2 m e L squared, where e is the total energy of the system which is constant, L square is the square of the angular momentum which is constant, m and k are anyway constants. So, and m and k are constant, so the rest of it is. So, as a result A squared is a constant.

And it is a constant that we have already encountered before; in the sense that it is related to constant that we already know this, it is not a new constant. So, the new constant really is the direction of A, is the direction of A that is a new constant. So, as a result you see that is the reason why in the last class I defined this peculiar quantity called alpha as the tan inverse of A y by A x.

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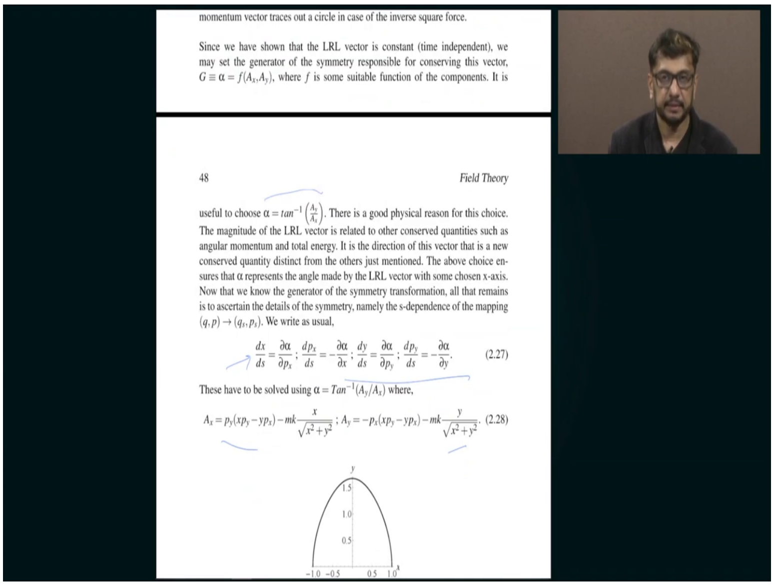

So, what that is basically it tells you the orientation of A; so that means, a vector is perpendicular to L. So, therefore, it is in the orbit of the planet. So, it is parallel to the orbit of the planet.

So, basically A points in some particular direction on the orbit of the planet; that means parallel to the orbit of the planet. So, what you do is that, you define the orientation of A relative to some chosen axis like x and y and you define the angle made by the A vector with the x axis as your alpha. So, if that is the case, then you see then tan inverse A y by A x is going to be your alpha clearly; because A x will be a cos alpha and A y will be a sin alpha and alpha is therefore, tan inverse A y by A x.

So, that basically, so the alpha is the new conserved quantity which specifies the direction of A; because the magnitude of a was already related to earlier constants, namely total energy and angular momentum.

So, that is exactly what we picked alpha as the generator of this new symmetry called dynamical symmetry. So, if you pick alpha as your generator, then you will be able to show that. So, the symmetry generated by alpha is the symmetry that is responsible for the conservation of the Laplace Runge Lenz vector, especially its direction, the magnitude is already conserved for other reasons. So, that is basically the long answer to the question, what is the symmetry that is responsible for the conservation of the Laplace Runge Lenz vector?

So, that is a new conserved quantity, which is only present for inverse square attraction central force; it is not there if it is central force of any other kind, ok. So, that was the missing part of my discussion, so which I have completed.

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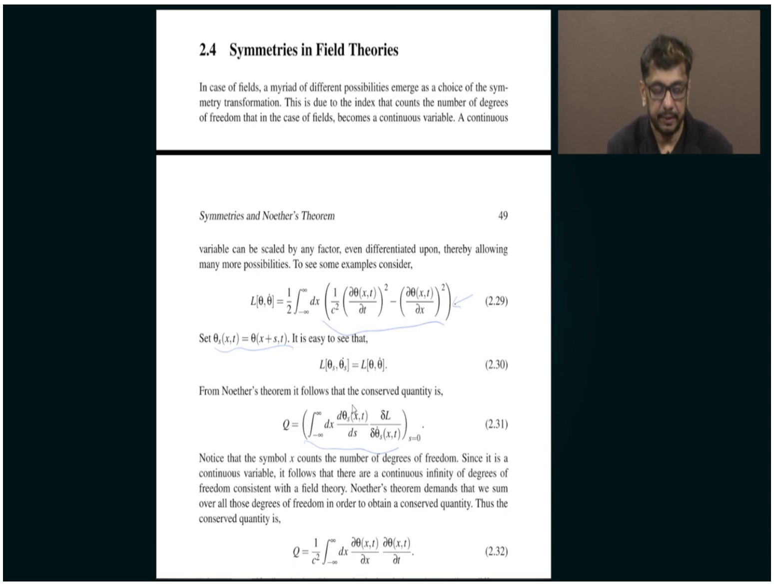

So, now, I am going to discuss the symmetries in field theories and specifically try to see if I can use Noether's theorem to write down conserved quantities in field theories. So, if you recall that I told you that every dynamical equation can be thought of as some kind of an Euler Lagrange equation of a suitable Lagrangian.

So, for example, if you think of a wave equation; so obviously a wave equation should also therefore, be imagined or should be you should be able to imagine the wave equation as the Euler Lagrange equation of some suitable Lagrangian. In fact, you will see that the suitable Lagrangian is this; I think we have encountered this before, so I will start from here. So, this is your Lagrangian, whose Euler Lagrange equations are the wave equation. So, now, I am going to identify a certain symmetry associated with this Lagrangian and that is a continuous symmetry.

So, notice that in order for me to derived conserve quantities using Noether's theorem, I should ensure that the symmetries are continuous; that means that it should not be a discrete symmetry like a reflection symmetry or that sort of thing. So, it has to be some continuous symmetry. So, this is an example that. So, I am going to first postulate a transformation that imagine that the original theta of x comma t is now replaced by theta subscript s, which is basically obtained by shifting x with or replacing x with x plus s. So, now, you can see that, when you do that your Lagrangian is unchanged. So, the theta itself changes.

So, theta changes from theta of x comma t to a theta of x plus s comma t. So, theta changes, but the Lagrangian does not. So, now, clearly since s is continuous Noether guarantees that there is a conserved quantity. And what is that conserved quantity? It is precisely this. So, notice that because it is a field theory, I am summing over x. So, if x takes on. So, in the case of say finite number of particles; this x would be your Q i, it is the number of generalized coordinates Q 1, Q 2, Q 3 like that.

So, now, you have infinitely many generalized coordinates. So, labelled by x, x is continuously infinite number of generalized coordinates. So, theta of x, theta of x dash they are all different generalized coordinates, but then x is continuous. So, all you have to do is rather than summing over those Q i's you have to integrate over x.

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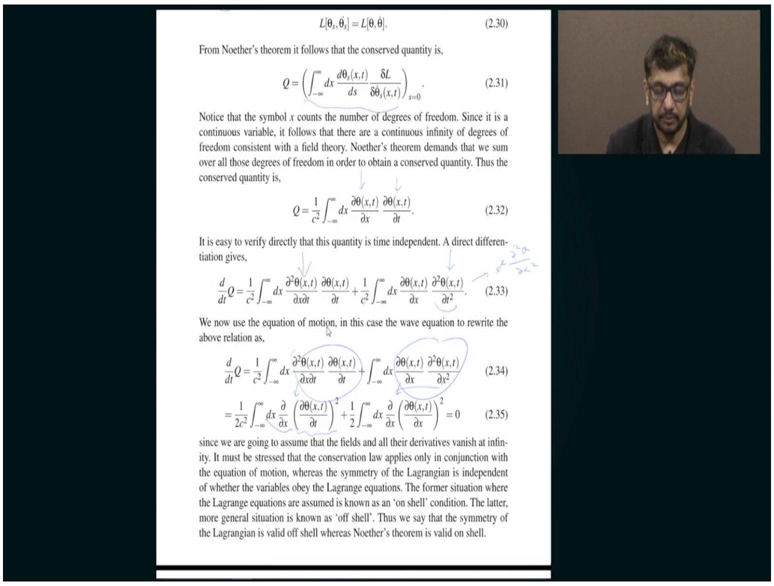

$$L[\theta_s,\dot{\theta}_s] = L[\theta,\dot{\theta}]. \quad (2.30)$$

From Noether's theorem it follows that the conserved quantity is,

$$Q = \left(\int_{-\infty}^{\infty} dx\, \frac{d\theta_s(x,t)}{ds}\, \frac{\delta L}{\delta\dot{\theta}_s(x,t)}\right)_{s=0}. \quad (2.31)$$

Notice that the symbol x counts the number of degrees of freedom. Since it is a continuous variable, it follows that there are a continuous infinity of degrees of freedom consistent with a field theory. Noether's theorem demands that we sum over all those degrees of freedom in order to obtain a conserved quantity. Thus the conserved quantity is,

$$Q = \frac{1}{c^2}\int_{-\infty}^{\infty} dx\, \frac{\partial\theta(x,t)}{\partial x}\, \frac{\partial\theta(x,t)}{\partial t}. \quad (2.32)$$

It is easy to verify directly that this quantity is time independent. A direct differentiation gives,

$$\frac{d}{dt}Q = \frac{1}{c^2}\int_{-\infty}^{\infty} dx\, \frac{\partial^2\theta(x,t)}{\partial x\partial t}\, \frac{\partial\theta(x,t)}{\partial t} + \frac{1}{c^2}\int_{-\infty}^{\infty} dx\, \frac{\partial\theta(x,t)}{\partial x}\, \frac{\partial^2\theta(x,t)}{\partial t^2}. \quad (2.33)$$

We now use the equation of motion, in this case the wave equation to rewrite the above relation as,

$$\frac{d}{dt}Q = \frac{1}{c^2}\int_{-\infty}^{\infty} dx\, \frac{\partial^2\theta(x,t)}{\partial x\partial t}\, \frac{\partial\theta(x,t)}{\partial t} + \int_{-\infty}^{\infty} dx\, \frac{\partial\theta(x,t)}{\partial x}\, \frac{\partial^2\theta(x,t)}{\partial x^2} \quad (2.34)$$

$$= \frac{1}{2c^2}\int_{-\infty}^{\infty} dx\, \frac{\partial}{\partial x}\left(\frac{\partial\theta(x,t)}{\partial t}\right)^2 + \frac{1}{2}\int_{-\infty}^{\infty} dx\, \frac{\partial}{\partial x}\left(\frac{\partial\theta(x,t)}{\partial x}\right)^2 = 0 \quad (2.35)$$

since we are going to assume that the fields and all their derivatives vanish at infinity. It must be stressed that the conservation law applies only in conjunction with the equation of motion, whereas the symmetry of the Lagrangian is independent of whether the variables obey the Lagrange equations. The former situation where the Lagrange equations are assumed is known as an 'on shell' condition. The latter, more general situation is known as 'off shell'. Thus we say that the symmetry of the Lagrangian is valid off shell whereas Noether's theorem is valid on shell.

So, then you see that what Noether says that this is conserved. So, now, let us calculate this generalized momentum as it were and you will see that it is basically this, ok.

So, now you can easily convince yourself that this is a conserved quantity, ok. So, how do you convince yourself of this? So, just take the time derivative of Q with respect to time and you will see that is in fact conserved. So, if I take the time derivative, you will see that I have to first differentiate Q with respect to time first here; then I have to second I have to differentiate this. So, first here, then second here; so but then you see this is nothing but the wave equation tells me that this is nothing but c squared d squared theta by d x squared.

So, that c squared and c squared cancel and I get this. But now you see this term can be written like this, ok. So, this is just square of 1 half of square of this. So, similarly this also can be written like this. So, basically you can write both these terms as the spatial derivative of some function, ok. So, this can be written like this, this can be written like this; but then you see once you do that the derivative the integral of a derivative is what you will get, so that means the value of the function at plus and minus infinity which we assume is 0, we have to assume that these functions vanish rapidly at infinity.

So, if that is the case, then these terms are actually 0, ok. So, you see that therefore, this is conserved quantity; so that means, Q is a conserved quantity as guaranteed by Noether, ok

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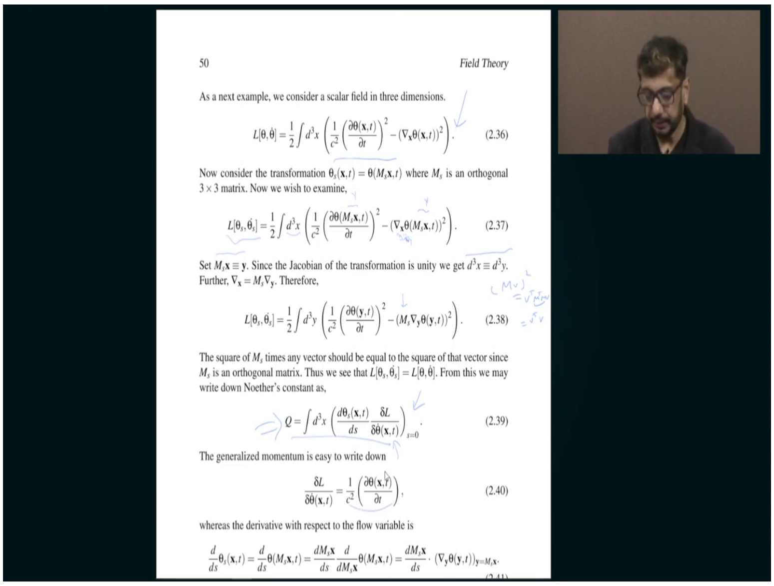

So, this is an example of a conserved quantity in field theory. So, the next example we can think of is, so that was in one dimension. So, I want to move to dependence on. So, that index x, remember that x takes on the role of an index. So, what was like a Q i the number of generalized coordinate was specified by that index i; now it is specified by this x.

So, you see the point is that, this index x is now continuous and not only it is continuous and now it is in three dimensions. So, if that is the case, then in addition to the what I had done earlier that is ability to translate x to x plus x; now I can do something even more interesting, namely I can rotate x to x dash, where x dash is some orthogonal transformation starting from x, right. So, I can rotate that vector to some other vector. So, that is what I have done here. So, I have said that let my continuous transformation be this rather than say x 2, x plus s that was the example I had already discussed earlier.

So, I want to discuss a different example. So, where I replace x by the rotated version of x which is M s into X. So, the M subscript s is an orthogonal matrix, ok. So, now, this orthogonal matrix you can see with clearly that, the earlier case it was obvious that the Lagrange is unchanged under this transformation, so therefore, it is a symmetry.

But here it is less obvious, so let us work it out. So, you see if I replace X by M s into X, you see I can do a change of variables call this Y; then the thing I have to do is I have to, so this will of course become some orthogonal matrix into inverse into y grad, y into inverse of an orthogonal matrix.

But then I am squaring it, so that goes away, right. So, square will involve the transpose and that sort of thing. So, M transpose M will come which is identity, but the only thing I have to make sure is this. So, this because x x going to y is an orthogonal transformation, the Jacobian is unity, ok. So, because of that the volume elements are the same, right. So, because they are the same, so that is what I have shown here. So, this M s comes out, but then its square is anyway 1 I mean square means like M transpose M is 1 that is what you will get.

So, if I M v squared is basically v transpose M transpose M v, which is basically v transpose v. So, that goes away, the M goes away. So, bottom line is that this transformation is in fact a symmetry. So, because it is a symmetry Noether again guarantees that there is a conserved quantity.

And what is that conserved quantity? It is precisely this; it is the by now you have probably already memorized this. So, it is the rate of change of the generalized coordinate with respect to that continuous symmetry parameter called s and then the generalized momentum, multiplied by generalized momentum.

And then I have to sum over all the degrees of freedom; that means that I have to sum over all the x values, which like I have repeatedly said it takes on the role of an index, which counts how many degrees of freedom there are which is a continuously infinity of them. So, bottom line is that this is what Noether tells you that, because this is a symmetry, this is a conserved quantity. So, you can see how powerful this theorem is that, it would have been impossible for you to guess this conserved quantity; but however it is extremely simple and natural for you to guess the symmetry.

You can see that this symmetry that you take x and you rotate it and your Lagrangian does not change is a fairly obvious thing just by visual inspection, you can easily see that that is a symmetry. So, that is why I have repeatedly told you that humans can spot symmetries in all kinds of places whether it is a visual symmetry, auditory symmetry all kinds of sensory organs can spot all kinds of symmetries; but however, no sensory organs can pick up a conserved quantity that easily.

So, what Noether's theorem allows you to do is; once you have spotted the symmetry, it immediately implies a conserved quantity and that is the real power of this technique. So, therefore, Q is a conserved quantity. So, let us proceed further and evaluate Q explicitly and you will see that this, the generalized momentum in this context is basically the rate of change of this theta variable with respect to time.

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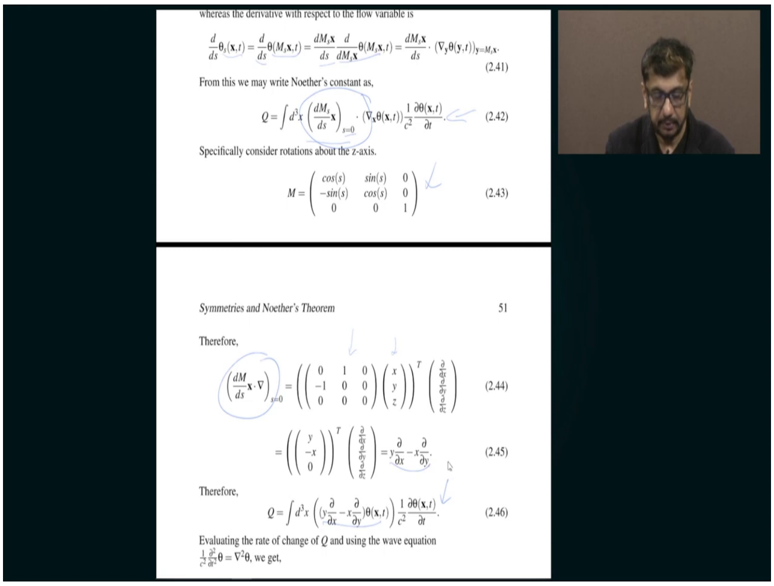

whereas the derivative with respect to the flow variable is

$$\frac{d}{ds}\theta_s(\mathbf{x},t) = \frac{d}{ds}\theta(M_s\mathbf{x},t) = \frac{dM_s\mathbf{x}}{ds}\frac{d}{dM_s\mathbf{x}}\theta(M_s\mathbf{x},t) = \frac{dM_s\mathbf{x}}{ds}\cdot(\nabla_{\mathbf{y}}\theta(\mathbf{y},t))_{\mathbf{y}=M_s\mathbf{x}}. \quad (2.41)$$

From this we may write Noether's constant as,

$$Q = \int d^3x \left(\frac{dM_s}{ds}\mathbf{x}\right)_{s=0}\cdot(\nabla_{\mathbf{x}}\theta(\mathbf{x},t))\frac{1}{c^2}\frac{\partial\theta(\mathbf{x},t)}{\partial t}. \quad (2.42)$$

Specifically consider rotations about the z-axis.

$$M = \begin{pmatrix} cos(s) & sin(s) & 0 \\ -sin(s) & cos(s) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2.43)$$

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Therefore,

$$\left(\frac{dM}{ds}\mathbf{x}\cdot\nabla\right)_{s=0} = \left(\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right)^T \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \quad (2.44)$$

$$= \left(\begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}\right)^T \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \quad (2.45)$$

Therefore,

$$Q = \int d^3x \left((y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})\theta(\mathbf{x},t)\right)\frac{1}{c^2}\frac{\partial\theta(\mathbf{x},t)}{\partial t}. \quad (2.46)$$

Evaluating the rate of change of Q and using the wave equation $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\theta = \nabla^2\theta$, we get,

So, this is easy, this just comes from the Lagrangian; but what is not so easy is this.

So, I have to evaluate the rate of change of the variable in question with respect to this flow parameter. So, remember that theta s is defined like this; it is the replacing of the original thetas X with M s into X. So, that gives you a new theta called theta subscript s. So, now, the derivative of this new thing with respect to s is going to look like this, this times this, ok.

So, now I evaluate this and then of course, I specialize to s equal to 0; because I have told you earlier that finally it does not matter what that s is, because it is going to be independent of s ok, finally Q is independent of s. So, I might as well call s equal to 0 as what I am looking for.

So, now specifically, so I have reduced the calculation of this to something simpler namely this. So, I have to evaluate such a quantity. So, now, you see how do you do this. So, specifically let us you cannot do it in general if you do not know what sort of rotations you are talking about. So, now, let us specifically focus on rotations that correspond to rotating about the z axis by some angle called s. So, in that case this matrix M is going to look like this, ok. So, if m is like that then you can see that it is clearly, right.

So, this much is what I have written there. So, this much can be written in this way, ok. So, there is a d M by d s evaluated at s equal to 0 is going to be like this and there is the gradient there. So, now, when you do this calculation, you will see that it is this into this, ok. So, now, this is a conserved quantity corresponding to this rotating x by some amount ok, that is what Noether guarantees.

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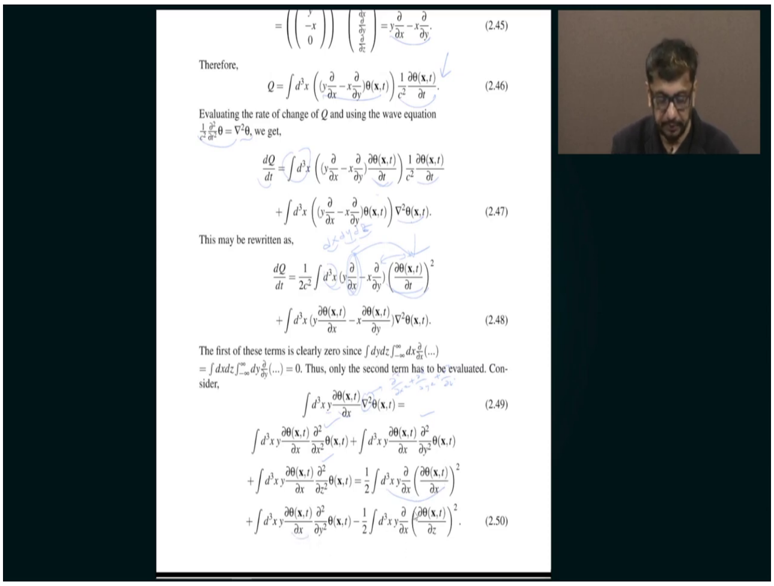

So, now, the question is it looks complicated, so it is not very obvious that it is a conserved quantity just. So, if you are the kind suspicious kind and you are not convinced.

So, then you can just go ahead and take the rate of change of Q with respect to time and you will be able to show and convince yourself that is in fact 0, so therefore Q is conserved. So, since we are somewhat suspicious, because this looks complicated; let us go ahead and find explicitly the rate of change of Q. So, if I do that, I have to do this, then I have to do the second derivative here; but then the second derivative is, because it theta weighs the wave equation it is this and like I told you if you have d by d theta squared, right.

So, that is what this is. So, you will have a d by d theta squared, right. So, now, bottom line is. So, the first term is 0, because you see you are supposed to integrate over x, y and so, you are integrating over a volume. So, x, y and z are independent. So, suppose I fix y and decide to integrate over x, ok. So, this term is going these are derivative with respect to x and I am integrating because this is going to be d x, d y, d z. So, if I fix y and try to integrate x, then I will be forced to I mean fix by means like I want to integrate y and z later.

So, I want to integrate x first; if I decide to integrate x first, then I will be integrating the derivative of something with respect to x. So, derivative with respect to x, integral with respect to x; so the two will cancel out and you will get this function, evaluated of course this is over all space. So, like I have repeatedly told you all these functions vanish rapidly at infinity at all the boundaries of the r 3 basically the space of all points. So, in which case this is basically a boundary term which goes away, this is a boundary term that goes away.

And so, it is only the second term which will remain ok. So, to evaluate the second term, so you will have to spend some effort; you will have to explicitly write this as a d squared by d x squared plus d squared by d y squared plus d squared by d z squared, because this is Cartesian, this is y and all that, so better work in Cartesian. So, it is going to be d squared by x this going to be d z squared. So, then you will see that each of these terms can be split up like this; you can write this as a derivative of the square, then this itself is derivative and derivative like this.

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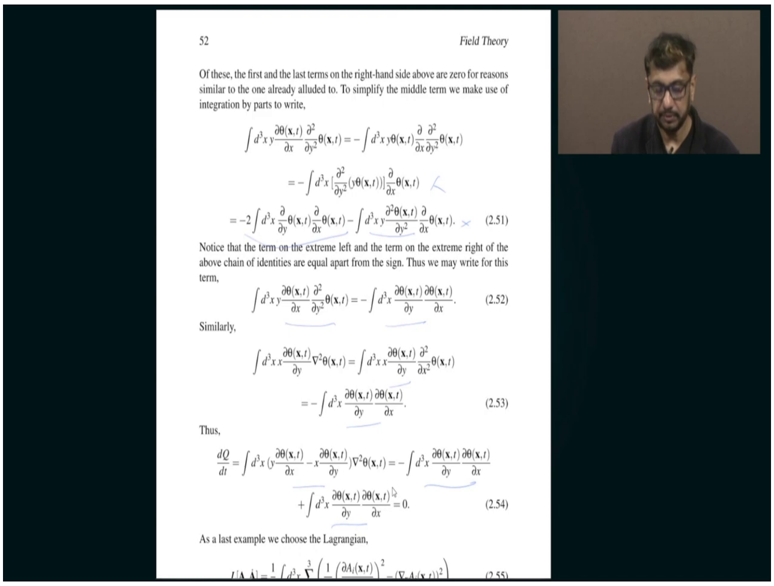

So, this is 0 for reasons that I already told you, ok.

So, the middle term you can simplify. So, this is less obvious, right. So, why this is 0? So, you have to split the, you do this integration by parts type of thing, right. So, you can throw this derivative on this side right and then you throw the derivative of this on that side. So, you will get this term. So, this is just integration by parts, ok. So, the bottom line is that, these terms together with these terms will cancel out. So, they will cancel out in pairs. So, you can just see that from these calculations, ok.

So, you will see that they cancel out in pairs; because it is hard for me to verbally explain things here, lot of things going on here. So, there are lots of terms and they all cancel out in pairs bottom line. So, eventually you will end up with this result, so that finally, they all pair up and cancel out and then towards the end you will be left with two terms, which also finally cancel out.

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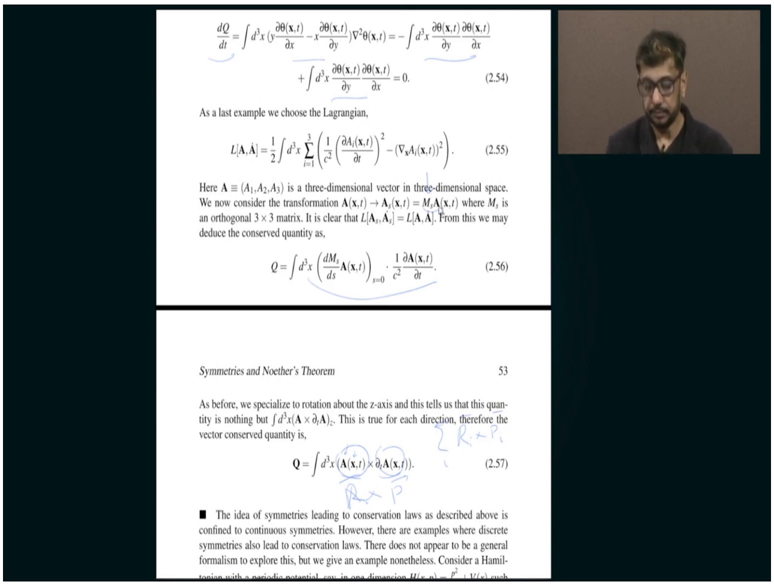

$$\frac{dQ}{dt} = \int d^3x \, (y\frac{\partial\theta(\mathbf{x},t)}{\partial x} - x\frac{\partial\theta(\mathbf{x},t)}{\partial y})\nabla^2\theta(\mathbf{x},t) = -\int d^3x \, \frac{\partial\theta(\mathbf{x},t)}{\partial y}\frac{\partial\theta(\mathbf{x},t)}{\partial x} + \int d^3x \, \frac{\partial\theta(\mathbf{x},t)}{\partial y}\frac{\partial\theta(\mathbf{x},t)}{\partial x} = 0. \quad (2.54)$$

As a last example we choose the Lagrangian,

$$L[\mathbf{A},\dot{\mathbf{A}}] = \frac{1}{2}\int d^3x \sum_{i=1}^{3}\left(\frac{1}{c^2}\left(\frac{\partial A_i(\mathbf{x},t)}{\partial t}\right)^2 - (\nabla_{\mathbf{x}}A_i(\mathbf{x},t))^2\right). \quad (2.55)$$

Here $\mathbf{A} \equiv (A_1, A_2, A_3)$ is a three-dimensional vector in three-dimensional space. We now consider the transformation $\mathbf{A}(\mathbf{x},t) \to \mathbf{A}_s(\mathbf{x},t) = M_s\mathbf{A}(\mathbf{x},t)$ where M_s is an orthogonal 3×3 matrix. It is clear that $L[\mathbf{A}_s,\dot{\mathbf{A}}_s] = L[\mathbf{A},\dot{\mathbf{A}}]$. From this we may deduce the conserved quantity as,

$$Q = \int d^3x \left(\frac{dM_s}{ds}\mathbf{A}(\mathbf{x},t)\right)_{s=0} \cdot \frac{1}{c^2}\frac{\partial\mathbf{A}(\mathbf{x},t)}{\partial t}. \quad (2.56)$$

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As before, we specialize to rotation about the z-axis and this tells us that this quantity is nothing but $\int d^3x(\mathbf{A} \times \partial_t\mathbf{A})_z$. This is true for each direction, therefore the vector conserved quantity is,

$$\mathbf{Q} = \int d^3x \, (\mathbf{A}(\mathbf{x},t) \times \partial_t\mathbf{A}(\mathbf{x},t)). \quad (2.57)$$

■ The idea of symmetries leading to conservation laws as described above is confined to continuous symmetries. However, there are examples where discrete symmetries also lead to conservation laws. There does not appear to be a general formalism to explore this, but we give an example nonetheless. Consider a Hamil-

So, bottom line is that with some effort you can show that, this Q is a conserved quantity; because if you explicitly evaluate the derivative, you get 0, ok.

So last example till now what we have done symmetry means, we have. So, that is the thing about field theories. So, there is no, there is no rule that you should only continuously modify x or anything; you can continuously modify anything so long as it is a continuous transformation, which leaves the Lagrangian invariant. So, we chose to rotate x, we chose to translate x; but now we did not touch theta, because earlier it was a scalar, but now I have generalized this concept to a vector dependence on the theta. So, instead of theta, I have a vector dependence.

So, your Lagrangian is a function of a vector field rather than a scalar field right, it is a function of a vector field. So, if that is the case, then that it affords a different transformation in addition to what we can still do what we did earlier; we can take x and replace x by x plus some constant vector, we will of course get back the same conserved quantity. But we can also take x and rotate it, we will get back the similar conserved quantity. But now in addition to that we can do something else when that instead of theta being a scalar, if that theta was replaced by A which is a vector, then you can do something more interesting that instead of rotating x, you can rotate a itself.

So, that is what I have done here. So, I rotate A, then clearly this is even more obvious that it is a symmetry; that means it leaves the Lagrangian invariant. So, if that is the case, then you see immediately we write down the conserved quantity as usual and then you will see that this basically amounts to something similar to angular momentum. So, this is like a X and this is like a P or like this is an R and this is a P. So, this is your generalized coordinate; notice that this A is your generalized coordinate X is like R i. So, R i cross P i summed over i. So, it has that flavour.

So, this is velocity, but basically time rate of change of R i is proportional to momentum. So, something like angular momentum. So, that is hardly surprising, because what this corresponds to is the rotation of the A vector continuously. So, if the Lagrangian is invariant, there should be something analogous to angular momentum which is conserved, which is what that is, ok. So, that is nice to know.

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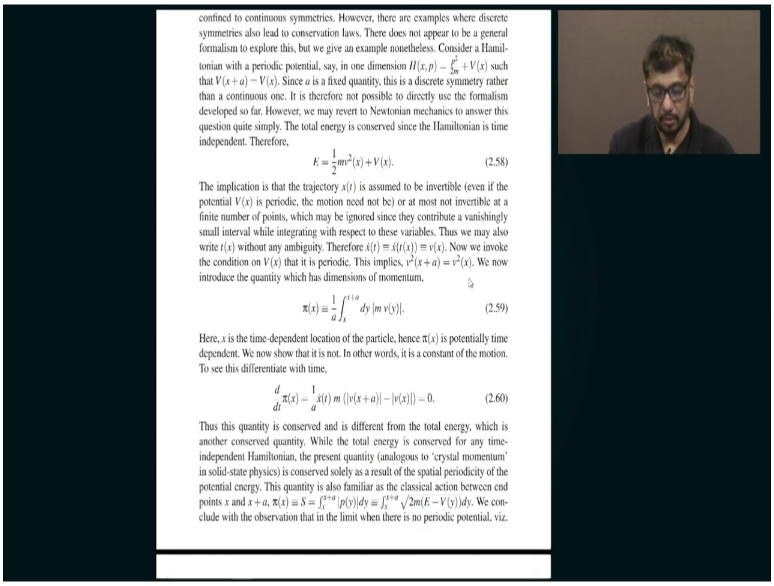

confined to continuous symmetries. However, there are examples where discrete symmetries also lead to conservation laws. There does not appear to be a general formalism to explore this, but we give an example nonetheless. Consider a Hamiltonian with a periodic potential, say, in one dimension $H(x,p) = \frac{p^2}{2m} + V(x)$ such that $V(x+a) = V(x)$. Since a is a fixed quantity, this is a discrete symmetry rather than a continuous one. It is therefore not possible to directly use the formalism developed so far. However, we may revert to Newtonian mechanics to answer this question quite simply. The total energy is conserved since the Hamiltonian is time independent. Therefore,

$$E = \frac{1}{2}mv^2(x) + V(x). \tag{2.58}$$

The implication is that the trajectory $x(t)$ is assumed to be invertible (even if the potential $V(x)$ is periodic, the motion need not be) or at most not invertible at a finite number of points, which may be ignored since they contribute a vanishingly small interval while integrating with respect to these variables. Thus we may also write $t(x)$ without any ambiguity. Therefore $\dot{x}(t) \equiv \dot{x}(t(x)) \equiv v(x)$. Now we invoke the condition on $V(x)$ that it is periodic. This implies, $v^2(x+a) = v^2(x)$. We now introduce the quantity which has dimensions of momentum,

$$\pi(x) \equiv \frac{1}{a}\int_x^{x+a} dy\, |m\, v(y)|. \tag{2.59}$$

Here, x is the time-dependent location of the particle, hence $\pi(x)$ is potentially time dependent. We now show that it is not. In other words, it is a constant of the motion. To see this differentiate with time,

$$\frac{d}{dt}\pi(x) = \frac{1}{a}\dot{x}(t)\, m\, (|v(x+a)| - |v(x)|) = 0. \tag{2.60}$$

Thus this quantity is conserved and is different from the total energy, which is another conserved quantity. While the total energy is conserved for any time-independent Hamiltonian, the present quantity (analogous to 'crystal momentum' in solid-state physics) is conserved solely as a result of the spatial periodicity of the potential energy. This quantity is also familiar as the classical action between end points x and $x+a$, $\pi(x) \equiv S = \int_x^{x+a} |p(y)| dy \equiv \int_x^{x+a} \sqrt{2m(E - V(y))} dy$. We conclude with the observation that in the limit when there is no periodic potential, viz.

So, I have shown some examples where. So, we just.

So, the point is that, while continuous symmetry is always guarantee conservation laws, but discrete symmetries do not always guarantee it.

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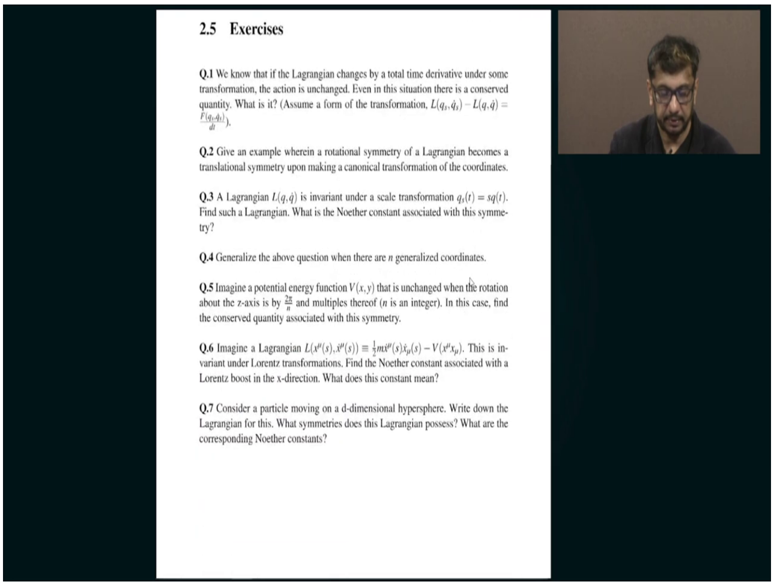

But then you can concoct peculiar examples, where in fact discrete symmetry does mean some conservation law. So, I do not want to spend too much time on that.

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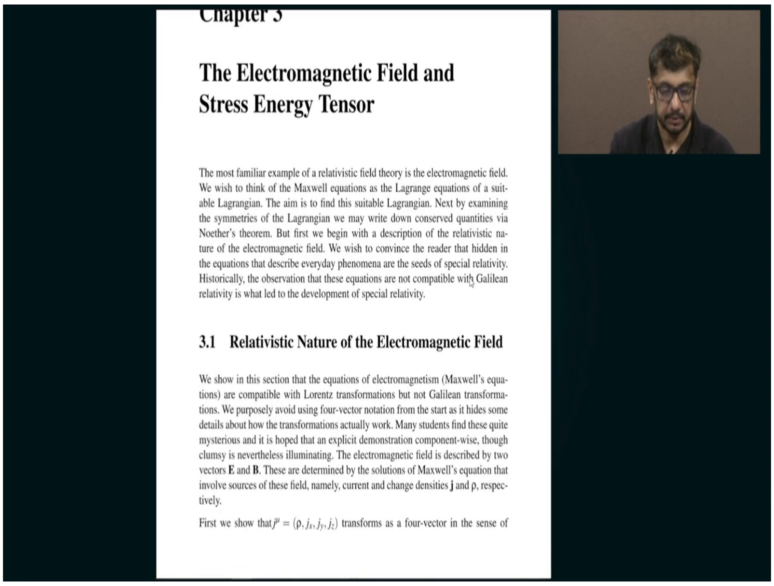

So, now, I want to start a different chapter namely. So, till now what I have done is, basically I have introduced the concept of a field; so that means remember what a field is. So, you have to first remember what a dynamical system is.

So, a dynamical system has some degrees of freedom, which we call generalized coordinates and they are function of time. So, you can write down a Lagrangian which is a function of the coordinates and the their rates of change. So, that will lead to some dynamical loss of motion. So, these generalized coordinates are finite in number typically when you encounter them; but in field theory what we do is we generalize this idea to infinitely many generalized coordinates.

So, instead of having Q 1, Q 2, Q 3; we are now have a Q bracket some X, where x is now continuous. So, that x is not position, but it basically takes on the role of i, where i is 1, 2, 3; now X is continuously changing from whatever to whatever. So, that is basically the idea behind a field theory, that is you are replacing generalized coordinates by their continuously varying counterparts, ok. So, having done that, we then were able to show that these sorts of continuously infinitely many degrees of freedom allows us many possibilities.

So, it allows us to do continuous transformations that leave Lagrangian in invariant in various ways. So, when Lagrangian's are invariant under continuous symmetries, Noether's theorem guarantees conservation laws. So, it guarantees that there are quantities that are independent of time.

So, basically we have succeeded in listing a plethora of conserved quantities, which correspond to these continuous symmetries, which are far more numerous in the context of field theories than in the context of systems with finite number of degrees of freedom; simply because of the really huge number of ways in which you can implement these transformations when you have all these infinitely many degrees of freedom. So, that we had stopped right there.

So, now, I want to discuss a new chapter which is of course, more practical; but it is going to use the pretty much the same ideas that I had started off with, namely that I am going to identify the electric and magnetic fields in the Maxwell's equation as a field a dynamical field. So, that means that you have instead of finite number of degrees of freedom, you have infinitely many degrees of freedom. Then I am going to show that the

Maxwell's equation which are the dynamical equations of the electromagnetic field, can be actually thought of as the Euler Lagrange equation of a suitable Lagrangian.

So, I am going to identify generalized coordinates, generalized momenta, so that will lead to the Euler Lagrange equations, which are precisely the Maxwell equations, ok. So, that is I am going to do that towards the end, then I am going to use symmetries and write down conserved quantities and so on. Before I do that I want to explain to you what the electromagnetic theory looks like and specifically the relativistic nature of the electromagnetic field.

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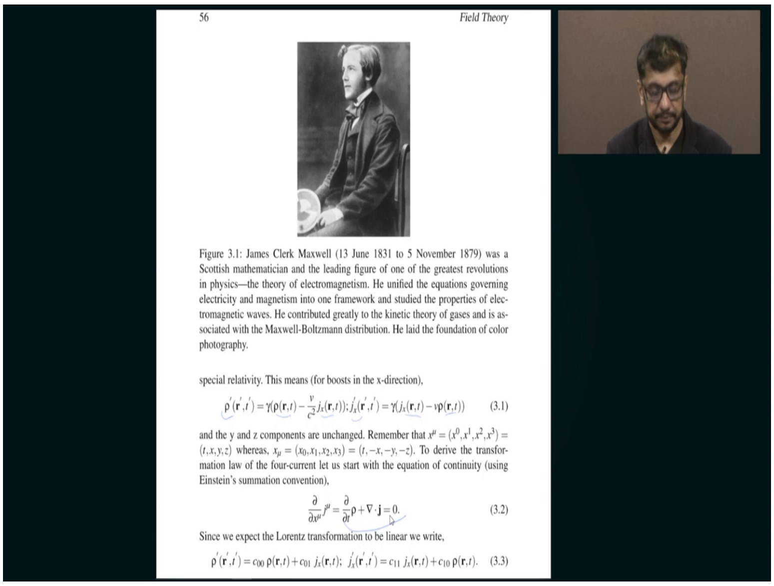

So, many students kind of do not really understand this why it is that Maxwell's equations are said to be consistent with special relativity, but not with Galilean relativity.

So, that is repeatedly pointed out, but not properly explained in many books. And of course, even when they explain it many books start with a four vector notation and make the whole proof so compact and that it becomes very opaque and students simply do not understand, the it feels like magic; because the four vector notation makes every statement so incredibly compact, that it is hard to see what assumptions have gone into the proof. And basically you would not be able to see that proof working out in all the steps and details; because many of the steps are swallowed up by the four vector notation.

So, the advantage of the four vector notation it makes proofs very quick. So, the brevity is a very big advantage; but it is also a disadvantage in this for a beginner, because they feel intimidated by the brevity, it seems a bit too quick for most students. So, just to avoid that I have decided to explicitly list out the steps involved in proving that Maxwell's equations are consistent with special relativity rather than Galilean relativity by expanding it out in components and looking at boosts along the x direction as it is customary in special relativity, ok.

So, when I do that you see. So, to do that first, I have to remind you what Maxwell's equations are all about. So, you see in Maxwell's theory you have sources of electromagnetic field which are basically the charge density and current densities. So, these sources lead to or they generate the electromagnetic field.

So, the idea is that, these charge densities and current densities obey what is called a continuity equation; that guarantees that the electric charge in any volume is conserved. So, that means that; so if the total electric charge is increasing or decreasing in a volume it is mainly because or it is only because the current is either flowing in or out.

So, if no current is flowing in or out, the continuity equation guarantees that the total charge in that volume is fixed. So, it will not spontaneously appear out of nowhere or suddenly disappear on its own. So, this is the conservation of electric charge. What I am going to show you now is that, I can think of this collection of four properties of the charge distribution as a relativistic four vector. So, I am going to collect these quantities in this form. So, I am going to define a vector, I mean basically a four vector ok; it is not right now it is not clear that it is a four vector in the relativistic sense, but, I am going to prove it.

But I am just going to collect them in the form of a ordered collection of four objects. So, the first one is the density of, I mean the electric charge density; the second is the x component of the current density, y component of this current density, the third component in this ordered collection of four objects and then lastly it is the z component of the current density. So, I have these three quantities. Now, I want to show you that these quantities obey under boost. So, if I boost you know what a boost is basically I move to a reference frame, where I travel relative to the original reference frame in the x direction with some velocity v.

So, that is called a boost in the x direction. So, what I am going to show you is that, when I do that and I measure the electric, I mean the charge density and the current density in the moving frame; that is going to be rho dash and j dash, that is going to be related to rho and j in this very interesting fashion which is very reminiscent of the original Lorentz transformation involving space time coordinates, ok. So, how do I prove this, ok. So, to prove that, I first start with the continuity equation. So, if you recall the continuity equation says that the rate of change of the charge density is basically the negative divergence of the current.

So, what that means is that, basically if charge density is increasing; that is because current is converging into the that region, right. So, conversely if the charge density is decreasing with time is, it is because current is diverging out of that region, ok.

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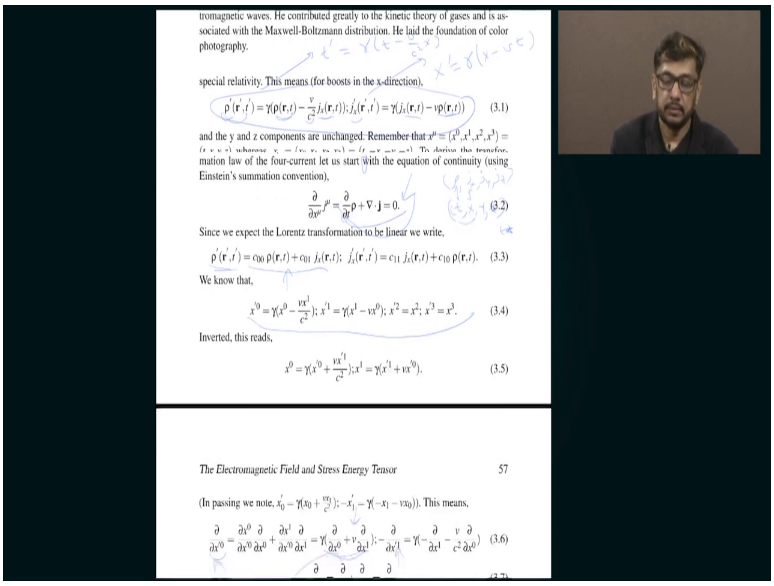

So, that is what continuity equation says. So, now, I am going to ask myself. So, the question the fundamental question that we have to ask ourselves is that, which transformation of rho and j leaves this invariant; that means that I want this continuity equation to have the same form in all reference frames.

So, I if I replace rho by rho dash, t by t dash, r vector by r vector dash, j j vector by j vector dash; I should get back the same continuity equation. So, the question is which is that transformation which does that? So, but for space time coordinates we know what that is and that is the Lorentz transformation; but for rho and j it is certainly not obvious that it is I mean. So, that is what we want to find. So, how should rho and j transform? So, is there a simple way in which rho and j transform, so that the continuity equation looks the same and you will see that in fact there is.

And the way to do that is you replace rho dash by linear combination of rho and x. So, you see this is a reminiscent of the rho j x, j y, j z. So, this is time, this is x, this is y, this is z, ok. So, the time corresponds to charge density, x corresponds to; so this is as far as transformations are concerned, so rho j x transforms the same way as t and x transform. So, t becomes t dash, x becomes x dash under Lorentz transformation and the exact you know exactly how x dash depends on x and d under Lorentz transformation.

So, in fact rho dash j x dash will depend in exactly the same way on j x and rho; just as x dash dependent on x and t in some particular way. So, in fact that is what we are going to prove now, that is not obvious at this stage. So, in order to prove that, first I am going to postulate that; because all these equations are linear, obviously the transformed quantities had better be linear, because if they are not it is at the outset clear that the whole thing is inconsistent. So, linearity is a must, because it was linear in one reference frame, it has to be linear in the other reference frame; because otherwise the equations do not look the same.

Bottom line is that if you assume linearity, then you will see that you can always rewrite this as linear combinations of so rho and x ok. So, you see this will involve derivative with respect to time and time in the new reference frame will get replaced by t dash and rho will get replaced by rho dash. So, you see now we know how derivative with respect to time in the new reference frame will look like with respect to the derivative with respect to time in the old reference frame and the spatial derivative.

So, remember that x 0 is c t and well I have said t itself. So, x 0 is t and x 1 is yeah the superscript is basically t, x, y, z the subscript is. So, these are the contravariant and covariant forms. So, I am using the four vector notation only in the context of trying to relate to special relativity; because you already know four vectors and how the spatial coordinates transform. So, the with respect to the electric magnetic field and currents and densities, it is not clear that they are four vectors. So, that is what we are trying to prove that they in fact are four vectors in the sense in which you know time and position form a four vector.

So, in order to do this, we have to transform to the new reference frame and then you will see that it is the derivative with respect to time in the new reference frame is writeable with in terms of the derivative with respect to time in the old reference frame and the spatial derivative with respect to old reference.

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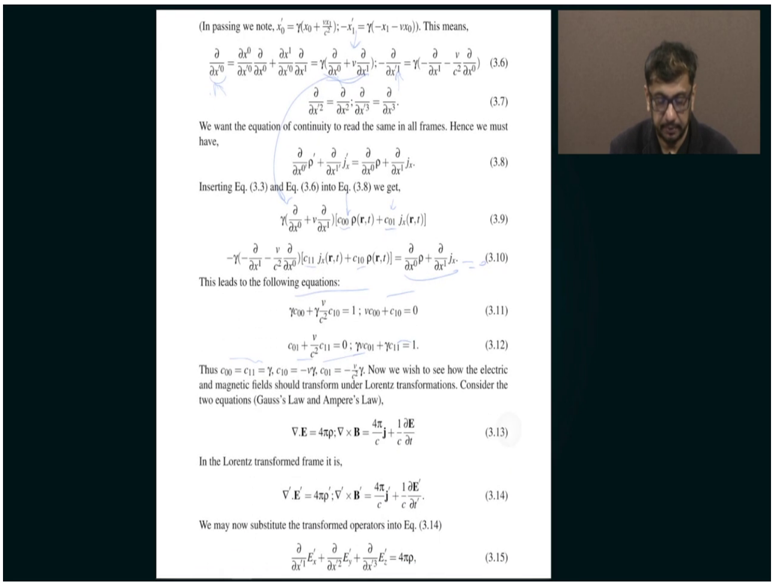

So, same with the spatial derivative with respect to the old reference; I mean, so same with spatial derivative. So, this was time derivative new reference frame, spatial derivative in the new reference frame.

So, now you go ahead and demand that this equation should be the same with just all the quantities replaced by primes. So, t gets replaced by t prime, rho gets replaced by rho prime, grad gets replaced by grad prime and so on. So, when you do that and you go ahead and substitute this here, you will see that. So, this is these constants or these coefficients, so what I do not know and trying to evaluate. So, I replace that by this and then I demand that this looks the same in, so this should look like this. So, because this is 0, this is also 0, ok.

So, then this will immediately lead to these equations, ok. And then when I solve them, I get this these answers ok and this is precisely telling you that the, so basically you will arrive at this. So, rho dash, so rho dash transforms exactly how t would have transformed. So, you see how does t transform; it is gamma t minus p by c squared into x. So, this is like t dash, this is like t, this is v by c squared, this is j x is like x. So, similarly here x dash is gamma x minus v t. So, it is v t, ok. So, they transform the same way.

So, therefore, rho j x, j y, j z are four vectors.

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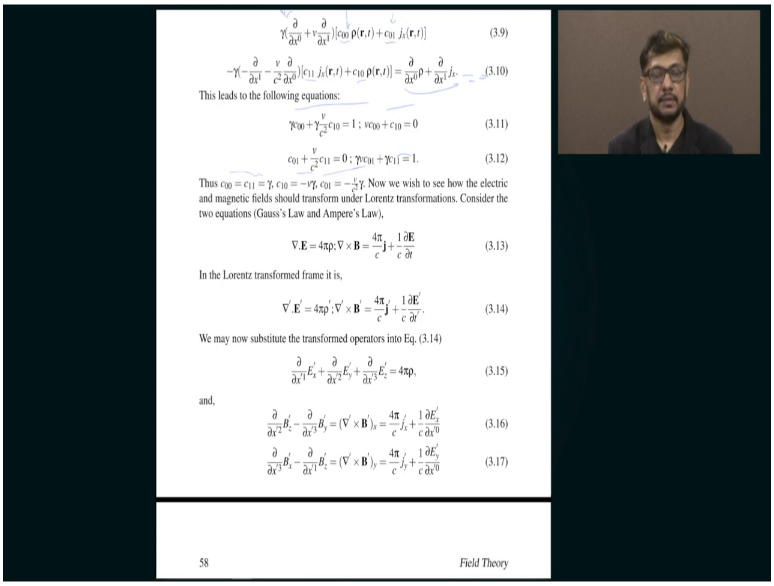
$$\gamma(\frac{\partial}{\partial x^0}+v\frac{\partial}{\partial x^1})[c_{00}\,\rho(\mathbf{r},t)+c_{01}\,j_x(\mathbf{r},t)] \quad (3.9)$$

$$-\gamma(-\frac{\partial}{\partial x^1}-\frac{v}{c^2}\frac{\partial}{\partial x^0})[c_{11}\,j_x(\mathbf{r},t)+c_{10}\,\rho(\mathbf{r},t)]=\frac{\partial}{\partial x^0}\rho+\frac{\partial}{\partial x^1}j_x. \quad (3.10)$$

This leads to the following equations:

$$\gamma c_{00}+\gamma\frac{v}{c^2}c_{10}=1\,;\; v c_{00}+c_{10}=0 \quad (3.11)$$

$$c_{01}+\frac{v}{c^2}c_{11}=0\,;\; \gamma v c_{01}+\gamma c_{11}=1. \quad (3.12)$$

Thus $c_{00}=c_{11}=\gamma$, $c_{10}=-v\gamma$, $c_{01}=-\frac{v}{c^2}\gamma$. Now we wish to see how the electric and magnetic fields should transform under Lorentz transformations. Consider the two equations (Gauss's Law and Ampere's Law),

$$\nabla.\mathbf{E}=4\pi\rho;\nabla\times\mathbf{B}=\frac{4\pi}{c}\mathbf{j}+\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} \quad (3.13)$$

In the Lorentz transformed frame it is,

$$\nabla^{'}.\mathbf{E}^{'}=4\pi\rho^{'};\nabla^{'}\times\mathbf{B}^{'}=\frac{4\pi}{c}\mathbf{j}^{'}+\frac{1}{c}\frac{\partial\mathbf{E}^{'}}{\partial t^{'}}. \quad (3.14)$$

We may now substitute the transformed operators into Eq. (3.14)

$$\frac{\partial}{\partial x^{'1}}E^{'}_x+\frac{\partial}{\partial x^{'2}}E^{'}_y+\frac{\partial}{\partial x^{'3}}E^{'}_z=4\pi\rho, \quad (3.15)$$

and,

$$\frac{\partial}{\partial x^{'2}}B^{'}_z-\frac{\partial}{\partial x^{'3}}B^{'}_y=(\nabla^{'}\times\mathbf{B}^{'})_x=\frac{4\pi}{c}j^{'}_x+\frac{1}{c}\frac{\partial E^{'}_x}{\partial x^{'0}} \quad (3.16)$$

$$\frac{\partial}{\partial x^{'3}}B^{'}_x-\frac{\partial}{\partial x^{'1}}B^{'}_z=(\nabla^{'}\times\mathbf{B}^{'})_y=\frac{4\pi}{c}j^{'}_y+\frac{1}{c}\frac{\partial E^{'}_y}{\partial x^{'0}} \quad (3.17)$$

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So, in the next class, I am going to stop here; because in the next class I am going to show that not only the continuity equations transform in a simple way and in fact they are so simple that they are identical to the position and time four vector. But then the there are other quantities in Maxwell's theory, namely it is not just the density charge density and current density; the other quantities are basically the electric and magnetic fields. So, it is a perfectly good question to ask how do they transform under Lorentz transformation.

So, for that you will see that they transform in a certain way which corresponds to actually basically a tensor; but I do not necessarily have to speak in that language, I will explicitly tell you how the components of the electric field transform and how the magnetic field transforms and they do not look too similar to position and time transformations, but they are still sufficiently similar that you will be able to see a pattern there, that also shows the relativistic nature.

So, that basically completes the description or the proof that the electromagnetic fields are in fact consistent with Lorentz transformation rather than Galilean transformation, ok. So, because anything which is consistent with Lorentz transformation is certainly not consistent with Galilean transformation; because in Lorentz transformation speed of light is unchanged, in Galilean transformation it is not. So, the moment was some set of equation is consistent with Lorentz transformation, it is automatically the case that they are not consistent with Galilean transformation.

So, I am going to leave that for the next class. So, I am going to show you, I just showed you how the current density and charge density transform under Lorentz transformation. So, the remaining parts of the Maxwell equations involve the electric and magnetic field vectors and I am going to show you how the components of those vectors transform under Lorentz transformation. So, that is for the next class. So, I hope to join you in the next class and then after that I will start talking about other subjects, ok.

Thank you.