Dynamics of Classical and Quantum Fields: An Introduction
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Symmetries
Lecture - 07
Dynamical Symmetries

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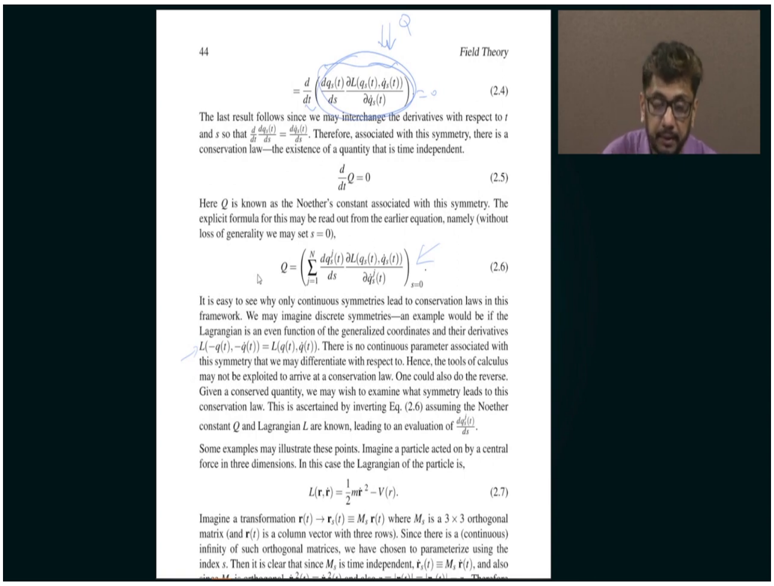

So, let us continue our discussion of Noether's theorem. So, we have if you recall we succeeded in showing that associated with a continuous symmetry of the Lagrangian, there is a conserved quantity and that conserved quantity is called Noether's constant and it may be written in this way.

So, we have successfully showed that Noether's constant has an expression in terms of the Lagrangian and how the generalized coordinates change with respect to the symmetry transformation. So, that is d q by d s would be that. Whereas, the Lagrangian the generalized velocities will be involved next to that rate of change.

So, bottom line is that the knowledge of the Lagrangian together with the transformation that leaves the Lagrangian unchanged is sufficient for you to, sufficient to allow you to construct explicitly the conserved quantity associated with that symmetry. So now, let me give you some examples that will convince you of the usefulness of this idea.

But before I do that, I want to point out that it is very necessary for the symmetry in question to be continuous. In the sense that there should be a continuous variable called s, when you continuously change that variable. So, the generalized coordinates continuously change with respect to that variable, it is only when that happens can you differentiate with respect to s.

So, remember that this Q involves d by ds of the generalized coordinate. So, implying therefore, that Q at the very least is a continuous function of s ok. So, bottom line is that there has to be a continuous symmetry. But the question is why cannot you have a discrete symmetry also leading to a conserved quantity?.

Well perhaps you can that will be an accident, but the fundamental reason why this particular formalism does not allow you to make such a statement is because you see if you take a discrete symmetry such as Lagrangian, which is unchanged if you flip the sign of the generalized coordinate of the generalized velocity, that is certainly a symmetry and it is a discrete symmetry.

Because we are just changing the sign of; so you either change or you do not change. So, that is a discrete symmetry, but then that does not obviously, lead to any conserved quantity because in this particular way of thinking about it you need something to differentiate with. So, you need to, you need to start with an assertion such as d by ds of L equals 0.

So, there is no continuous parameter there. So, because of that discrete symmetries are not part of this discussion ok. So, let me go ahead and explain to you some or point out some interesting applications of this Noether's theorem.

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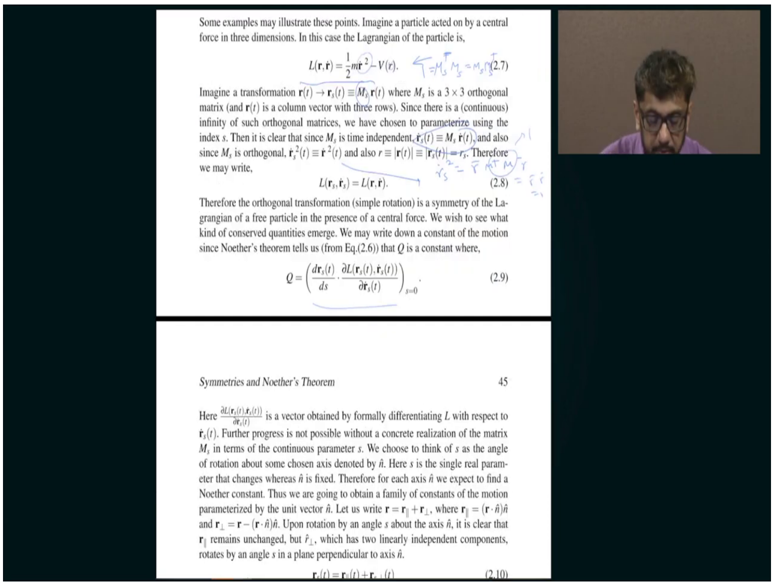

So, imagine I have a Lagrangian of a point particle subject to a central force. So, that is what I have described in this equation. So, you have a Lagrangian subject to a central force directed towards the origin. So, now, I want to ask myself, what is the symmetry of this Lagrangian there are many symmetries, one of them is what I am going to discuss and the specific symmetry I have in mind is the symmetry that rotates the position vector by a certain angle about some axis.

So, basically that sort of a rotation is obviously, captured by this kind of an orthogonal transformation. So, that means, its replaced by r subscript s where s is continuous and that r subscript s is basically an orthogonal matrix called capital M subscript s right. So, times position vector. So, I am going to assume this is an orthogonal matrix. So, if you recall an orthogonal matrix is something whose transpose is it is inverse, is not it. So, I have an orthogonal matrix here ok.

So, this would be the most general sort of rotation that you can think of. So, for some general orthogonal matrix this would correspond to some general rotation. So, now, obviously, this M s is a function of s, but not of time. So that means, at a given time you rotate all the position vectors that may be, if you have just one particle there is nothing else, but if you have more than one particle you are supposed to rotate all of them.

So, in this particular example there is only 1 position vector. So, therefore, the velocity vector is just going to be the time derivative which is given by this. And since M is orthogonal the velocity changes, but the speed does not change ok, because the speed is the square of the velocity vector and the square of the velocity vector involves you know. So, if you remember r s dot square is nothing but r M transpose M r ok. So, it is r M transpose M r, is not it.

So, this is nothing but 1, so this is r r; so which is r squared. So, in other words velocity changes r dot becomes r dot subscript s, but the square of the velocity does not change, because it is an orthogonal transformation. So, given that the square does not change and the magnitude of r s for similar reason does not change, which is which is r without a bold face is basically the magnitude of the position vector.

So, that does not change and the velocity square does not change, which is the speed square. So, that means, the Lagrangian is unchanged with respect to this transformation. So now, you see that this is a continuous transformation which preserves the Lagrangian. So, if it does then you can see that there is Noether's theorem guarantees that there is a conserved quantity and that conserved quantity is precisely this.

So, it is rate of change of the position vector with respect to the continuous parameter s times the generalized momentum as it T L by d r dot is the generalized momentum right.

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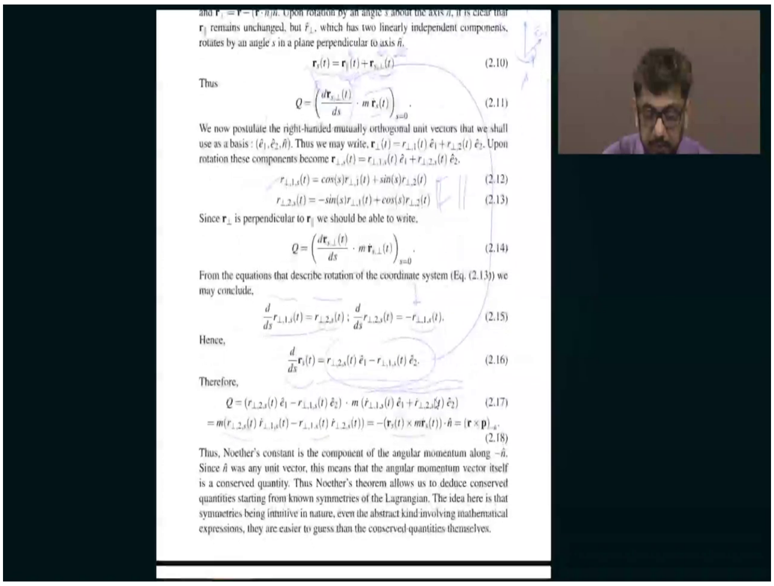

So, now the question is, well this is not particularly illuminating. So, I want to given that this L has this particular specific form, I want to be able to explicitly write this in terms of the functions that are present in L. So, in order to do that so you see d L by d r dot by definition and construction it only is just M times r dot ok. So, there is nothing important there. So, that is why I have written M times r dot, it is just the momentum.

So, now what I am going to do is that, I have to now commit myself to a specific form of this orthogonal transformation so that orthogonal transformation now is going to be represented by a rotation by a certain angle about some fixed axis. So, I am going to assume that this axis that of rotation that I am going to think of is fixed and that is determined by a unit vector called n hat.

So, the n hat is the fixed axis about which I am going to rotate. And the, but the angle through which I rotate is my variable that continuous transformation that I was talking about, that continuous variable that I am going to select is basically the angle of rotation about this fixed axis ok. So, the axis is fixed, but the amount of rotation is a variable which is s ok.

So, if that is the case then obviously, I can write this r vector as a component a projection of r along n cap and this is the rest of it. So, r r parallel is defined as the projection of r vector along the unit vector n hat. And r perpendicular is whatever else is remaining in r. So, once you subtract out r parallel from r whatever is remaining is r perpendicular.

So, if that is the case then you can see that, as I do the rotation r parallel will not change, because it is parallel to the axis of rotation so that vector will not change. So, if I put a subscript s, it is the perpendicular vector which changes, but the parallel vector does not change. But now so, as a result if I since Noether's theorem calls for the finding, the derivative of r s with respect to s you see that this derivative with respect to s drops out because it does not depend on s.

So, there is only a derivative of r perpendicular with respect to s. So, you see you have a axis and you are rotating by some angle s ok and you have a r perpendicular which is pointing like this ok. So, that r perpendicular is your function of s and that is going round and round as you rotate it.

So, now how do you describe a vector that is going round and round by some angle s? And this is how you would describe it right. So, if s is the amount by which it this vector twists in the plane perpendicular to n cap. So, that twisting is determined by this obvious orthogonal transformation. So, given this construction it is easy to see that the rate of change of r.

So, you see the plane perpendicular to n hat is a plane. So, vectors in a plane has to be described using a basis and I have selected some arbitrary basis called e 1, e 1 hat and e 2 hat. And so, these are mutually perpendicular basis vectors which are lying in the plane perpendicular to n hat. So, and r 1 is basically the component of r perpendicular, r perpendicular comma 1 means the component of r perpendicular along this e 1 cap which is arbitrarily chosen.

So, bottom line is that having chosen it, then this sort of a rotation implies that the rate of change of r perpendicular which is parallel to e 1 is basically r perpendicular parallel to e 2 and so on. Similarly, for other derivative with respect to r 2. Now, we go back and try to write down the rate of change of this with respect to s. But, notice that this continues to drop out because the rate of change does not involve s and only this is there ok.

So, r s perpendicular is nothing but r 1 perpendicular e 1 hat plus r 2 perpendicular e 2 hat so that is what I have done, but then the rate of changes will involve this with a minus sign ok. So, you can easily convince yourself that this is what it is. Now, I go ahead and substitute this expression all the way there ok and I go ahead and substitute this here ok.

So, as usual you see because r s perpendicular is in the plane, which is perpendicular to n cap. So, and it is being dot producted with r s dot, I have to ensure that only a components of r s which are perpendicular to n cap or involved. So, which is why there is no n cap component. So, it is r s is of will have n cap also, but then I have ignored that the r parallel part will drop out because that; so even though it is in principle there, but when you take dot product with respect to this which is in a plane perpendicular to n cap it drops out.

So, I have just decided not to include that for brevity, alright. So, the moment you take this dot product you end up getting this and this is nothing but the component of the angular momentum right in the minus n cap direction. So, you can easily convince yourself that this Q, the Noether constant is nothing but the component of the angular momentum of the particle in the direction of minus n cap.

But then keep in mind that n cap was any its fixed, but then it is arbitrary it can be anything, anything which is fixed. So, if it is anything then; obviously, if q is constant that is going to happen only if r cross p itself is constant not necessarily along any particular axis, but in general. So, its only when r cross p is actually fully constant this makes sense ok. So, bottom line is that you see Noether's theorem very beautifully connects the concept of a rotational symmetry to a conserved quantity namely angular momentum.

So, we all know that angular momentum is conserved in a central force situation. But what this theorem very elegantly shows is that this conservation law has a deeper origin. Namely, that deep origin is the fact that your Lagrangian is unchanged with respect to a certain continuous symmetry and this continuous symmetry is the rotational symmetry, alright. So, this is one very nice example.

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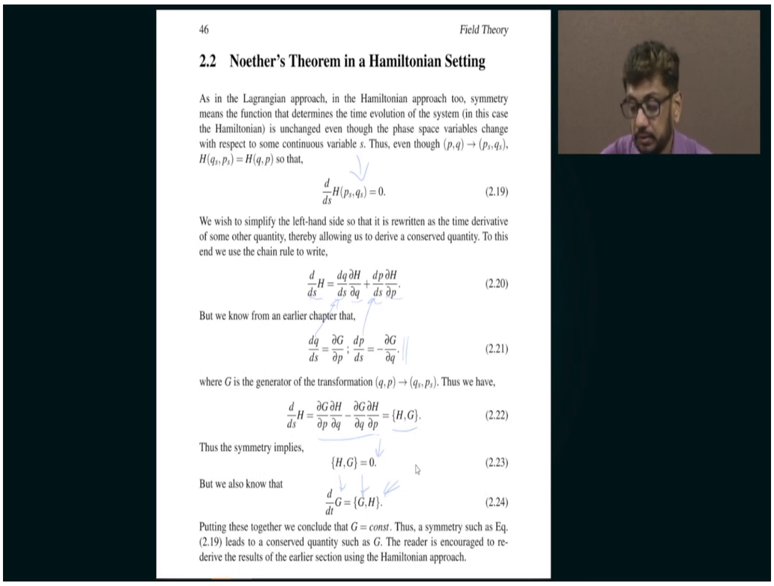

So, now I am going to discuss, I am just going to mention that it is s possible to rework this idea in the context of Hamiltonian's, because if you recall we discussed something called flows earlier and that is going to be useful now. So, that the concept of flows in the context of Hamiltonian mechanics was introduced that here. So, I am going to make use of that now.

So, now, just like it was in the Lagrangian case, a continuous symmetry is postulated which leaves, now the Hamiltonian unchanged just as the earlier case it was a Lagrangian that was unchanged.

So, now I am going to postulate that there is a symmetry that leaves the Hamiltonian unchange. So, if the Hamiltonian is unchanged, you can see that the d by d s of H changes so; that means, even though the generalized momentum p and generalized coordinate q change with respect to the continuous parameter s, the Hamiltonian itself does not. So, now, how would you go about evaluating the rate of change of the Hamiltonian with respect to this continuous parameter?

So, you would basically do it successively. So, H depends on s through p and q. So, you first differentiate H with respect to q then differentiate q with respect to s, then you differentiate H with respect to p and then differentiate p with respect to s. But then now comes the really important step where we make use of the earlier idea of flows. So, remember that we had decided that I can think of this change with respect to s as a kind of a flow.

So, if that is a flow then flow has a generator which I call G. So, the generator of the flow obeys these Hamilton's equations. So, the Hamilton's equations were usually if s was time ok. So, but in this particular case s is that any continuous variable which corresponds to a symmetry. So, given that fact the you can see that the Hamilton's equation for this flow can be written like this for a suitable generator.

So, if that is the case, then you can go ahead and substitute that here right and then you will see that this is writable as. So, the rate of change of H with respect to s is nothing but the Poisson bracket of H with the generator of the symmetry. So, given that H is unchanged under the symmetry, what this implies is that the Poisson bracket of the Hamiltonian and the generator is 0.

So, what this statement says is that, if there is a symmetry there should be a generator which has a 0 Poisson bracket with the Hamiltonian, but now keep in mind that the rate of change of G with respect to time, now time means the actual time, the dynamical parameter which describes the sequence of events. So, as opposed to s which can be some abstract continuous parameter which correspond to some symmetries.

So, now the time rate of change of the generator or any other operator for that matter or any other state function is always writeable as rate of change of that with respect to time is the Poisson bracket of that quantity with the Hamiltonian. Now, we have successfully shown that for a symmetry the that Poisson bracket is actually 0, but if it is 0 so what combining these two ideas allows us to conclude that this generator of the symmetry is in fact, not only does it generate those symmetries it is also independent of time.

So, as a result it is the constant, it is a constant of the motion. So now, I am going to allow you to convince yourself that in the case of this rotations that we discussed in the earlier example, the generator of rotations is nothing but the appropriate component of the angular momentum. So, then you will be able to show that just like we did earlier, is the angular momentum about any axis which is a constant of the motion.

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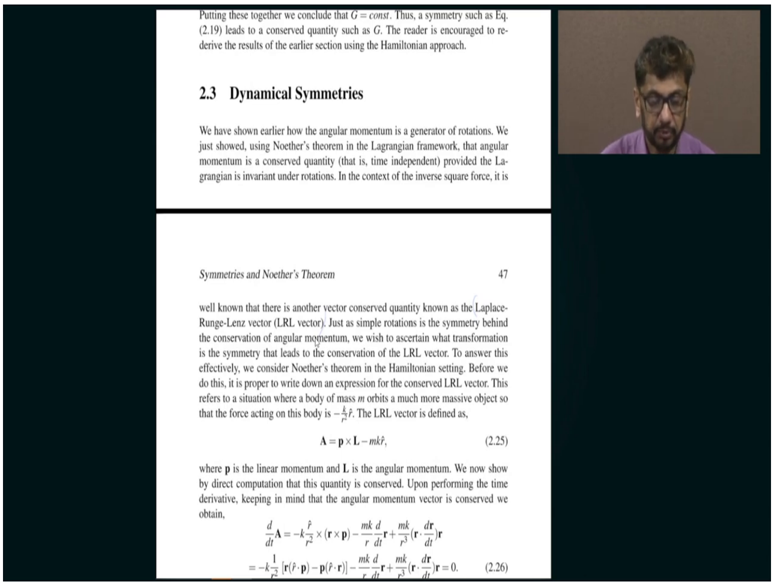

So, in the Hamiltonian setting it appears as the generator of the rotational symmetry. The angular momentum appears or makes its presence felt as the generator of rotational symmetry. So, now, I am going to discuss something which is somewhat less familiar to many people.

And certainly, it was unfamiliar to me when many years ago I learned it on my own and it is an idea that is not often discussed properly in many books. And that is the concept of dynamical symmetries and the reason is of course, because the number of such examples are also somewhat limited. And in fact, I can only think of one example in the present situation and that is the conservation of what is called the Runge-Lenz vector right. So, it is called R L L R Laplace Runge-Lenz vector. So, I will tell you what that is.

So, you see the idea is that if you recall earlier just a while back, we were talking about central forces. So, central forces are basically forces acting on a given particle which are directed towards some origin. So, bottom line is that the functional form of that force or potential is can be anything.

So, even though that is not specified irrespective of that, we were successful in concluding that implies that there is a vector quantity that is conserved and that is the angular momentum. So, regardless of what the nature of the force is. So, long as it is central, the angular momentum vector is in fact, a conserved quantity.

But now, if specifically in addition to the central nature of the force imagine it had some other quality namely that it was actually a columbic force; that means, that it obeyed inverse square law the force dies off as inversely proportional to the square of the distance.

So, if it is not just central force, but also a coulomb force or a you know the Newtonian gravitational force, both of which have this 1 by r squared form. So, if that is the case then what we are going to show is that in addition to the angular momentum, there is another conserved quantity called the Laplace Runge-Lenz vector. And that is very unique to the inverse squared type of force, its not a conserved quantity if the force had any other functional dependence.

So, imagine that instead of 1 by r, if the force was 1 instead of the potential being 1 by r, the potential was 1 by r squared, then immediately Laplace Runge-Lenz vector or there is no other conserved quantity other than angular momentum and total energy. So, bottom line is that. So, this is this symmetry is called an dynamical symmetry for reasons I am going to get too shortly.

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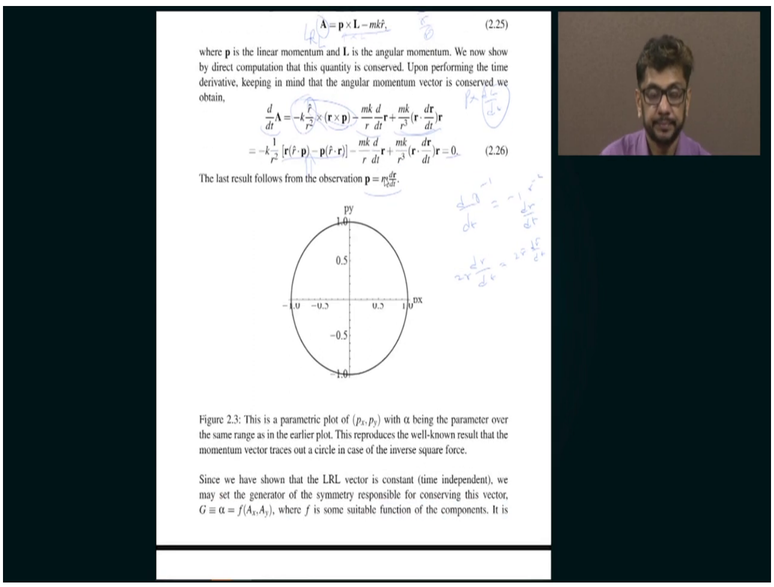

So, bottom line what we are trying to find out is that given the fact that you can easily verify that this peculiar object called the Laplace Runge-Lenz vector which is defined in this way. So, you see A is called the Laplace Runge-Lenz vector LRL vector. So, that is defined in this peculiar way. So, imagine that the force acting on the particle is minus some constant called k divided by r squared and it is directed towards the origin so; that means, k k by r squared is the strength of the force and the direction is minus r hat.

So, it is directed towards the origin then I am entitled to define a quantity called L r L vector which is defined in this fashion. So, it is linear momentum cross product angular momentum minus mass times that constant k in the r hat direction. So, this is a very peculiar rather arbitrary looking definition of a yet to be understood vector called LRL vector.

So, it is not at all clear why we would invoke something like this. So, the reason for that is that it is easy to show that this is a conserved quantity. Firstly, even before that it is immediately obvious that this A has nothing to do with L, in the sense that A and L are not the same at least, I mean it has something to do with L, but A is completely different from L; it is not.

So, the conservation of L does not necessarily imply conservation of A for example, because there is a p sitting next to the L is certainly not conserved, linear momentum is not conserved because there is a force acting which is the central force. So, there is no guarantee that A has to be conserved simply because L is conserved. This L is conserved because of rotational symmetry.

So, there is no guarantee A should be conserved, but in fact, for the specific Coulomb force or the newtons 1 by r squared force A is indeed conserved. So, in order to prove that what we do is we find the rate of change of A with respect to time and you will be able to see that you can write that rate of change in this fashion.

See firstly, you see the rate of change of p cross L is the rate of change of p with respect to time which is the force which is this one and L which is r cross p, but then I am not going to write the p cross d L by d t because you all know that 0 because d L by d t is 0 because L is conserved, but then I also have to write this. So, you see r hat is nothing but r divided by r vector ok, r vector divided by r. So, that is what I have done I have done differentiate one by one and I will get this ok.

So, if I differentiate r and if I differentiate r I get this, if I differentiate r raised to this right. So, this will become minus 1 r raised to minus 2 d r by d t and, but then d r by d t is nothing but 2 r d r by d t is nothing but write 2 r dot d r by d t. So, like that I will be able to rewrite it in this way ok. So, bottom line is that, if I take this out right so that there is a triple product, this is a vector triple product which you can expand like this.

So, yeah there are some steps you have to go through, it is not that obvious. In fact it is not obvious that is the reason why I am discussing it, because it is not at all obvious that there should be A vector like a which is conserved. Because it is pretty much most of the time not a conserved quantity, it is conserved quantity only for central forces that have the form of 1 by r squared force if that was anything else it is not conserved.

So, now this vector triple product is writable in this way and you can see that, when you expand this out all terms cancel out in pairs and you get 0 ok. And the last result follows from the observation that this p is nothing but mass times velocity. So, bottom line is now so I have succeeded in convincing you that their exists a vector called A which is unrelated to L, in the sense that the conservation of L does not guarantee conservation of A. So, that these two distinct vectors L and A and both of which are conserved.

So, now the question is we have successfully pinpointed the symmetry that is responsible for the conservation of L and that is the rotational symmetry. Now, similarly we want to know what is the symmetry responsible for the conservation of A, because a perfectly valid question which we are now going to answer ok.

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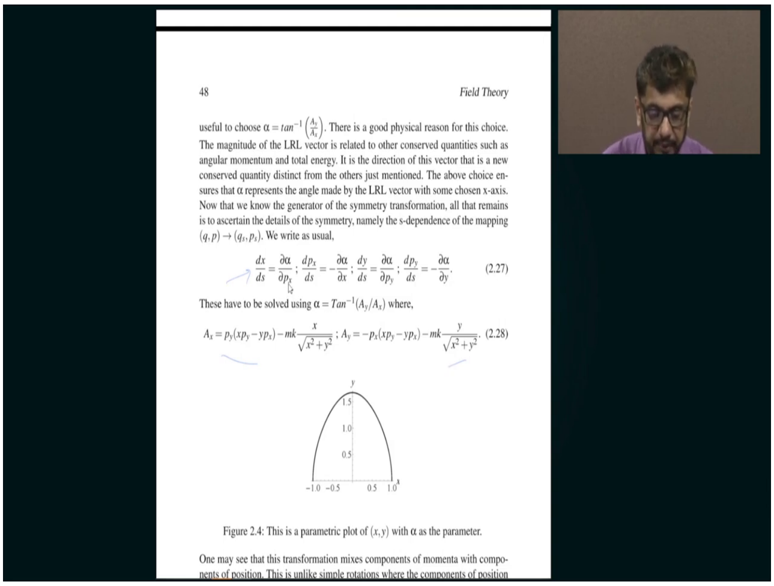
48 Field Theory

useful to choose $\alpha = tan^{-1}\left(\frac{A_y}{A_x}\right)$. There is a good physical reason for this choice. The magnitude of the LRL vector is related to other conserved quantities such as angular momentum and total energy. It is the direction of this vector that is a new conserved quantity distinct from the others just mentioned. The above choice ensures that α represents the angle made by the LRL vector with some chosen x-axis. Now that we know the generator of the symmetry transformation, all that remains is to ascertain the details of the symmetry, namely the s-dependence of the mapping $(q,p) \rightarrow (q_s, p_s)$. We write as usual,

$$\frac{dx}{ds} = \frac{\partial \alpha}{\partial p_x}; \frac{dp_x}{ds} = -\frac{\partial \alpha}{\partial x}; \frac{dy}{ds} = \frac{\partial \alpha}{\partial p_y}; \frac{dp_y}{ds} = -\frac{\partial \alpha}{\partial y}. \quad (2.27)$$

These have to be solved using $\alpha = Tan^{-1}(A_y/A_x)$ where,

$$A_x = p_y(xp_y - yp_x) - mk\frac{x}{\sqrt{x^2+y^2}}; A_y = -p_x(xp_y - yp_x) - mk\frac{y}{\sqrt{x^2+y^2}}. \quad (2.28)$$

Figure 2.4: This is a parametric plot of (x,y) with α as the parameter.

One may see that this transformation mixes components of momenta with components of position. This is unlike simple rotations where the components of position

So, the answer to that question is the following. So, I am going to think of the generator of the symmetry responsible for conserving A to be this some function of this A x and A y, where A x and A y are the x and y components of this Runge-Lenz vector ok. So, we might be wondering what is x what is y. So, of course, the answer is the following, that you see the angular momentum right points in a fixed direction because it is conserved its points in a fixed direction.

Now, I think of that fixed direction as my z axis. So, the x and y directions are perpendicular to that angular momentum direction ok. So, given that you can define x and y in that precise way ah. So, it make sense to talk of A x and A y. So, a x and a y are the x and y components of the Laplace Runge-Lenz vector ok. So, I am further going to say that this alpha is defined as tan inverse A y by A x.

So, in other words what I have done here is that I have postulated that G is the generator of some symmetry. So, where G is defined as inverse tan of A y by A x. Now, given that this is a that definition of this generator. Now, I want to see what sort of a conserved quantity this generator entails, means what conserved quantity does it lead to. So, because G is a generator of a symmetry, it should the transformation that is the putative symmetry that we are talking about the symmetry that this transformation implies should certainly obey this these equations.

So, these are the flow equations with respect to this generator. So, d x by d s is therefore, equal to the generators derivative with respect to p x. Similarly, p x itself will evolve with respect to that continuous parameter and so on and so forth. So, now, given that alpha is tan inverse A y by A x ,we can go ahead and keep in mind that A x and A y can be explicitly written out because we know that A we know the definition of A is p cross L minus M, k r cap.

So, we can explicitly write down A x and A y which is what I have done here. So, this is the explicit construction for A x and A y and given this we can go ahead and evaluate this these flow transformations ok.

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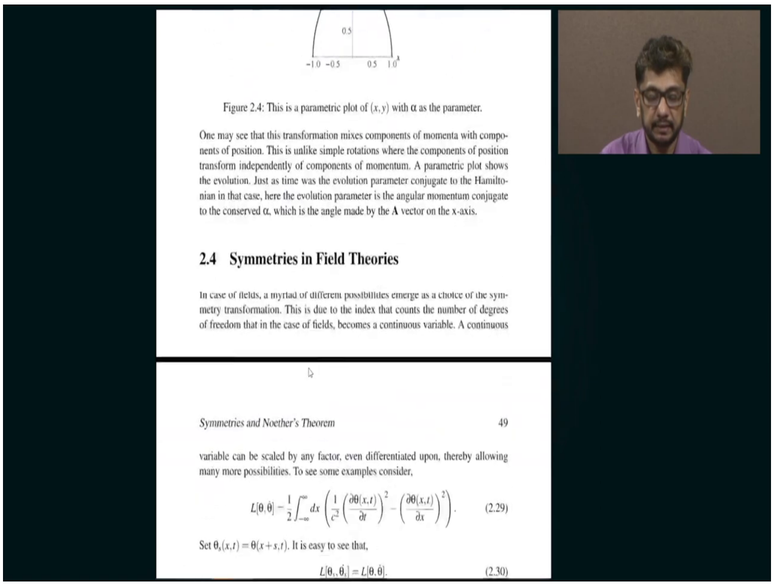

So, you can see that ok. So, you can see that first of all that Hamiltonian description of Noether's theorem guarantees that this G is a conserved quantity ok. So, if G is a conserved quantity then its clear that the symmetry which is responsible for conserving G ok is basically this dynamical symmetry, it is called a dynamical symmetry because you see this symmetry actually mixes momentum and position.

So, see in the earlier case the you could at a given time just rotate the position and it really does not do anything the momentum components do not get mixed up with the position components, where the different position components get mixed up when you rotate. But here it is not like that, the value of the position after rotation depends upon not only the components of the position before rotation, they also depend on the components of the momentum before rotation.

So, see earlier in the case of just ordinary physical rotation, the components after rotation, the position components after rotation depended only on the position components before rotation. But here, it is not its dynamical because the position components after rotation not only depend on the position component before rotation, like it is the case in the case of physical rotation.

But in the case of this dynamical type of symmetry that the momentum components also get mixed up. So, it is a whole mix of the components of momentum, linear momentum position components and so on and so forth. So, they all get mixed up and give you the answers for the position vector and momentum vector after rotation.

So, that is the reason why it is called dynamical symmetry ok. So, I will allow you to think about this and in the next class.

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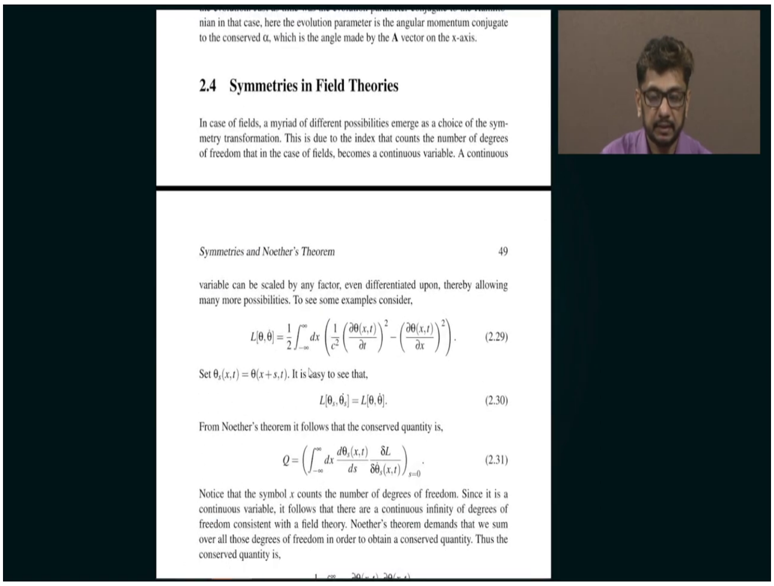
nian in that case, here the evolution parameter is the angular momentum conjugate to the conserved α, which is the angle made by the **A** vector on the x-axis.

2.4 Symmetries in Field Theories

In case of fields, a myriad of different possibilities emerge as a choice of the symmetry transformation. This is due to the index that counts the number of degrees of freedom that in the case of fields, becomes a continuous variable. A continuous

Symmetries and Noether's Theorem 49

variable can be scaled by any factor, even differentiated upon, thereby allowing many more possibilities. To see some examples consider,

$$L[\theta,\dot{\theta}] = \frac{1}{2}\int_{-\infty}^{\infty} dx \left(\frac{1}{c^2}\left(\frac{\partial\theta(x,t)}{\partial t}\right)^2 - \left(\frac{\partial\theta(x,t)}{\partial x}\right)^2 \right). \quad (2.29)$$

Set $\theta_s(x,t) = \theta(x+s,t)$. It is easy to see that,

$$L[\theta_s,\dot{\theta}_s] = L[\theta,\dot{\theta}]. \quad (2.30)$$

From Noether's theorem it follows that the conserved quantity is,

$$Q = \left(\int_{-\infty}^{\infty} dx \frac{d\theta_s(x,t)}{ds} \frac{\delta L}{\delta\dot{\theta}_s(x,t)} \right)_{s=0}. \quad (2.31)$$

Notice that the symbol x counts the number of degrees of freedom. Since it is a continuous variable, it follows that there are a continuous infinity of degrees of freedom consistent with a field theory. Noether's theorem demands that we sum over all those degrees of freedom in order to obtain a conserved quantity. Thus the conserved quantity is,

I will discuss the symmetries in field theory. So, in other words; so till now I have only talked about systems with just one I mean few degrees of freedom, like 3 degrees of freedom or 2 degrees of freedom just point particle in a central force. But this title of this course is dynamics of classical quantum fields. So, I have to quickly come back to discussing fields. So, I am that is the intention right now.

So, in the next class I am going to tell you how to invoke Noether's theorem in the context of fields. So, you can also invoke them Noether's theorem in the context of fields and go ahead and derive conserved quantities in field theory by staring at symmetry. And that is actually even more powerful than it is in the case of point particles because conserved quantities in field theories are even harder to guess compared to point particle theories.

So, but; however, symmetries continue to be somewhat easy to guess both in the case of point particles as well as field theories. So, that is a bone because then we can use that to generate conserved quantities even in field theory. So, I am going to stop here and I hope you will join me for the next class, which is all about symmetries and conserved quantities in field theories ok.

Thank you.