Dynamics of Classical and Quantum Fields: An Introduction
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Symmetries
Lecture - 06
Symmetries and Noether's Theorem

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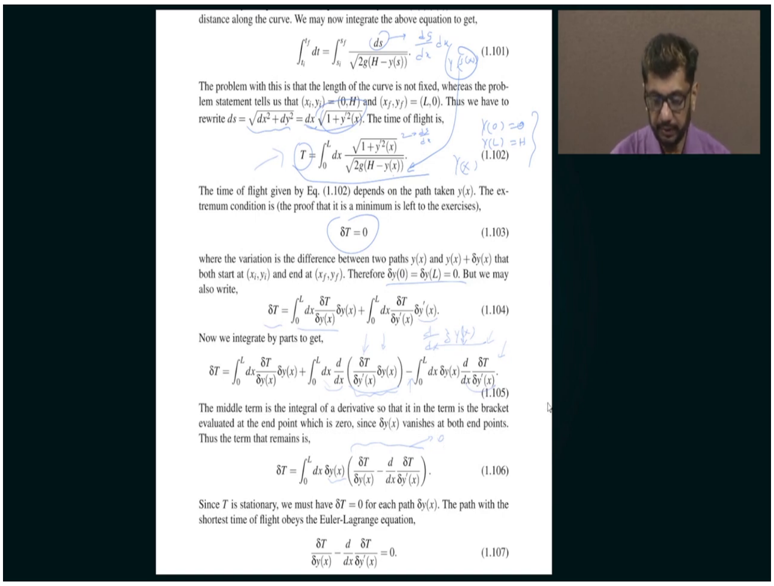

So, let us begin this session by recalling what we did last time, so where we left off. So, if you remember I was discussing the brachistochrone problem which is basically the problem of finding the path in a uniform gravitational field of an object, such that it takes the minimum amount of time for the particle to reach the starting point which is at an elevation and the end point which is at a lower level.

So, your nave guess that it should be a straight line connecting those two points is going to be wrong, because that would of course, be correct if there was no gravitational field. So, when there is a gravitational field, it is not at all clear what that should be. So, let us try and see what is the correct answer to this question.

So, if you remember we spent a lot of time deriving the time taken for the mass to reach the end point starting from the elevation. So, brachistochrone problem this equation 1.102 right tells you what that time taken is. So, the left-hand side capital T is the duration the mass spends on its journey. So, x is the horizontal displacement of the mass. So, x equal to 0 corresponds to the starting horizontal position and x equal to L is where it ends up. So, and then y of x is correspondingly the path the particle takes. So, for every x between 0 and L there is a y which will determine what the path is.

So, we just showed that the time taken is basically given by this formula, where H is the y value when x is 0 ok, so in other words the height with which it starts off. So, bottom line is that this has to be minimized. So, then if you change, so this is a functional. So, capital T is a functional of y, the path. So, you change the function y you get a different answer. So, T is a number y is a function. So, you change the function y, you get a different number for T.

So therefore, T is a functional of y and then you keep changing y until you reach a situation then T is the minimum possible value. So, to determine that we make the assertion that the variation in T. So, basically if you remember your calculus a function is minimum if its first derivative is 0 at some point. So, similarly a function null is minimum if its variation with respect to changes small changes in the function itself is 0 for that particular desirable function which minimizes that time duration.

So, in other words, if y star is the function which minimizes the time duration then delta T equals 0 whenever y is exactly that desirable function y star, which basically is the path the particle takes. So, that the time taken is minimum. So, we use the conventional rules of calculus to find delta T, just like you would find delta of x you take the derivative, but here you take the functional derivative with respect to y.

But keep in mind here that you have to even though you are varying the function y, you are finding different functions that minimize T. But then you have to keep in mind that those different functions should start at the same location and end up at the same location, it is only the shape of the function that changes. So, the question we are asking is that given the end points are fixed.

So, this the end points at the start is x equal to 0, y equal to H, finishes x equal to L and y equal to 0. So, if that is the case, then you see variations in the path will be significant only far away from the end points. Because in end points all the different possible paths converge.

So, that is the reason why I have written that the delta of ys. So, the variation in y, so delta y is the difference between the y values for two different paths, which are close to each other. So, the delta of y is basically 0, it is exactly 0 at the starting point and at the ending point. So, keep in mind; so keeping this in mind we can go ahead and find the variation in T.

So, remember that T depends on two unrelated variables, namely y itself and its derivative. So, y and y dash can be very different from one another especially for some arbitrary curve, at this stage we do not know what y is. So, y can be anything so y and y dash are completely unrelated. So, if you want to find the variation in T, if you have to first differentiate with respect to y and then find the derivative with respect to the with respect to y dash and multiply with respect to the multiply with the variation in y dash, because y and y dash are at this stage unrelated ok.

So, now, I am going to do the following. I am going to rewrite this, so basically I am going to think of this as d by dx of delta y, then I am going to take this derivative outside, but then I will be making a mistake by doing that because then I will get a spurious derivative with respect to this, which I will cancel out by putting in a minus sign. So, I got this spurious term which I am by hand cancelling out.

So, if I really want to take the derivative outside for obvious reasons, then this becomes the integral of a derivative which I know how to do. But then the price I have to pay is that it introduces this term, which was not there earlier which I have to necessarily get rid of by subtracting it out this way.

So, bottom line is that when I do this that you see this is easy because the integral of the derivative is this function itself and this function itself is being evaluated at x equal to 0 and x equal to L. I just told you that delta y is 0 at x equal to 0 and it is also 0 at x equal to L, because that is the change in y for different values of x.

So, having gotten rid of this middle term, we can just go ahead and combine this and you know separate out or you know take this delta y as common factor outside. And then we get this nice looking formula for the variation in the time duration or the time taken for the mass to end up at this end point starting from its initial location. So, keep in mind that, so this is going to be 0 ok. So, the bottom line is that this delta T should be 0 for any change in y.

So that means, that regardless of what this delta y is this should still be 0. So, for any delta y obeying the constraint delta y at x equal to 0 equals delta by at x equal to L equals 0. So, so long as that is obeyed this delta T should be 0 for every delta y, which obeys those end point constraints. So, that is going to happen only if this itself is 0 ok.

So, that is precisely what will determine, that the path that the particle will take in order to minimize the time duration, the time it takes for it to reach the end point.

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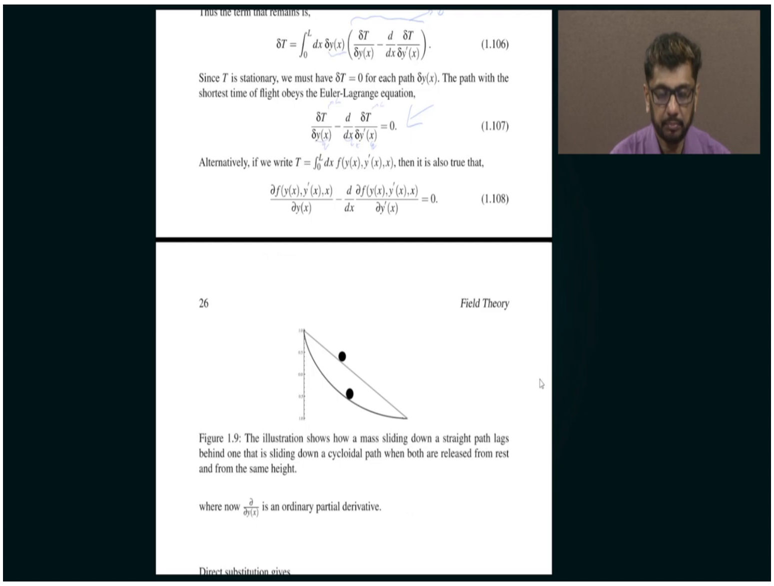

So, this is analogous to the Lagrange equation. So, if you recall, so there was a situation where this was the; so if instead of x you imagine it was time and you imagine this capital T is your Lagrangian; then what would this correspond to?

This is $\frac{d}{dt}(\frac{dL}{d\dot{q}}) = \frac{d}{dq}(L)$. So, imagine this is q, imagine this is L, imagine this is t and this is $\dot{q}$. So, this is going to be $\frac{d}{dt}(\frac{dL}{d\dot{q}}) = \frac{d}{dq}(L)$. So, Lagrange equations, this is not surprising because Lagrange equation themselves are consequence of an extremum principle ok. So, bottom line is that we really have to work this out ok.

So, now let us work this out. So, if you work this out delta T by delta y dash is just this. So, you think of y and y dash has two independent variables, because now we know what T is, T is this. So, just work out the derivatives ok.

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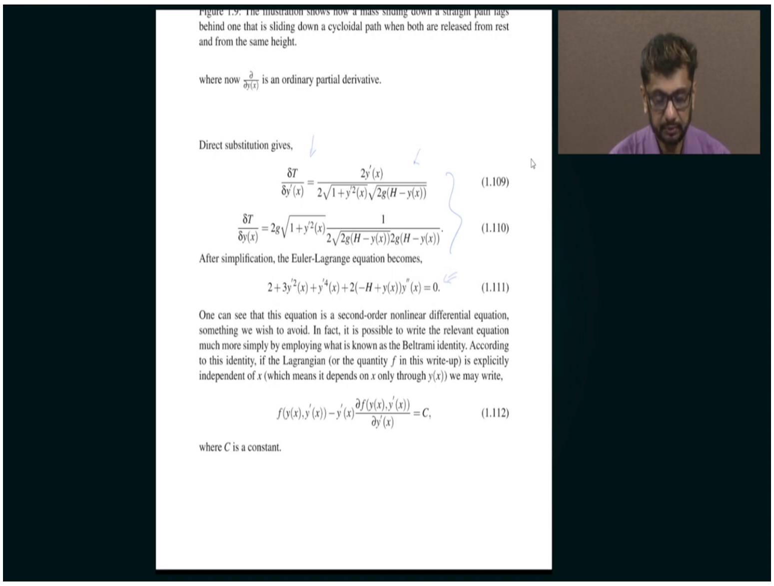

So, delta T by delta y dash is this and delta T by delta y is this. So, then you just substitute these two into your Euler Lagrange equation which is this one. So, which will then tell you the equation that y of x has to obey this equation in order for T to be minimum.

So, if T has to be a minimum y has to obey this equation. Well, there are some technical issues which mean that you can avoid the second derivative type of equation, which is somewhat complicated to deal with. So, you can actually work with a; just like you know you can avoid solving the second order Newton's second law equation by; so that is what the Lagrange equations would be it, would be second order in time.

So, you can avoid that by using the Hamiltonian approach, where the Hamiltonian you know is a constant and that is basically equal to you know p and q and p is itself a first order in the generalized coordinate. The first-time derivative of the generalized coordinate. So, similarly even in this case you can do this and analogous idea in the, in this context is called the Beltrami identity, ok.

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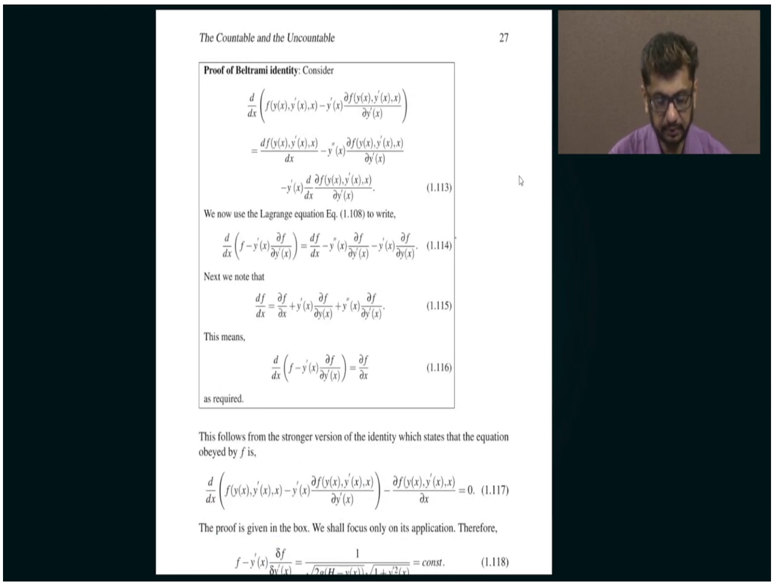

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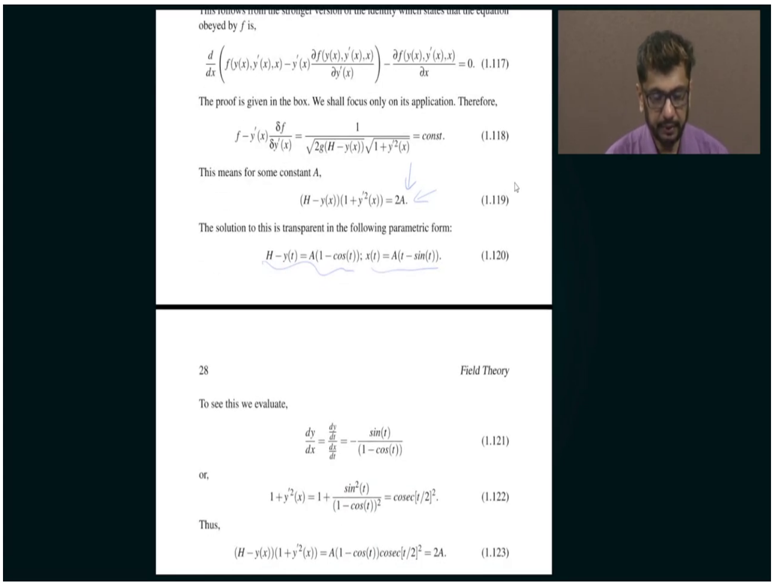

So, that is analogous to using energy conservation. So, that is; so when you do that you get this sort of an equation. So, I will allow you to go through the details yourself from the notes and the book. So, I otherwise it is a little bit technical I do not want to bore you with the details. So, bottom line is that you can you know introduce the analogue of the Hamiltonian and make the assertion that that Hamiltonian is a constant.

So, here also there is some analogue of that and that is going to be a constant; and then. So, the equation is going to simplify a lot. So, instead of being second order like it was earlier, it is going to be first order ok. So, the first order equation that I am going to be required to solve is 1.119, which is this one ok. So, that is not difficult to solve because you can easily see that the solution that we are looking for has this parametric form ok.

So, you can just go ahead and substitute this and you will see that it is obeyed ok. So, of course, you might think that that is a little too quick. So, well you can do it the long way, but bottom line is this is a very standard problem and everybody knows the answer to this. So, nobody spends too much time understanding how these answers were arrived at, but bottom line is that, you know worst case you can just assume this is the answer and substitute this back into the equation and then convince yourself that this is indeed the solution.

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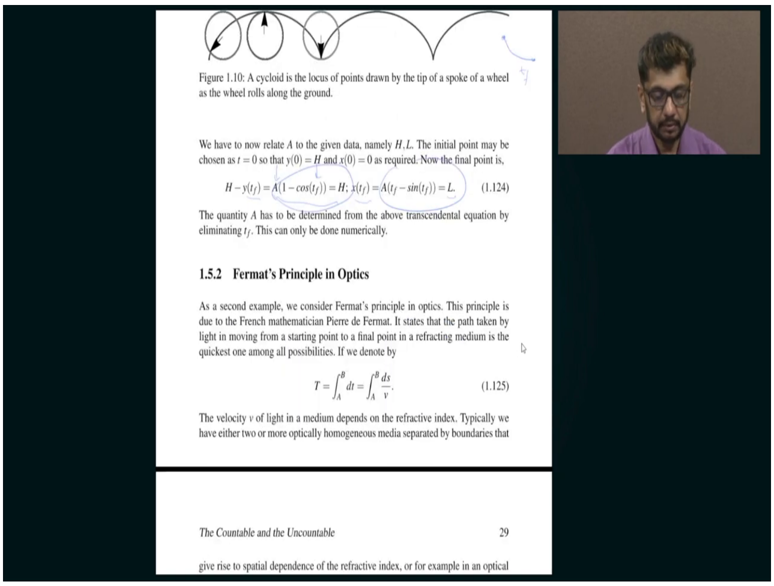

So, you see it also obeys; so, we have to of course, this these it involves some integration constants like capital A and all that. So, we have to relate that to some other things that we are more familiar with. So, now, assume that t f is the time taken for the mass to reach the end point, in that case you can see that this t f and A are linked in this manner.

So, this is going to be this because remember at the end point y is 0 and in the beginning. So, well at the end point x is L and y is 0. So, that is how it looks ok. So, that is going to indirectly tell you what the time duration is and you can be guaranteed that this time duration that you arrive at by solving these equations so; that means, by L. So now, there are two unknowns t f and A, but then there are two equations this and this. So, you can eliminate them and get both.

Of course, t f is more interesting. So, that will tell you the time duration t f is the time, the mass takes to slide from its starting point to its ending point. And the path it takes is what is called a catenary right. So, that is the path it takes.

So, and the time duration is t f and is guaranteed to be a minimum ok. So, this is one nice application of the variational method and you can see that it involves the use of functional derivatives and that is basically the infinite dimensional version of the ordinary derivative, which is why it deserves to be under this chapter which is titled countable and uncountable.

So, I am just trying to introduce you to the concept of fields, through an example where the number of degrees of freedom are infinite and this is one such example where you have a functional, it depends on infinitely many variables in a continuous way ok.

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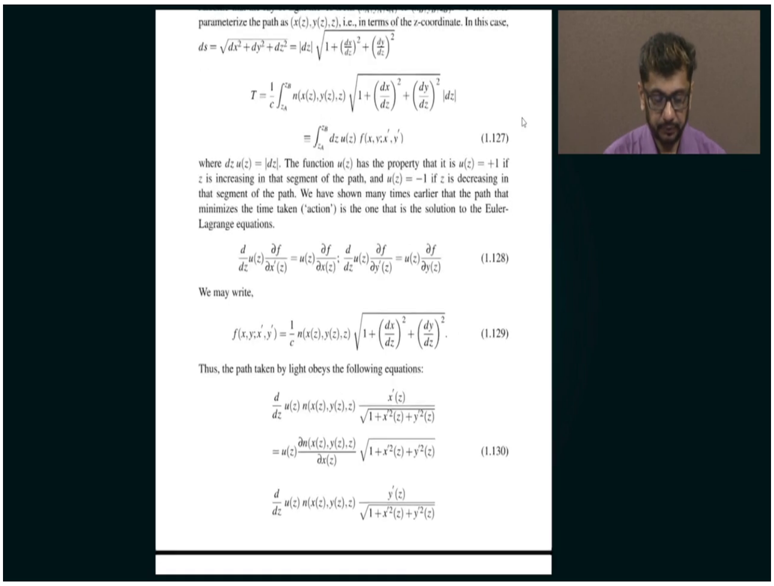

So, the other example of functional calculus so, the earlier one was the brachistochrone problem the other example is Fermat's principle in optics. So, even Fermat's principle in optics is similar, he stated that the time duration to go from. So, if you imagine a beam of light passing through refractive medium, the time duration for the light to reach from its starting to ending point is the one which minimizes basically the time taken. So, but the time taken is basically the distance travel divided by speed and then speed is of course, function of the refractive index.

So, then if you go ahead and write v as c by n, you are going to get this as your time taken, but then your refractive index can be path dependent. So, basically it depends on what path your light takes. So, imagine that your path taken is determined by x. So, its path is by definition a one parameter family of points and that parameterization can be done by choosing z as your parameter. So, z is the z coordinate of the point.

So, that itself can serve as a parameter in which case x and y depend on z. So, therefore, it gives you a one parameter family of points which is basically a path. So, now, what you are going to do is you are going to be called upon to minimize a function like this. Now, this is a functional of not just one unknown function like x or y, but it is a function of both of them x and y both.

So, the question is how do you do that? Of course, you do it the same way and you keep repeating your approaches, means you differentiate with respect to one of them and you differentiate with respect to the other subsequently.

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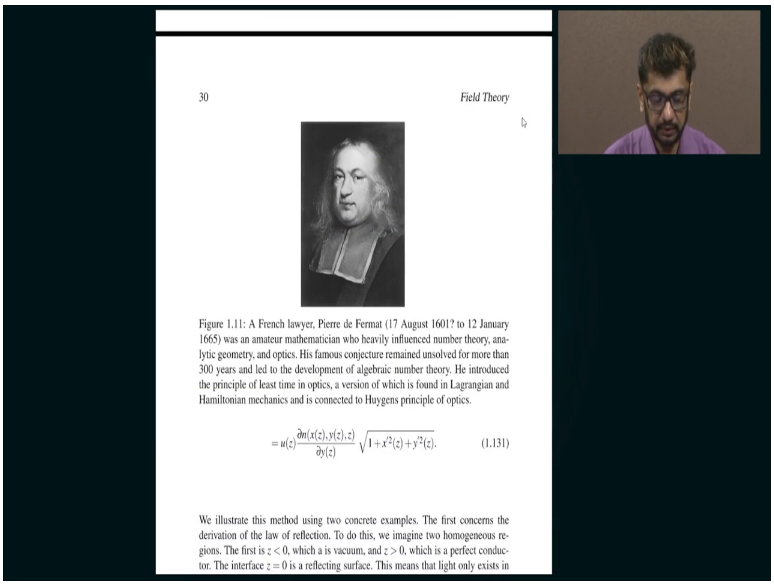

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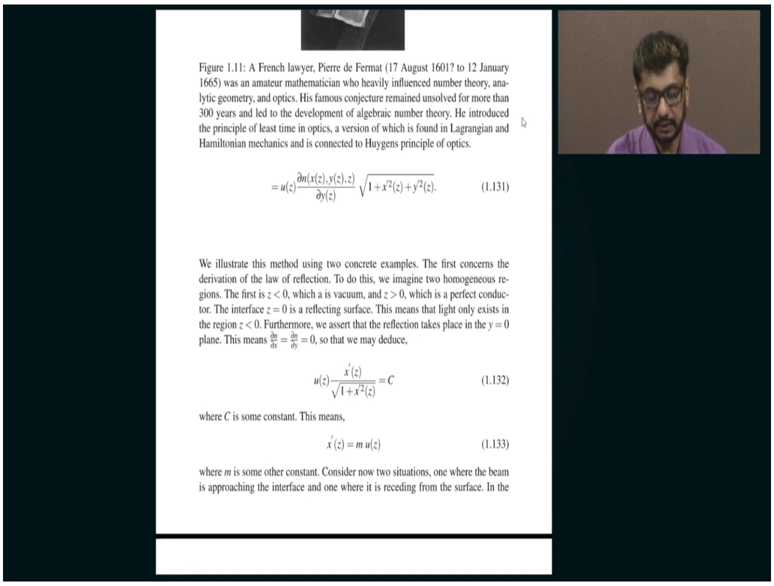

So, I am not going to bore you with the details, but bottom line is that just like that Beltrami type of idea, you can apply it the similar type of ideas here also and you can then derive some analogous equations ok. So, I have applied this idea to study reflection ok. So, for example, the; so this idea can be used to study various or not just study, but derive various laws which are well known in optics, such as the law of reflection, then Snell's law of refraction and so on.

So, I have used this idea to derive the law of reflection so; that means, angle of incidence is same as angle of reflection. So, that can be proven by, so you do not have to assume that you can think of that as a consequence of Fermat's principle ok.

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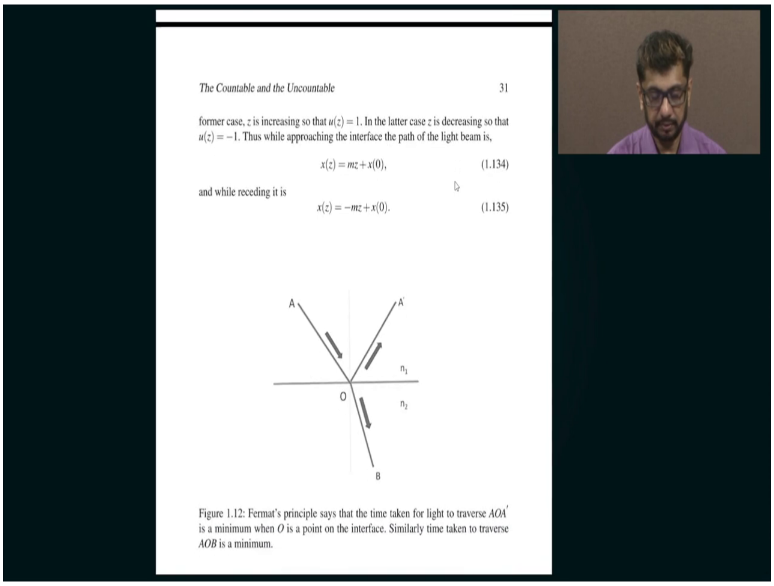
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former case, z is increasing so that $u(z) = 1$. In the latter case z is decreasing so that $u(z) = -1$. Thus while approaching the interface the path of the light beam is,

$$x(z) = mz + x(0), \tag{1.134}$$

and while receding it is

$$x(z) = -mz + x(0). \tag{1.135}$$

Figure 1.12: Fermat's principle says that the time taken for light to traverse AOA' is a minimum when O is a point on the interface. Similarly time taken to traverse AOB is a minimum.

So, I have just gone ahead and do it, I do not want to spend too much time discussing the details you can read it yourself.

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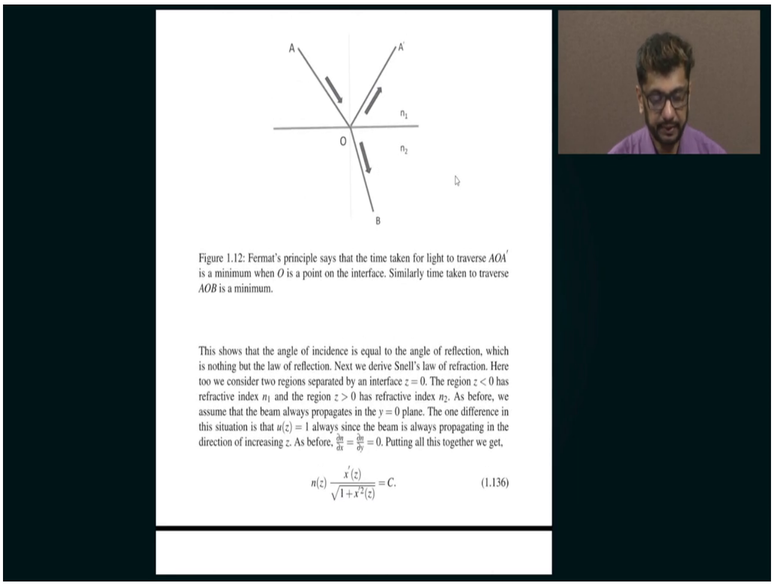

So, similarly, I have I have studied refraction. So, you have two media, one of them is homogeneous with refractive index n 1, but then there is a surface beyond which there is another medium homogeneous with refractive n 2. So, the in homogeneity is abrupt at the surface. So, when light is incident on the surface interface between these two media, then you will see that there is not only refraction which you expect there is also an reflection.

So, whenever you have an interface between two media, you also have a reflection. So, then you can work out, you can derive both these. So, Snell's law rather Fermat's principle directly tells you that not only is there a refraction that is going on that obeys Snell's law. But Fermat's principle also for the same effort tells you that there is a reflection going on that obeys the laws of reflection, namely the angle of incidence is equal to the angle of reflection.

But of course, as you very well know the angle of refraction is certainly not equal to the angle of incidence, because that is going to be determined by Snell's law and the relative refractive index across the medium, across the two media ok.

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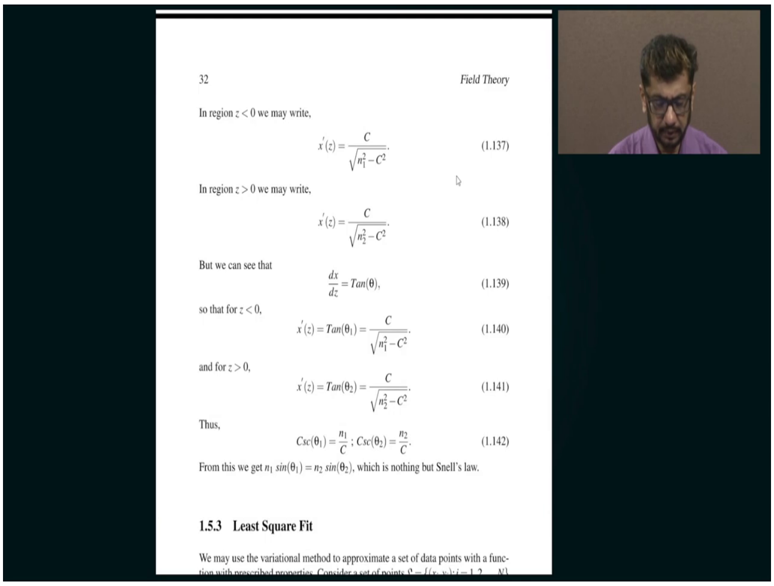

So, that is the bottom line. So, you get this famous Snell's law, n 1 sin theta 1 equals n 2 sin theta 2. So, this is a consequence of Fermat's principle.

So, even in Fermat's principle you can see that the derivation of Snell's law and the law of reflection involves first writing down the time taken as the minimum and as an extremum of some functional and then deriving the form of that functional in different situations ok. So, this is another example of functional calculus which is basically another way of thinking about systems with infinitely many degrees of freedom.

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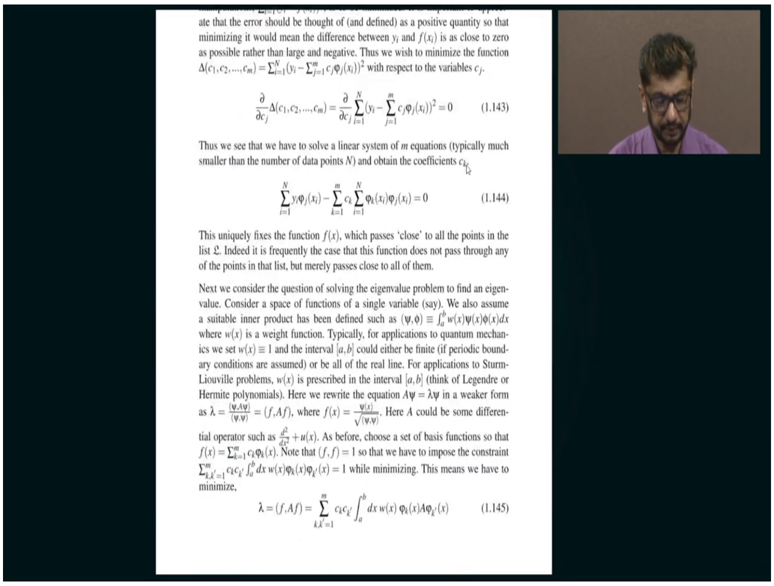

So, the last example is the least square fit which of course, is more obvious and I think I will skip this, because it has less to do with infinitely many degrees of freedom; it is closer to what we were basically it is just another way of minimizing the error and so on.

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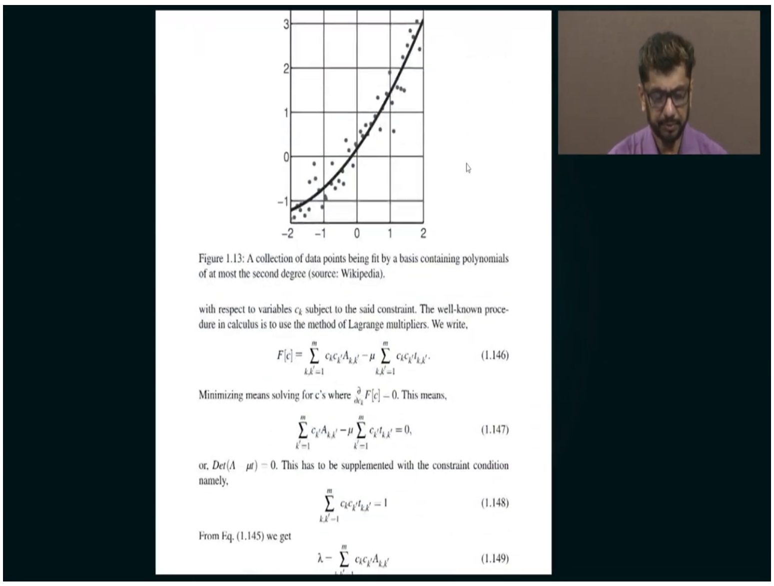

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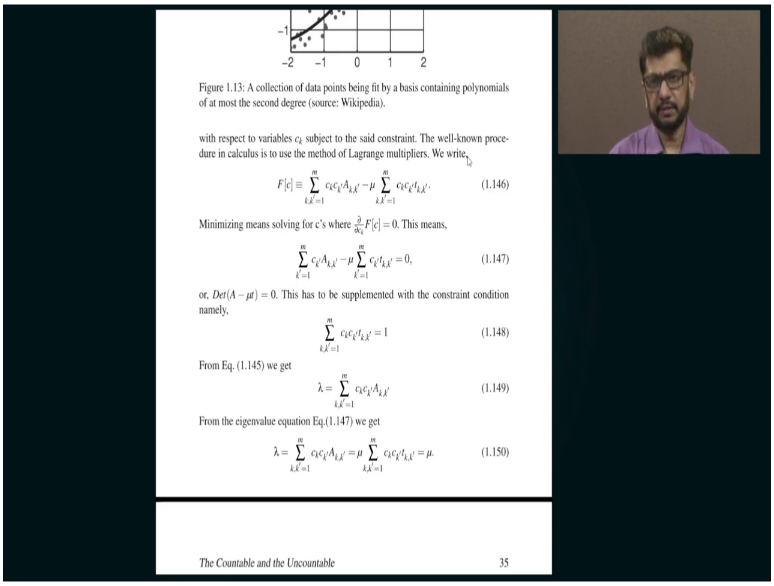

But it, since it has less relevance to systems with infinitely many degrees of freedom I am going to skip it, but I included it just because you know it has wide applications in many areas of physics and engineering ok.

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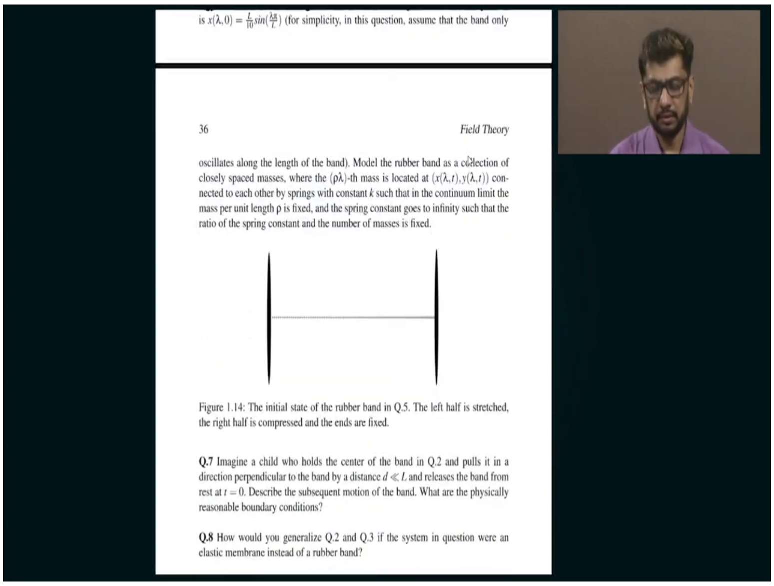

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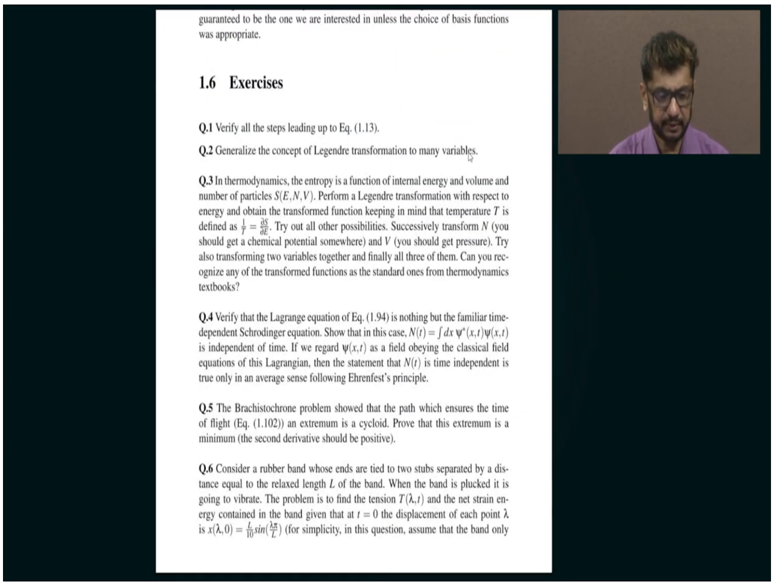

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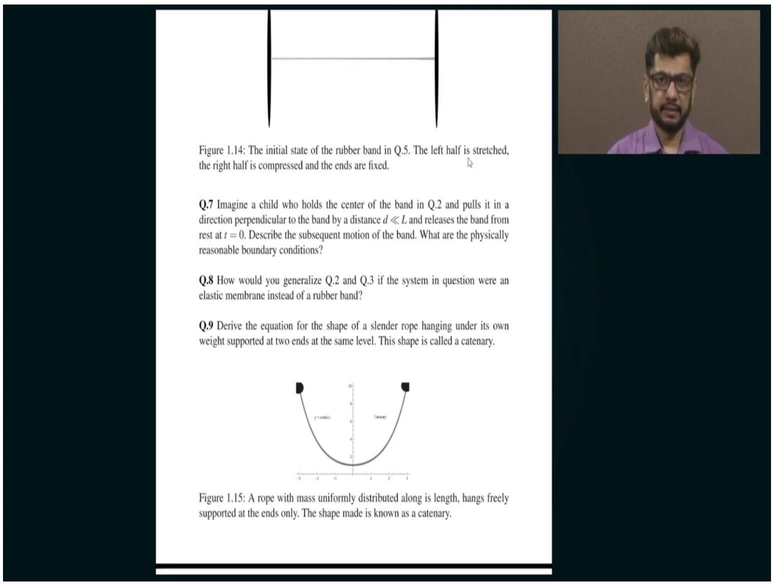

So, basically you know the end of this chapter has several exercises, as you can see here in page 35, there are several exercises which are very important and I have spent considerable amount of effort in creating these exercises. And as publisher insists of course, that there should also be detailed solutions to each of these questions and which is available with the publisher.

So, it is not like these questions I have just been thrown at you and there are no solutions possible. So, I have made sure that each of them can be solved and there are sensible solutions available if you are interested in knowing what they are. So, I strongly urge you to work out these questions on your own and contact me if you have any doubts about any of those answers ok.

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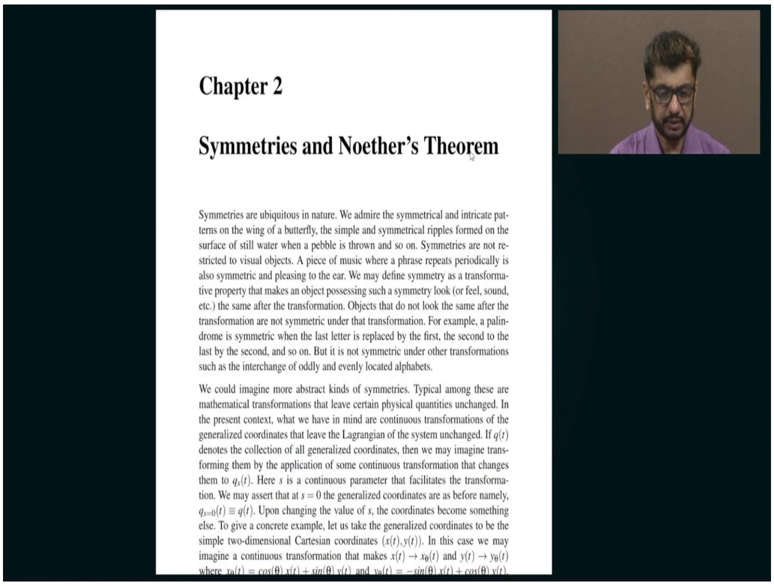

So, now, I am going to go ahead and jump to another topic which is extremely important and that topic is basically the idea of symmetries and how they lead to conservation laws. So, we can see that you know the word symmetry is a very familiar one and we use it in our everyday conversations. So, symmetries are ubiquitous in nature. So, we admire for example, the symmetrical an intricate patterns on the wings of a butterfly the simple and symmetrical I am just reading of this paragraph.

So, the simple and symmetrical ripples formed on the surface of still water which is a simpler type of symmetry and then yeah so there are several such visual symmetries that you can think of, that you encounter constantly in your everyday life, but then symmetries are not restricted to visual phenomena. So, there are other types of experiences which are also which also lend themselves to description in terms of symmetries.

For example, in music. So, in music typically pleasing composition will have a phrase that repeats frequent or periodically and that is pleasing to the ear and even drums. So, you have drums when they are played, you know there is a pattern which repeats and that pattern before it repeats it can be quite intricate and complicated, but then the same pattern repeats and that is typically a hallmark of some melodious piece of music.

So, if there is no repetition of any kind, we usually do not think of such auditory experiences as being musical. So, bottom line is that some sort of a symmetry is there in all our day-to-day experiences and it looks like the human biology is uniquely tuned to be sensitive and pick up on these symmetries. So, they are very good at picking up on these symmetries.

So, we can actually define mathematically the notion of symmetry can be made quite rigorous by defining symmetry as the property of an object, that is unchanged under some transformation. So, I define symmetry as the property of an object which remains unchanged if I do a certain transformation. So, I have given this example, it is not an example that you usually encounter in other places, but I found this interesting.

For example, a palindrome. So, palindrome is basically a word which looks the same if you read it backwards, but then that is exactly the symmetry that I am discussing. So, then you flip the first letter with the last, the second with the penultimate one and so on you get back the same word. So, there is the palindromic words are symmetrical or invariant under this particular transformation, but they are not symmetrical under other types of transformations.

For example, you replace the odd letters with the even ones and so on. So, the message there is that symmetries; so in other words objects are symmetrical only under certain specific set of transformations, they may not be under other sets of transformation. So, we have to be, we cannot make a blanked statement this object is symmetrical, you have to say under what transformation.

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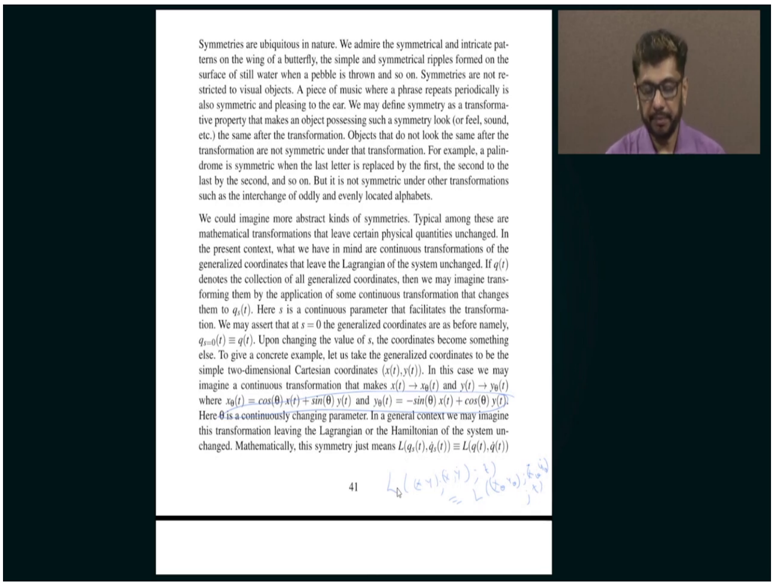

So, we can of course, so these are all visual type of symmetries which are easy to appreciate. But in mathematics and physics we encounter more abstract kinds of symmetries and these symmetries are the mathematical symmetries which leave certain physical quantities unchanged.

So, for example, if; so if you have a Lagrangian which depends upon some generalized coordinates and if q denotes the collection of all generalized coordinates, we may imagine a transformation which changes all those generalized coordinates to some other coordinates. So, q gets mapped to q of s where s is some continuous variable.

So, you I continuously deform each of those generalized coordinate to some other some other one now. So, suppose I can do that and yet if the Lagrangian remains the same, even after I do that so that is somewhat surprising. So, you should be able to find a continuous transformation that transforms your generalized coordinates into some other set of generalized coordinates continuously. So, that means, all the intermediate set of coordinates are also legitimate generalized coordinates.

And then you reach the end set of generalized coordinates and you ask yourself that the Lagrangian of the final set of generalized coordinates is that the same as what it was earlier. If the answer is yes, then that is a kind of symmetry and that is a kind of continuous symmetry that that means, that the Lagrangian is unchanged under a continuous symmetry. So, you might be thinking this is very unusual and very hard to find perhaps, but you will be surprised that it is not hard to find. In fact, it occurs very frequently.

For example, imagine I have a particle in 2-dimensions described by position coordinate x comma y. So, x bracket t represents the position x position of the particle at time t and y bracket t is the position y position at time t. So, that is the position vector of the particle in 2-dimensions at time t. So, now, I am going to ask myself what if I rotate my coordinate system?

So, at every time I rotate my coordinate system by an angle theta. So, then I am going to, what I am going to do is, when I do that I will be describing the particle not in terms of x and y, but rather in terms of x theta and y theta. And x theta and y theta are basically linear combinations of x and y ok. So, this is an example of a continuous transformation ok.

So, this is an example of a continuous transformation. Now, the question is that if you are able to find a Lagrangian of a system, which is unchanged under a continuous transformation. So, in other words your L x y x dot y dot is same as L x theta y theta x dot theta y dot theta. So, remember that Lagrangian is a function of q and q dot. So, in my case in this present case q is nothing but x comma y and q dot is nothing but x dot comma y dot.

So, if your L of x comma y x dot comma y dot is same as x theta comma y theta and x dot theta comma y dot theta so; that means, you replace x by x theta y by y theta, your Lagrangian is unchanged then you call this as continuous symmetry. So that means, it is a symmetry of the Lagrangian, the Lagrangian is unchanged by this transformation. So, why am I even mentioning this? So, what means sure you can perhaps find such Lagrangian’s which do this ok.

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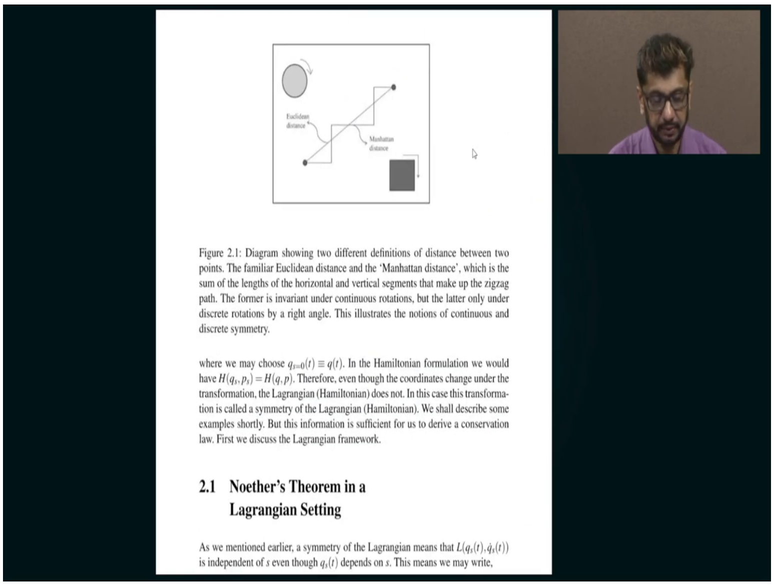

So, now the main point here is that if; so this is the famous Noether's theorem. So, what Noether's theorem says is that, if such a transformation is exists then you will be able to find a quantity which of the dynamical system that you are looking at which is described by that particular Lagrangian, you will be able to find a quantity that is conserved. That means, it does not change with time.

So, that is the remarkable statement, because you see conserved quantities are very important in dynamical systems and they are often hard to guess, except very obvious ones like energy and so on. But there are other quantities which are hard to guess. So, it is nice to know that you do not have to guess, rather what you have to do is you have to look for symmetries of the Lagrangian and make sure those symmetries are continuous.

And if they are continuous you are guaranteed to be presented with courtesy Noether's, you will you are guaranteed to be presented with a quantity that is conserved for each of those symmetries that you have uncovered. So, that is amazing and it is really worth knowing how that comes about ok.

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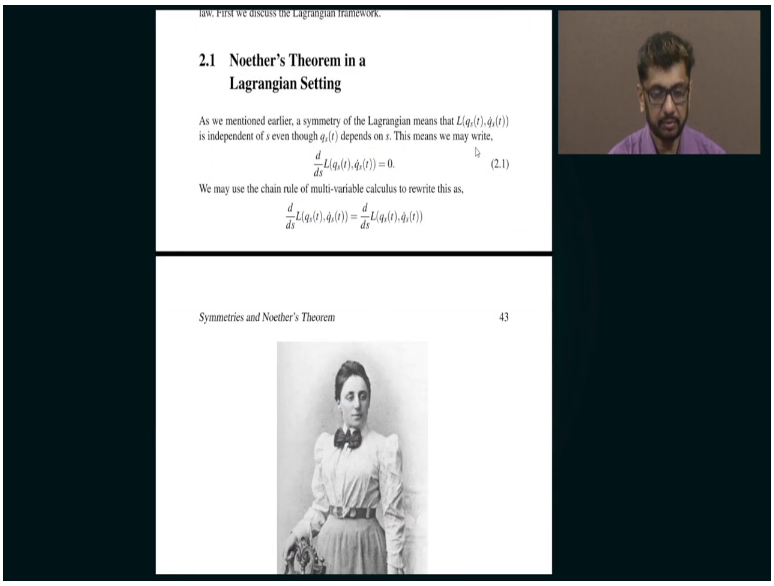

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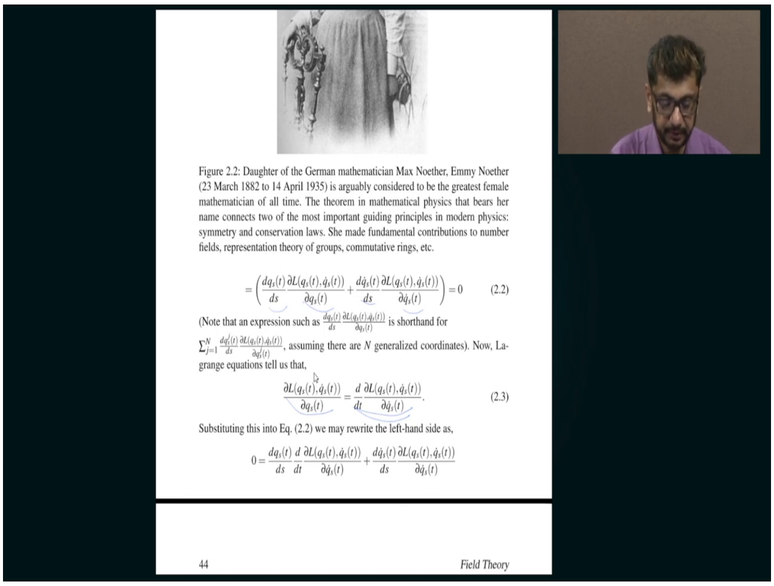

So, let me tell you how that comes about. So, this is before I tell you how that comes about, let us admire this wonderful portrait by the great mathematician Emmy Noether, whose name is associated with this famous theorem. And she was the person mathematician who proved this theorem and Einstein regarded her as one of the greatest mathematicians of her generation.

So, let us now go back and see how she proved this and the way she proved this is. First let us start with the obvious assertion that; that means, So, we have said that there exists a continuous symmetry of the Lagrangian. So, what does that mean?

So, if I change q to q subscript s. So, s is continuous right, so q has been deformed into q of s. So, even though q has deformed into q of s which is completely different from q and therefore, q dot also has deformed into q dot subscript s, which is also completely different from q dot. So, even though q and q dot have completely changed now, but the Lagrangian itself does not change. So, that is what we mean by symmetry.

So, if such a situation is possible so; that means, we have to assume that this is possible. So, if it is possible; so now, the question is what are the consequences? So, first let us write down the assertion of the statement that it is in fact, possible to do that. So, if that is a symmetry of the Lagrangian what; that means, is that the Lagrangian does not depend on s.

So, regardless of how you have deformed whether or not you have deformed this q into q of s, the Lagrangian does not care. So, it is derivative with respect to s is 0. So, the Lagrangian remains unchanged even though you have deformed the qs and q dots. So, now, let us see what is the consequence right of the statement. So, the consequence of this statement is that, unfortunately I have done this twice and this is redundant ok.

So, how do you find the derivative of L with respect to s. So, now, L depends on s through its dependence of its dependence on q and q dot, each of which depend on s. So obviously, you have to invoke the chain rule which now says that in order to find the rate of change we have to first differentiate with respect to q of s, and then we differentiate with respect to s; that means, we differentiate q of s with respect to s.

Similarly, with q dot, but now keep in mind that the Lagrange equations will tell you that you can make this statement; that means, you can write dL by d q s as d by dT of dL by dq dot ok, so that is Lagrange equations. So, I am going to make use of that and go ahead and substitute. So, instead of this I am going to write this ok. So, I am going to write this, replace this with this ok.

So, when I do that this equation which is 0, because remember what that is, that is d by d s of L which is 0. So, that is going to be rewriteable as this ok.

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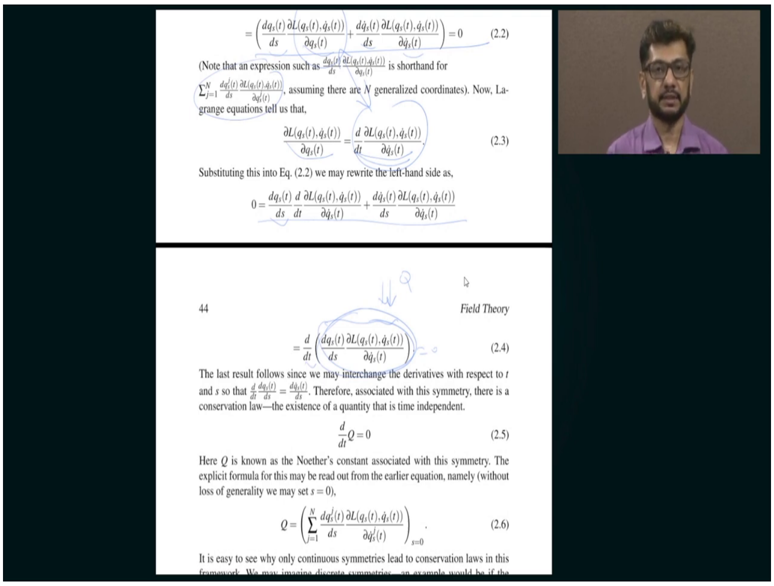

So, now this is nothing but. So, I can pull out this time derivative outside and this becomes just this ok. So, the last result follows from the fact that the derivative with respect to time and derivative with respect to s are interchangeable. So, this means that, since this is 0 what this means is that this quantity is unchanged ok.

So, remember that q is a shorthand for several generalized coordinates. The system can have several generalized coordinates, typically most interesting physical systems have several more than 2 3 or more ok. So, in that case you see so this is; so that is the reason why I have pointed this out. So, it is just summation over all the possible generalized coordinates. So, what Noether’s theorem, so this is Noether’s theorem. So, we have just successfully proved that the moment there is a continuous symmetry it implies a conservation law.

And this conservation law is basically not only does it guarantee a conservation law. So, it also tells you what is conserved, what is conserved is basically this quantity. So, this quantity I have called as q and this is called the Noether’s constant ok. So, what so what we have succeeded now in doing is that we have explicitly been able to construct a quantity, which is conserved, which does not change with time, just because there is a continuous symmetry in the problem. That means, the moment you are able to spot a symmetry in the problem that immediately means there is a conserved quantity.

So, you see the bottom line is that you know humans are biologically perfectly tuned to spot symmetries. So, for some reason, it is probably an evolutionary adaptation that we can recognize symmetries faster than most other creatures perhaps or maybe as well as other creatures. But bottom line is that it is something very innate and intrinsic to us and we readily appreciate symmetries.

So, even when the symmetries are abstract, like they are in the Lagrangian it is not that difficult for us to spot them. But however, it is incredibly hard for us to spot conserved quantities in a dynamical system so that is the, that is the bottom line here. So, the moment you are able to spot a continuous symmetry which is easy to spot, because we are humans and the moment you are able to spot a continuous symmetry Noether guarantees you that not only there is a conserved quantity, she even tells you what it is that is conserved and that is amazing.

So, and conserved quantities are really important in physics as you very well know. So, in the next class I will discuss the application of Noether's theorem to various dynamical systems and you will see that in the case of central forces, well Hamiltonian of the system is of course, conserved, but we will even identify what is the symmetry that is responsible for the conservation of Hamiltonian.

So, in fact, the converse is also true we just proved that for every conserved quantity there is a or rather for every continuous symmetry there is a conserved quantity. So, the question is the natural question is the converse true; that means, that for every conserved quantity is there a continuous symmetry; the answer is yes.

So in fact, we will be able to identify them. So, we will be able to identify the symmetry which makes the Hamiltonian a constant of the motion, we will be able to identify the symmetry that makes the Lagrangian a constant of the motion. And for inverse square forces not just central force, but you know the Coulomb's law type of force there is a third conserved quantity which is called the Runge Lenz vector.

And there is of course, a symmetry associated continuous symmetry which leads to the conservation of Runge Lenz vector, but that is more subtle. So, that is what is called a dynamical symmetry, which I am going to discuss later ok. So, but bottom line is it is still a symmetry.

So, it is a symmetry which leads to the conservation of the Hamiltonian, is a symmetry that leads to the conservation of angular momentum for the case of free particles it is a symmetry, the translational symmetry which leads to the conservation of linear momentum. In the case of inverse square forces central force is the dynamical symmetry, which leads to the conservation of the Runge Lenz vector.

So, the moral of this lecture is that, pretty much behind every conserved quantity there exists a symmetry, a continuous symmetry ok. I am going to stop here and in the next class I will give you all these examples which will convince you about the power and usefulness of this theorem ok.

Thank you, see you in the next lecture.